

MS. L - PRIVATE
DEPOSITORY

SHL
WITHDRAWN



PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY

SECOND SERIES

VOLUME 31

LONDON :

PRINTED BY C. F. HODGSON & SON, LTD.,

2 NEWTON STREET, KINGSWAY, W.C.2;

AND PUBLISHED FOR THE SOCIETY BY

FRANCIS HODGSON, LTD., 15 ST. BRIDE STREET, E.C.4.

1930



C O N T E N T S.

Papers published in the *Proceedings*, pp. 1-553.

Index of Papers published in this Volume arranged according to the names of Authors, p. 554.

Index according to Subjects, p. 556.

Corrigenda, p. 556.



Digitized by the Internet Archive
in 2026

London Mathematical Society.

MEMBERS ELECTED DURING THE SESSION 1929-30.

- AITKEN, ALEXANDER CRAIG, M.A. New Zealand, D.Sc. Edinburgh, F.R.S.E.; Lecturer in Statistics and Actuarial Mathematics, University of Edinburgh; 2 *Sycamore Terrace, Corstorphine, Edinburgh.* [March 13, 1930]
- BAKHOO, NAGUIB GHALI, B.Sc. Leeds; 137 *Harehills Lane, Roundhay, Leeds.* [Feb. 6, 1930]
- BARTER, JOHN DOUGLAS, B.A., Ph.D. Toronto; *The University, California, U.S.A.* [Dec. 12, 1929]
- BESICOVITCH, ABRAM SAMOILOVITCH, M.A. Cambridge; Cayley Lecturer in Mathematics in the University of Cambridge; *Trinity College, Cambridge.* [Jan. 16, 1930]
- BIGGERSTAFF, JOHN NICHOLS, M.Sc. Washington; *State College of Washington, Pullman, Washington, and 2045 E. Conch Street, Portland, Oregon, U.S.A.* [Dec. 12, 1929]
- BISHARA, SADEK BISTOUROS, B.Sc. Leeds; 137 *Harehills Lane, Roundhay, Leeds.* [Feb. 6, 1930]
- BRAHMACHARI, INDU BHUSHAN, M.Sc. Calcutta; 110/2 *Lake Road, Kalighat, Calcutta, India.* [Nov. 14, 1929]
- CATON, CYRIL THOMAS LEAR, M.A. Cambridge, M.Sc. London; *Warwick School, Warwick.* [Feb. 6, 1930]
- CHOWLA, SARV DAMAN SINGH; *Trinity College, Cambridge.* [Dec. 12, 1929]
- DAVIES, EVAN TOM, M.Sc. Wales, Dottore in Matematica, Rome; *Brynaeson, Pencader, Carmarthenshire.* [April 24, 1930]
- ESTERMANN, THEODOR, D.Sc. London; 142 *King Henry's Road, London, N.W.3.* [Nov. 14, 1929]
- GERGEN, J. J., Ph.D. Harvard; *Harvard University, Cambridge, Mass., U.S.A.* [June 19, 1930]
- GRIMSHAW, MARGARET ELEANOR, B.A. Cambridge; *Newnham College, Cambridge.* [Dec. 12, 1929]
- HARTREE, DOUGLAS RAYNER, Ph.D. Cambridge; Beyer Professor of Applied Mathematics, University of Manchester; *The University, Manchester.* [Nov. 14, 1929]
- ICHIDA, ASAJIRO; c.o. Prof. T. Kubota, *Tôhoku University, Sendai, Japan.* [Dec. 12, 1929]
- IZUMI, SHIN-ICHI, 32 *Kitazaimoku chô, Sendai, Japan.* [Dec. 12, 1929]
- LITTLEWOOD, DUDLEY ERNEST, M.A. Cambridge; *University College, Dundee, Angus.* [March 13, 1930]
- MAUNSELL, FREDERICK GEORGE, M.A., Ph.D. Cambridge; Lecturer in Mathematics, University College, Southampton; 123 *University Road, Southampton.* [April 24, 1930]
- NAYLOR, VERNON, M.Sc. Liverpool; 11 *Lower Lyndhurst Road, Plymouth, S. Devon.* [Dec. 12, 1929]
- OWEN, PETER MACAULAY, B.A. Oxford; *Jesus College, Oxford.* [Dec. 12, 1929]
- PAGE, ARTHUR, M.Sc. Manchester; *Borough Road Training College, Isleworth, Middlesex.* [Jan. 16, 1930]
- PALEY, RAYMOND EDWARD ALAN CHRISTOPHER, B.A. Cambridge; Fellow of Trinity College, Cambridge; *Trinity College, Cambridge.* [Jan. 16, 1930]
- PRICE, ALBERT THOMAS, M.Sc. Manchester; Lecturer, City and Guilds (Engineering) College, South Kensington; 121 *The Grove, Wandsworth, S.W.18.* [Nov. 14, 1929]
- RIGBY, CONSTANCE MARY, B.A. Oxford; 121 *London Road, Coalville, Leicester.* [Mar. 13, 1930]

MEMBERS ELECTED DURING THE SESSION 1929-30.

ROSS, SIR RONALD, K.C.B., K.C.M.G., F.R.S.; 2 *Heath Mansions, Putney Heath Lane, S.W.15*,
and *Ross Institute and Hospital of Tropical Diseases, Putney Heath, S.W.15*.

[Jan. 16, 1930]

RUSE, HAROLD STANLEY, B.A. Oxford; Lecturer in Mathematics, University of Edinburgh;
Gowan House, George Square, Edinburgh.

[Dec. 12, 1929]

SADDLER, WILLIAM, B.A. Cambridge, D.Sc., M.A. St. Andrews, F.R.S.E.; Professor of Mathematics;
Canterbury College, Christchurch, New Zealand.

[Feb. 6, 1930]

TAMARKIN, JACOB DAVID, Master of Applied Mathematics, University of Petrograd; Professor
of Mathematics; *Brown University, Providence, Rhode Island, U.S.A.*

[Dec. 12, 1929]

TEMPLE, GEORGE FREDERICK JAMES, Ph.D. London; 15 *Aldenburg Avenue, London, W.13*.

[Nov. 14, 1929]

WALKER, GEORGE, M.Sc. Sidney, Ph.D. Cambridge; *Emmanuel College, Cambridge*.

[April 24, 1930]

WHITTAKER, JOHN MACNAGHTEN, B.A. Cambridge, D.Sc. Edinburgh, F.R.S.E.; Fellow of Pembroke
College, Cambridge; *Pembroke College, Cambridge*.

[Dec. 12, 1929]

WILSON, ALAN HERRIES, B.A. Cambridge; Fellow of Emmanuel College, Cambridge;
Emmanuel College, Cambridge.

[Feb. 6, 1930]

WRIGHT, EDWARD MAITLAND, B.A. Oxford and London; *Jesus College, Oxford*.

[Dec. 12, 1929]

YOUNG, SYDNEY GEORGE, B.Sc. London; Mathematical Master, Queen Elizabeth's Grammar
School, Barnet; 87 *Bedford Avenue, Chipping Barnet, Herts*.

[Dec. 12, 1929]

ZYGMUND, ANTONI; *Mathematical Seminary, The University, Wilno, Poland*.

[Jan. 16, 1930]

Members deceased during the Session 1929-30.

MAURICE ALFRED GIBLETT.

JAMES HAMMOND.

PERCY ALEXANDER MACMAHON.

FRANK PLUMPTON RAMSEY.

P A P E R S

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

CONGRUENCE PROPERTIES OF RAMANUJAN'S FUNCTION $\tau(n)$

By J. R. WILTON.

[Received 19 November, 1928.—Read 13 December, 1928.]

1. The function $\tau(n)$ is the coefficient of x^n in the expansion of the modular function

$$f(x) = x \{(1-x)(1-x^2)(1-x^3) \dots\}^{24} = \sum \tau(n) x^n.$$

It was stated by Ramanujan* that

$$(1.1) \quad \tau(5n) \equiv 0 \pmod{5},$$

$$(1.2) \quad \tau(7n) \equiv \tau(7n+3) \equiv \tau(7n+5) \equiv \tau(7n+6) \equiv 0 \pmod{7},$$

and that

$$(1.3) \quad \tau(23n+\nu) \equiv 0 \pmod{23},$$

if ν is any one of the quadratic non-residues of 23, namely,

$$5, 7, 10, 11, 14, 15, 17, 19, 20, 21, 22.$$

Of these relations (1.1) is a consequence† of the result that

$$\tau(5) = 4830 \equiv 0 \pmod{5},$$

and $\tau(7n) \equiv 0 \pmod{7}$ is a consequence of

$$\tau(7) = -16744 \equiv 0 \pmod{7};$$

and the remaining congruences have been proved by Mordell‡.

* *Proc. London Math. Soc.* (2), 18 (1920), *Records* for 13th March, 1919. Ramanujan's *Collected papers*, No. 28.

† The formulae (5.1) and (5.2) show this immediately.

‡ L. J. Mordell, *Proc. London Math. Soc.* (2), 20 (1922), 408–416 (413). In the last two lines of p. 413 for “2, 4, 6” read “3, 5, 6”. The relation (1.1) was proved by H. B. C. Darling, *Proc. London Math. Soc.* (2), 19 (1921), 350–372, Equation (98).

In this note I give straightforward algebraic proofs of (1.2) and (1.3), and I have little doubt that the method is that by which Ramanujan established all these congruences*. I prove, in fact, a good deal more than (1.2) and (1.3). In the case of the modulus 23 the results obtained may be summarised in the following theorem:

Let p, q, r typify, respectively, (p) prime quadratic non-residues of 23, (q) prime quadratic residues of 23 not expressible in the form $a^2 + 23b^2$, and (r) primes expressible in the form $a^2 + 23b^2$; let

$$n = 23^k \prod_p p^k \prod_q q^l \prod_r r^m.$$

Then $\tau(n) \equiv 0 \pmod{23}$ if any k is odd, or if any $l \equiv 2 \pmod{3}$, or if any $m \equiv 22 \pmod{23}$. If none of these conditions is satisfied

$$\tau(n) \equiv (-1)^\lambda \prod (m+1) \pmod{23},$$

where λ is the number of the indices l which are congruent to 1 (mod 3).

The only point at which the theorem is incomplete is that there seems to be no simple rule by means of which to distinguish between the primes q and r . In Table III are given the twenty-five primes $r < 1000$ (together with the two which lie between 3800 and 3900).

Proof of (1.2)†.

2. Let

$$\begin{aligned} \psi(x) &= \{(1-x)(1-x^2)(1-x^3) \dots\}^3 \\ &= 1 - 3x + 5x^3 - 7x^6 + 9x^{10} - \dots, \end{aligned}$$

* Cf. the posthumous paper, "Congruence properties of partitions", *Math. Zeitschrift*, 9 (1921), 147-153. *Collected papers*, No. 30.

† A Referee remarks "The method employed can probably be extended to find congruence properties among the coefficients of the terms of the various positive integral powers of a power series (of certain types) with integral coefficients, though the number of moduli which can be used for any given power of the series appears to be very limited." In agreement with this remark it may be observed that, if

$$\sum \tau_k(n) x^n = \{\sum \tau(n) x^n\}^k,$$

the congruences

$$\begin{aligned} \tau_2(n) \pmod{47}, \quad \tau_3(n) \pmod{23}, \quad \tau_4(n) \pmod{71}, \quad \tau_5(n) \pmod{31}, \\ \tau_6(n) \pmod{47}, \quad \tau_7(n) \pmod{167}, \quad \tau_8(n) \pmod{191}, \quad \tau_9(n) \pmod{71}, \\ \tau_{10}(n) \pmod{79}, \quad \tau_{11}(n) \pmod{239}, \text{ etc.}, \quad \tau_{12}(n) \pmod{11}, \text{ etc.}, \end{aligned}$$

where $\tau_k(x)$ is defined by $\sum \tau_k(n) x^{2n-1} = \{f(x)\}^k$, have very remarkable properties which may be established by the method of this paper. It is, moreover, evident that, in the case of any power series with integral coefficients, the method will determine, among the coefficients of the $(p+1)$ -th power of the series (p prime), congruence relations (mod p)—which may or may not be interesting.

then, since

$$\{\psi(x)\}^7 - \psi(x^7) \equiv 0 \pmod{7},$$

we have

$$f(x) = x\{\psi(x)\}^8 \equiv x\psi(x)\psi(x^7) \pmod{7}.$$

But

$$x\psi(x)\psi(x^7) = x \sum_0^\infty b_n x^n \sum_0^\infty b_n x^{7n} = \sum_1^\infty d_n x^n,$$

where

$$b_n = (-)^m(2m+1) \text{ if } n = \frac{1}{2}m(m+1),$$

and otherwise $b_n = 0$; and

$$d_n = b_{n-1} + b_1 b_{n-8} + b_2 b_{n-15} + \dots + b_h b_{n-1-7h},$$

where $h = [(n-1)/7]$. Now let

$$n = 7k + l \quad (k \geq 0, \quad n \geq 1);$$

then (i) if $l > 0$, $b_{n-1} = 0$ unless there are positive integers l and m such that

$$7k + l - 1 = \frac{1}{2}m(m+1),$$

i.e.

$$(2m+1)^2 = 56k + 8l - 7 \equiv l \equiv n \pmod{7}.$$

Hence $d_n = 0$ if n is a quadratic non-residue of 7.

(ii) If $l = 0$, we have $b_{7n-1} = 0$ unless

$$7n - 1 = \frac{1}{2}m(m+1), \quad \text{i.e.} \quad (2m+1)^2 = 56n - 7,$$

in which case

$$b_{7n-1} = (-)^m(2m+1) \equiv 0 \pmod{7}.$$

Hence

$$\tau(7n) \equiv d_{7n} \equiv 0 \pmod{7}.$$

This completes the proof of (1.2).

A simple application of the method of § 4 will show that, when p is a prime quadratic residue of 7,

$$\tau(p) \equiv 2p \pmod{7};$$

and equations (5.1) and (5.2) may then be employed to prove the following theorem analogous to that stated in § 1 for modulus 23.

If p typifies prime quadratic non-residues of 7, and q typifies prime quadratic residues of 7, and if

$$n = 7^h \prod_p p^k \prod_q q^l,$$

then $\tau(n) \equiv 0 \pmod{7}$ if $h > 0$, or if any k is odd, or if any $l \equiv 6 \pmod{7}$; in all other cases

$$\tau(n) \equiv n \prod (l+1) \pmod{7}.$$

Proof of (1.3).

3. Let $\phi(x) = (1-x)(1-x^2)(1-x^3) \dots$, then

$$(3.1) \quad \phi(x) = 1 - x - x^2 + x^5 + x^7 - \dots = 1 + \sum_{n=1}^{\infty} (-1)^n \{x^{\frac{1}{2}n(3n-1)} + x^{\frac{1}{2}n(3n+1)}\},$$

and, as in the case of modulus 7,

$$(3.2) \quad f(x) = x \{ \phi(x) \}^{24} \equiv x \phi(x) \phi(x^{23}) \pmod{23}.$$

$$\text{But} \quad x \phi(x) \phi(x^{23}) = x \sum_0^{\infty} a_n x^n \sum_0^{\infty} a_n x^{23n} = \sum_1^{\infty} c_n x^n,$$

where

$$(3.3) \quad a_n = (-1)^n \quad \text{if} \quad n = \frac{1}{2}m(3m \pm 1),$$

and otherwise $a_n = 0$; and

$$(3.4) \quad c_n = a_{n-1} + a_1 a_{n-24} + a_2 a_{n-47} + \dots + a_h a_{n-1-23h},$$

where $h = [(n-1)/23]$. It follows from (3.2) that

$$\tau(n) \equiv c_n \pmod{23}.$$

Let $n = 23k + l$ ($k \geq 0$, $n \geq 1$); then $a_{n-1} = 0$ unless, for some positive integers l and m ,

$$23k + l - 1 = \frac{1}{2}m(3m \pm 1),$$

$$\text{i.e.} \quad 36m^2 \pm 12m = 552k + 24l - 24,$$

$$\text{i.e.} \quad (6m \pm 1)^2 \equiv l \equiv n \pmod{23}.$$

Hence $c_n = 0$ if n is a quadratic non-residue of 23. This completes the proof of (1.3).

Determination of c_p .

4. When the prime p is a quadratic residue of 23, we have from (3.3) and (3.4)

$$(4.1) \quad c_p = \sum a_{\frac{1}{2}m(3m \pm 1)} a_{p-1-23[\frac{1}{2}m(3m \pm 1)]} = \sum (-1)^{m+n},$$

summed over those values of m and n for which

$$p-1-23[\frac{1}{2}m(3m \pm 1)] = \frac{1}{2}n(3n \pm 1),$$

i.e.

$$(4.11) \quad (6n \pm 1)^2 + 23(6m \pm 1)^2 = 24p,$$

so that the value of c_p depends upon the character of the integers u and v which satisfy the Diophantine equation

$$(4.2) \quad u^2 + 23v^2 = 24p.$$

Let ω denote $\sqrt{-23}$, then in the corpus $K(\omega)$ of algebraic numbers every prime p which is a quadratic residue of 23 splits into two prime ideal factors

$$\pi, \pi' = \left(p, \frac{2r+23 \pm \omega}{2} \right),$$

where r is a solution of the congruence

$$(4.31) \quad (2r+23)^2 \equiv -23 \pmod{4p}.$$

Interchanging the two values of r simply interchanges π and π' .

In $K(\omega)$ there are three ideal classes*, the principal class typified by the unit ideal, and two classes which may be typified by the ideal prime factors of 2,

$$(4.32) \quad \pi_2 = (2, \tfrac{1}{2} + \tfrac{1}{2}\omega), \quad \pi'_2 = (2, \tfrac{1}{2} - \tfrac{1}{2}\omega).$$

We shall also require the ideal prime factors of 3,

$$(4.33) \quad \pi_3 = (3, \tfrac{1}{2} - \tfrac{1}{2}\omega), \quad \pi'_3 = (3, \tfrac{1}{2} + \tfrac{1}{2}\omega),$$

where the notation has been so chosen that π_2, π_3 belong to the same ideal class, $\pi_2 \sim \pi_3$.

There are now two possibilities with respect to π and π' . (i) Both π and π' may be principal ideals, and (ii) π and π' may be non-principal ideals belonging to different ideal classes.

(i) If π is a principal ideal

$$\pi = (\tfrac{1}{2}a + \tfrac{1}{2}b\omega) \quad (a-b \text{ even}),$$

and

$$4p = \pi\pi' = a^2 + 23b^2,$$

so that

$$a^2 - b^2 = 4(p - 6b^2),$$

which shows that a and b must both be even, since if they were both odd $a^2 - b^2$ would be $\equiv 0 \pmod{8}$. Hence

$$(4.4) \quad p = h^2 + 23k^2,$$

* In the case of a prime $p \equiv 7 \pmod{8}$ the class number is $\sum_{r=1}^{\frac{1}{2}(p-1)} \left(\frac{r}{p} \right) = 7 - 4 = 3$, in the case $p = 23$. See also E. Hecke, *Theorie der algebraischen Zahlen* (1923), 177-178.

and since decomposition into prime ideal factors is unique there is only one solution in positive integers of (4.4). Thus in this case

$$24p = (1+\omega)(1-\omega)(h+k\omega)(h-k\omega),$$

and (4.2) has the two solutions*

$$u = h+23k, \quad v = |h-k|; \quad u = |h-23k|, \quad v = h+k.$$

In both cases one of the two congruences

$$u+v \equiv 0 \pmod{24}, \quad u-v \equiv 0 \pmod{24}$$

is satisfied; hence, in (4.11), m and n are either both even or both odd, and it follows from (4.1) that

$$(4.5) \quad c_p = 2.$$

(ii) If π and π' are non-principal ideals we may choose that solution r of (4.31) for which

$$\pi_2 \sim \pi_3 \sim \pi, \quad \pi'_2 \sim \pi'_3 \sim \pi'.$$

It then follows that $\pi'_2\pi$, $\pi'_3\pi$, $\pi_2\pi_3\pi$ are principal ideals, so that $8p$, $12p$, and $24p$ are expressible in the form u^2+23v^2 , each in one way only. The case which is of interest here is

$$\pi_2\pi_3\pi = \left(\frac{3+3\omega}{2}, 1-\omega\right)\left(p, \frac{2r+23+\omega}{2}\right) = \left(\frac{u+v\omega}{2}\right),$$

where u and v satisfy (4.2). It follows that integers h and k , h' and k' exist such that

$$\frac{3p+3p\omega}{2} = \frac{h+k\omega}{2} \frac{u+v\omega}{2}, \quad p-p\omega = \frac{h'-k'\omega}{2} \frac{u+v\omega}{2},$$

whence, using (4.2),

$$(u-v\omega)(1+\omega) = 4(h+k\omega),$$

$$(u-v\omega)(1-\omega) = 6(h'-k'\omega),$$

from which it follows that

$$u-v = 4k, \quad u+v = 6k',$$

* Since $h^2-k^2 = p-24k^2$, it follows that both $h+k$ and $h-k$, and therefore also $h-23k$ and $h+23k$, are prime to 6; that is to say both pairs of values of u and v are of the form $6n \pm 1$.

so that either*

$$u = 6n+1, \quad v = 6m-1, \quad \text{or} \quad u = 6n-1, \quad v = 6m+1,$$

and

$$3n-3m \pm 1 = 2k.$$

Thus $m+n$ is odd, and from (4.1)

$$(4.6) \quad c_p = -1.$$

Proof of the theorem.

5. So far the proof has been purely algebraic. We have now to make use of the relations

$$(5.1) \quad \tau(mn) = \tau(m)\tau(n), \quad \text{if } m \text{ is prime to } n,$$

$$(5.2) \quad \tau(np^2) - \tau(p)\tau(np) + p^{11}\tau(n) = 0, \quad \text{if } p \text{ is prime,}$$

which were conjectured by Ramanujan and proved by Mordell[†] by means of the theory of the modular functions.

(i) If p is a quadratic non-residue of 23,

$$\tau(p) \equiv 0 \quad \text{and} \quad p^{11} \equiv -1 \pmod{23};$$

hence, if n is prime to p ,

$$(5.31) \quad \tau(np^{2k}) \equiv \tau(n), \quad \tau(np^{2k+1}) \equiv 0 \pmod{23}.$$

(ii) When $p = 23$, since $c_{23} = 1$, it follows from (5.1) and (5.2) that

$$(5.32) \quad \tau(23^k n) \equiv \tau(n) \pmod{23}.$$

(iii) If p is a quadratic residue of 23, so that $p^{11} \equiv 1$, then either $\tau(p) \equiv -1$ or $\tau(p) \equiv 2 \pmod{23}$.

When $\tau(p) \equiv -1$ and n is prime to p , it follows from (5.1) and (5.2) that

$$(5.33) \quad \tau(np^{3k}) \equiv \tau(n), \quad \tau(np^{3k+1}) \equiv -\tau(n), \quad \tau(np^{3k+2}) \equiv 0 \pmod{23}.$$

When $\tau(p) \equiv 2$ and n is prime to p ,

$$(5.34) \quad \tau(np^k) \equiv (k+1)\tau(n) \pmod{23}.$$

* From (4.2) we have

$$u^2 - v^2 = 24(p - v^2),$$

from which it is obvious that u and v must both be prime to 6.

† *Proc. Camb. Phil. Soc.*, 19 (1920), 117-124.

The congruences (5.31)–(5.34), together with (4.5) and (4.6), are equivalent to the theorem stated.

The relation (5.34) is interesting because it shows that every number from 0 to 22 is a possible residue of $\tau(n) \pmod{23}$. In particular

$$\tau(59^k) \equiv k+1 \pmod{23}.$$

The three tables which follow give: I, the least value of n , in each case, for which $\tau(n) \equiv k \pmod{23}$ ($k = 0, 1, \dots, 22$); II, the primes less than 1000 for which $\tau(p) \equiv -1 \pmod{23}$; and III, the primes less than 1000 which are expressible in the form $a^2 + 23b^2$, together with the corresponding values of a and b . For these primes $\tau(p) \equiv 2 \pmod{23}$.

TABLE I.

k	n	k	n	k	n	k	n
0	4	6	$59^2.101$	12	$59^2.101.167$	18	2.59^4
1	1	7	$2.59.101.167.173$	13	$2.59^4.101$	19	11918
2	59	8	$59.101.167$	14	$2.59^2.101^2$	20	6962
3	3481	9	$59^2.101^2$	15	$2.59.101.167$	21	118
4	5959	10	$59^4.101$	16	$59.101.167.173$	22	2
5	59^4	11	$2.59^2.101.167$	17	$2.59^3.101$		

TABLE II.

2	3	13	29	31	41	47	71	73	127	131
189	151	163	179	193	197	233	239	257	269	277
311	331	349	353	397	409	439	443	461	487	491
499	509	541	547	577	587	601	647	653	673	683
739	761	811	823	857	859	863	887	929	947	967

TABLE III.

p	a	b	p	a	b	p	a	b
59	6	1	347	18	1	821	27	2
101	3	2	449	9	4	829	1	6
167	12	1	463	16	3	853	5	6
173	9	2	593	15	4	877	7	6
211	2	3	599	24	1	883	26	3
223	4	3	607	20	3	991	28	3
271	8	3	691	22	3	997	13	6
307	10	3	719	12	5	3821	39	10
317	15	2	809	21	4	3853	55	6

Other congruence relations.

6. Precisely the same method may be applied in the case of the modulus 11. For let

$$\begin{aligned}\chi(x) &= \{(1-x)(1-x^2)(1-x^3) \dots\}^2 \\ &= \Sigma \Sigma (-)^n x^{\frac{3}{2}(m^2+n^2)+\frac{1}{2}m},\end{aligned}$$

taken over all (positive and negative) integers such that $m+n$ is even. Then, as before,

$$f(x) \equiv x\chi(x)\chi(x^{11}) \pmod{11}.$$

But, although it is easy to compile a table of residues* of $\tau(n) \pmod{11}$ for small values of n , there does not seem to be a general formula of any interest. The values of $n \leq 30$ for which $\tau(n) \equiv 0 \pmod{11}$ are 8, 19, 24, 29.

It was known to Ramanujan† that

$$\tau(n) \equiv n\sigma(n) \pmod{5},$$

where $\sigma(n)$ is the sum of the divisors of n ; in particular, if p is prime,

$$\tau(p) \equiv p(p+1) \pmod{5}.$$

And it is easy to deduce‡ from a formula which he gives that

$$\tau(p) \equiv p^{11} + 1 \pmod{691}.$$

* The computation of such a table is much facilitated by using an explicit formula for the coefficients in the expansion of $x\chi(x^{12})$, given by Ramanujan, *Trans. Camb. Phil. Soc.*, 22 (1916), 159–184, Equation (119) (*Collected papers*, No. 18), and proved by Mordell, *Proc. Camb. Phil. Soc.*, 19 (1920), 117–124 (121). [November, 1929. In a brief note, “Congruence properties of Ramanujan’s function $\tau(n)$ to the modulus 11”, which will appear in Dr. Baidaff’s *Boletín Matemático* early in 1930, I have given a table of residues of $\tau(p) \pmod{11}$ for prime $p \leq 257$. The only primes in the table for which $\tau(p) \equiv 0$ are 19, 29, and 199.]

† This is evident from a quotation made by Miss G. K. Stanley, *Journal London Math. Soc.*, 3 (1928), 232–237 (§ 3), from an unfinished manuscript by Ramanujan.

‡ J. R. Wilton, “On Ramanujan’s arithmetical function $\Sigma_{r,s}(n)$ ”, *Proc. Camb. Phil. Soc.*, 25 (1929), 255–264. In this paper, proofs are given of the two congruences

$$\tau(n) \equiv n\sigma(n) \pmod{5},$$

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691},$$

where $\sigma_{11}(n)$ is the sum of the eleventh powers of the divisors of n .

Thus, if p is a quadratic non-residue of $\mathbb{Z}_3$,

$$\tau(p) \equiv p^{11} + 1 \pmod{15893}.$$

The information given by the congruences which have been established is quite considerable. For example*, in the case of $\tau(19)$, we have

$$\tau(19) \equiv 0 \pmod{5}, \quad \equiv 0 \pmod{7}, \quad \equiv 0 \pmod{11}, \quad \equiv 0 \pmod{23},$$

$$\tau(19) \equiv -19 \pmod{691}.$$

Thus

$$\tau(19) \equiv 4542615 \pmod{6118805}.$$

Ramanujan gives

$$\tau(19) = 10661420.$$

* I have chosen $n = 19$ as being particularly favourable for calculation, but for every n for which the residue of $\tau(n) \pmod{11}$ has been calculated, the theorems which have been proved are sufficient to determine the residue of $\tau(n) \pmod{6118805}$.

AN APPROXIMATE FUNCTIONAL EQUATION FOR THE
PRODUCT OF TWO ζ -FUNCTIONS*By* J. R. WILTON.

[Received 16 April, 1929.—Read 25 April, 1929.]

The product of two Riemann ζ -functions, $\zeta(s)\zeta(s')$, where $s = \sigma + it$, $s' = \sigma' + it'$, may be expressed in the form of a convergent Dirichlet series if $\sigma > 1$, $\sigma' > 1$; and, by employing the functional equations, convergent series may be obtained when either $\sigma < 0$, or $\sigma' < 0$, or both σ and σ' are negative; but if either $0 \leq \sigma \leq 1$ or $0 \leq \sigma' \leq 1$, such series do not exist. In this note I obtain, in the critical regions, an approximate expression for the product analogous to the approximate functional equation for $\zeta^2(s)$ obtained by Hardy and Littlewood*.

The method lies entirely in the field of real analysis, and is very simple. It therefore seems worth recording even if it will not yield so much as the more powerful and more elaborate method employed by Hardy and Littlewood. The most interesting case of their theorem on $\zeta^2(s)$ is :

$$\text{If} \quad -\frac{1}{2} \leq \sigma \leq \frac{3}{2}, \quad \tau = |t|/(2\pi) > A > 1,$$

$$\chi(s) = 2(2\pi)^{s-1} \sin \frac{1}{2}s\pi \Gamma(1-s),$$

and if $d(n)$ is the number of divisors of n , then

$$(1.0) \quad \zeta^2(s) = \sum_{n \leq \tau} n^{-s} d(n) + \chi^2(s) \sum_{n \leq \tau} n^{s-1} d(n) + O(\tau^{\frac{1}{2}-\sigma} \log \tau).$$

Here and throughout the paper A denotes a positive constant, not the same at different occurrences. The special case $s' = s$ of my theorem is equivalent to (1.0), but with a remainder term

$$O(\tau^{\frac{1}{2}-\sigma} \log^2 \tau).$$

* Hardy and Littlewood 2, Theorem 2.

The unnecessary factor $\log \tau$ is a blemish which cannot be removed by the method here used*.

Let $\sigma_z(n)$, where z is an arbitrary (complex) number, denote the sum of d^z over the divisors d of n , so that the function usually denoted by $\sigma(n)$ is that in which $z=1$; then we have

$$(1.1) \quad \sigma_{s-s'}(n) = \sum_{d|n} d^{s-s'} = \sum_{d|n} \left(\frac{n}{d}\right)^{s-s'} = n^{s-s'} \sigma_{s'-s}(n).$$

THEOREM. If

$$-\frac{1}{2} \leq \sigma \leq \frac{3}{2}, \quad -\frac{1}{2} \leq \sigma' \leq \frac{3}{2}, \quad \sigma' > \sigma - 1,$$

$$\tau = \frac{|t|}{2\pi} \geq \tau' = \frac{|t'|}{2\pi} > .1 > 1,$$

and if (1) $\tau' > A\sqrt{\tau}$ when $\sigma' \leq \sigma$, (2)† $\tau' > A\tau^{\frac{1}{2}-\frac{1}{2}(\sigma'-\sigma)}$ when $\sigma < \sigma' < 1+\sigma$, then

$$(1.2) \quad \zeta(s)\zeta(s') = \sum_{n \leq \tau} \sigma_{s-s'}(n) n^{-s} + \chi(s)\chi(s') \sum_{n \leq \tau'} \sigma_{s'-s}(n) n^{s-1} \\ + \mu \log \tau \{ O(\tau^{\frac{1}{2}-\sigma}) + O(\tau^{\frac{1}{2}-\sigma'}) \},$$

where $\mu = \text{Min}(|\sigma - \sigma'|^{-1}, \log \tau)$.

The condition $\sigma' > \sigma - 1$ is unnecessary if $\tau' = \tau^+$. For, if the theorem has been established when $\sigma' \geq \sigma$, it follows that, when $\sigma' \leq \sigma$,

$$(1.21) \quad \zeta(1-s)\zeta(1-s') = \sum_{n \leq \tau} \sigma_{s'-s}(n) n^{s-1} + \chi(1-s)\chi(1-s') \sum_{n \leq \tau'} \sigma_{s-s'}(n) n^{-s} \\ + \mu \log \tau \{ O(\tau^{\sigma-\frac{1}{2}}) + O(\tau^{\sigma'-\frac{1}{2}}) \}.$$

Since

$$(1.3) \quad \zeta(s) = \chi(s)\zeta(1-s), \quad \chi(s)\chi(1-s) = 1, \quad \chi(s) = O(\tau^{\frac{1}{2}-\sigma}),$$

it follows, on multiplying (1.21) by $\chi(s)\chi(s')$, that, if $\tau = \tau'$, the theorem is true when $\sigma' \leq \sigma$. And it is easy to verify that it remains true if only

$$\tau - \tau' = O(\sqrt{\tau}).$$

* In fact it is easy to see that, in the sum corresponding to (4.1) when $s' = s$,

$$\sum_{n \leq \tau} \frac{d(n)}{\tau - n + \sqrt{\tau}} = O(\log^2 \tau).$$

† The range of values of τ' in case (2) may be extended by using deep-lying theorems on the order of $\zeta(s)$ when $0 < \sigma < 1$.

‡ More generally, if $\tau - \tau' = O(\sqrt{\tau})$.

COROLLARY 1. If $-\frac{1}{2} \leq \sigma \leq \frac{3}{2}$, $\tau > A > 1$,

$$\frac{1}{2}\zeta(s)\zeta(1-s) = \sum_{n \leq \tau} \sigma_{2s-1}(n) n^{-s} + \mu \log \tau \{O(\tau^{\frac{1}{2}-\sigma}) + O(\tau^{\sigma-\frac{1}{2}})\},$$

where $\mu = \text{Min}(|\frac{1}{2}-\sigma|^{-1}, \log \tau)$,

Here $s' = 1-s$, so that $\tau' = \tau$, and we employ (1.1).

COROLLARY 2. If $-\frac{1}{2} \leq \sigma \leq \frac{3}{2}$, $\tau > A > 1$,

$$\frac{1}{2}|\zeta(s)|^2 = \Re \chi(s) \sum_{n \leq \tau} \sigma_{1-2\sigma}(n) n^{s-1} + O(\log \tau) + O(\mu \tau^{1-2\sigma} \log \tau),$$

where μ is defined as in Corollary 1.

Here $s' = 1-\sigma+it$, and therefore $\tau = \tau'$, and we multiply (1.2) by $\chi(\sigma-it)$.

The proof of the theorem consists in a simple application of known theorems, which I state as lemmas.

LEMMA 1. If $\sigma > -1$, $s \neq 1$, and $k \geq 1$ is a positive integer, then

$$\zeta(s, k) + \frac{k^{1-s}}{1-s} - \frac{1}{2}k^{-s} = 2(2\pi)^{s-1} \sum_{h=1}^{\infty} h^{s-1} s \int_{2\pi h k}^{\infty} u^{-s-1} \sin u du,$$

where $\zeta(s, 1) = \zeta(s)$, and, when $k > 1$,

$$\zeta(s, k) = \zeta(s) - 1 - 2^{-s} - \dots - (k-1)^{-s}.$$

This is a trivial modification of a lemma due to Hardy and Littlewood*.

LEMMA 2†. If $\sigma + \sigma' > 0$, then

$$\begin{aligned} \zeta(s)\zeta(s') &= \zeta(s+s'-1) \left(\frac{1}{s-1} + \frac{1}{s'-1} \right) \\ &+ \sum_{k=1}^{\infty} \left[k^{-s} \left\{ \zeta(s', k) + \frac{k^{1-s'}}{1-s'} - \frac{1}{2}k^{-s'} \right\} + k^{-s'} \left\{ \zeta(s, k) + \frac{k^{1-s}}{1-s} - \frac{1}{2}k^{-s} \right\} \right]. \end{aligned}$$

Combining the two lemmas we have, if $\sigma > -1$, $\sigma' > -1$, $\sigma + \sigma' > 0$,

$$\begin{aligned} &\zeta(s)\zeta(s') - \zeta(s+s'-1) \left(\frac{1}{s-1} + \frac{1}{s'-1} \right) \\ (2.1) \quad &= \sum_{k=1}^{\infty} k^{1-s-s'} \cdot 2(2\pi)^{s-1} \sum_{h=1}^{\infty} (hk)^{s-1} s \int_{2\pi hk}^{\infty} u^{-s-1} \sin u du + (s \leftrightarrow s') \end{aligned}$$

$$(2.2) \quad = 2(2\pi)^{s-1} \sum_{n=1}^{\infty} \sigma_{1-s-s'}(n) n^{s-1} s \int_{2\pi n}^{\infty} u^{-s-1} \sin u du + (s \leftrightarrow s'),$$

* Hardy and Littlewood 1, Lemma 1, or 2, Lemma a.

† Wilton 3, 330.

where $(s \rightleftharpoons s')$ indicates a repetition of the previous term with s and s' interchanged. The rearrangement in (2.2) is legitimate since the double series (2.1) is absolutely convergent.

LEMMA 3. If $-\frac{1}{2} \leq \sigma \leq \frac{3}{2}$, $n \geq 1$ is a positive integer, and

$$I = I(n, s) = s \int_{2\pi n}^{\infty} u^{-s-1} \sin u \, du,$$

then

$$(3.1) \quad I = \Gamma(1-s) \sin \frac{1}{2}s\pi + O\{n^{1-\sigma}(\tau-n)^{-1}\} \quad (n < \tau),$$

$$(3.21) \quad I = \Gamma(1-s) \sin \frac{1}{2}s\pi + O(\tau^{\frac{1}{2}-\sigma}) \quad (\tau - \sqrt{\tau} < n \leq \tau),$$

$$(3.22) \quad I = O(\tau^{\frac{1}{2}-\sigma}) \quad (\tau < n < \tau + \sqrt{\tau}),$$

$$(3.3) \quad I = O\{n^{1-\sigma}(n-\tau)^{-1}\} \quad (\tau < n < A\tau),$$

$$(3.4) \quad I = O(n^{1-\sigma}\tau). \quad (\tau < A\tau < n).$$

This is Lemma 12 of Hardy and Littlewood 1*, slightly modified. As in the case of Lemma 1, the modification depends essentially on the fact that n is an integer.

In all that follows a and b denote fixed numbers with $a > 1 > b > 0$.

The series (2.2) may now be reduced by means of Lemma 3. It will be sufficient to consider half the series (2.2), i.e.,

$$(3.5) \quad 2(2\pi)^{s-1} \sum_{n=1}^{\infty} \sigma_{1-s-s'}(n) n^{s-1} s \int_{2\pi n}^{\infty} u^{-s-1} \sin u \, du = \chi(s) \sum_{n \leq \tau} \sigma_{1-s-s'}(n) n^{s-1} + R_1 + R_2 + S,$$

where

$$(3.51) \quad R_1 = O\left(\sum_{n < \tau - \sqrt{\tau}} \frac{|\sigma_{1-s-s'}(n)|}{\tau - n}\right), \quad (3.1)$$

$$(3.52) \quad R_2 = O(\tau^{\frac{1}{2}-\sigma}) \sum_{\tau - \sqrt{\tau} \leq n \leq \tau} \frac{|\sigma_{1-s-s'}(n)|}{n^{1-\sigma}}, \quad (3.21)$$

$$\text{and} \quad S = 2(2\pi)^{s-1} \sum_{n > \tau} \sigma_{1-s-s'}(n) n^{s-1} s \int_{2\pi n}^{\infty} u^{-s-1} \sin u \, du$$

$$(3.61) \quad = O(\tau^{\frac{1}{2}-\sigma}) \sum_{\tau < n < \tau + \sqrt{\tau}} \frac{|\sigma_{1-s-s'}(n)|}{n^{1-\sigma}} \quad (3.22)$$

$$(3.62) \quad + O\left(\sum_{\tau + \sqrt{\tau} \leq n < a\tau} \frac{|\sigma_{1-s-s'}(n)|}{n - \tau}\right) \quad (3.3)$$

$$(3.63) \quad + O(\tau) \sum_{n \geq a\tau} \frac{|\sigma_{1-s-s'}(n)|}{n^2}. \quad (3.4)$$

* There is no gain in using the refined form given as Lemma 3 in Hardy and Littlewood 2, except indeed that I have adopted the condition $\sigma \leq \frac{3}{2}$ there given, in place of $\sigma < 1$ as in the earlier paper.

LEMMA 4. If $\alpha = 1 - \sigma - \sigma'$, so that $-2 \leq \alpha < 1$, then

$$\sum_{n \geq \tau} \frac{|\sigma_{1-s-s'}(n)|}{n^2} = O\left(\frac{1}{\tau}\right)(\tau^\alpha \log \tau + 1),$$

and therefore (3.63) is $O(\tau^\alpha \log \tau + 1)$.

This is almost trivial. We have

$$\begin{aligned} \sum_{n \geq \tau} \frac{|\sigma_{1-s-s'}(n)|}{n^2} &\leq \sum_{h \geq 1} h^{\alpha-2} \sum_{k \geq \tau h^{-1}} k^{-2} \\ &= \sum_{h \leq \tau} h^{\alpha-2} \sum_{kh \geq \tau} k^{-2} + \sum_{h > \tau} h^{\alpha-2} \sum_{k \geq 1} k^{-2} \\ &= \sum_{h \leq \tau} h^{\alpha-2} \left\{ \frac{h}{\tau} + O\left(\frac{h^2}{\tau^2}\right) \right\} + \frac{\pi^2}{6} \sum_{h > \tau} h^{\alpha-2} \\ &= \tau^{-1} O(\tau^\alpha \log \tau + 1) + O(\tau^{\alpha-1} \log \tau) + O(\tau^{\alpha-1}). \end{aligned}$$

The remaining terms of (3.5) may be grouped together, for it is clear that (3.51), (3.52), (3.61), and (3.62) are together less than some positive multiple of

$$(4.1) \quad \sum_{n \leq a\tau} \frac{|\sigma_{1-s-s'}(n)|}{|\tau - n| + \sqrt{\tau}}.$$

LEMMA 5. If $R_1 + R_2 + S$ is defined by (3.5), and if

$$\mu = \text{Min}(|a|^{-1}, \log \tau),$$

then

$$(5.1) \quad |R_1 + R_2 + S| = O(\mu \tau^\alpha \log \tau) + O(\log \tau).$$

On account of Lemma 4 it will be sufficient to prove that the series (4.1) satisfies the implied inequality of (5.1). The proof is not quite so simple as that of the preceding lemma, but it follows easily by a well known process due to Dirichlet.

Let $\theta = a\tau$, $\phi = (|\tau - hk| + \sqrt{\tau})^{-1}$; then (4.1) does not exceed

$$\sum_{h \leq \sqrt{\theta}} h^\alpha \sum_{k \leq \theta h^{-1}} \phi + \sum_{h \leq \sqrt{\theta}} \sum_{h \leq \theta k^{-1}} h^\alpha \phi - \sum_{h \leq \sqrt{\theta}} h^\alpha \sum_{k \leq \sqrt{\theta}} \phi = \Sigma_1 + \Sigma_2 - \Sigma_3,$$

say. Since ϕ has a single maximum $\tau^{-\frac{1}{2}}$ at $hk = \tau$, the inner sum in either Σ_1 or Σ_3 is equal to

$$\int \phi dk + O(\tau^{-\frac{1}{2}}) = O(h^{-1} \log \tau) + O(\tau^{-\frac{1}{2}}).$$

Hence

$$\Sigma_1 - \Sigma_3 = O(\log \tau \sum_{h \leq \sqrt{\theta}} h^{\alpha-1}) + O(\tau^{\frac{1}{2}\alpha})$$

$$(5.2) \quad = O(\log \tau)(\mu \tau^{\frac{1}{2}\alpha} + 1).$$

In Σ_2 , if $hk \leq \tau$,

$$(h^\alpha \phi)' = h^{\alpha-1} \phi^2 \{a(\tau + \sqrt{\tau}) + (1-a)hk\},$$

so that $h^a \phi$ is constantly increasing if $a \geq 0$, and has a single minimum in the interval

$$1 \leq h \leq \tau k^{-1}$$

if $a < 0$. In either case the corresponding part of the inner sum of Σ_2 is equal to

$$(5.3) \quad \sum_{h \leq b\tau k^{-1}} h^a \phi + \int_{b\tau k^{-1}}^{\tau k^{-1}} h^a \phi dh + O\left(\frac{\tau^{a+\frac{1}{2}}}{k^a}\right) + O(\tau^{-\frac{1}{2}}).$$

The series in (5.3) is

$$(5.4) \quad O(\tau^{-1}) \Sigma h^a = O(\tau^a k^{-1-a} \log \tau) + O(\tau^{-1}).$$

When summed over $k \leq \sqrt{\theta}$, (5.4) and the O -terms of (5.3) are together

$$(5.5) \quad \begin{aligned} & O(\mu \tau^{\frac{1}{2}a} \log \tau) + O(\tau^a \log \tau) + O(1) \\ & = O(\log \tau)(\mu \tau^a + 1). \end{aligned}$$

If $hk > \tau$,

$$(h^a \phi)' = h^{a-1} \phi^2 \{-a(\tau - \sqrt{\tau}) - (1-a)hk\} < -h^{a-1} \phi^2(\tau - \sqrt{\tau}),$$

so that $h^a \phi$ decreases throughout the interval

$$\tau k^{-1} < h \leq \theta k^{-1}.$$

It follows that, without altering the error term (5.5), the inner sum Σ_2 may be replaced by

$$\begin{aligned} \int_{b\tau k^{-1}}^{\theta k^{-1}} h^a \phi dh &= \left[-\frac{h^a}{k} \log(\tau - hk + \sqrt{\tau}) \right]_{b\tau k^{-1}}^{\tau k^{-1}} + \left[\frac{h^a}{k} \log(kh - \tau + \sqrt{\tau}) \right]_{\tau k^{-1}}^{\theta k^{-1}} \\ &+ \int_{b\tau k^{-1}}^{\tau k^{-1}} \frac{a h^{a-1}}{k} \log(\tau - kh + \sqrt{\tau}) dh - \int_{\tau k^{-1}}^{\theta k^{-1}} \frac{a h^{a-1}}{k} \log(kh - \tau + \sqrt{\tau}) dh \\ &= O(\log \tau) \left(\frac{\tau^a}{k^{a+1}} + \int_{b\tau k^{-1}}^{\theta k^{-1}} \frac{|a| h^{a-1} dh}{k} \right) \\ &= O(\tau^a k^{-1-a} \log \tau). \end{aligned}$$

And the sum over $k \leq \sqrt{\theta}$ is

$$(5.6) \quad O(\log \tau)(\mu \tau^{\frac{1}{2}a} + \tau^a) = O(\log \tau)(\mu \tau^a + 1).$$

The lemma follows from (5.2), (5.5), and (5.6).

The proof of the theorem is obtained from (2.2), (3.5), and Lemma 5 when we write $1-s$ for s in (2.2) and multiply through the resulting equation by $\chi(s)$, if it is shown that

$$\xi(s'-s) \left(\frac{1}{s'-1} - \frac{1}{s} \right) = O(\log \tau) + O(\tau^{\sigma-\sigma'} \log \tau),$$

i.e., since $\tau' \leq \tau$, if

$$(5.7) \quad \zeta(s'-s) = O(\tau' \log \tau)(1 + \tau^{\sigma-\sigma'}).$$

Now, (1) when $\sigma' \leq \sigma$, $\zeta(s'-s) = O(\tau^{\frac{1}{2}+\sigma-\sigma'} \log \tau)$, and (5.7) is satisfied if $\tau' > A\sqrt{\tau}$;

(2) when $\sigma < \sigma' < 1+\sigma$,

$$\zeta(s'-s) = O\{\tau^{\frac{1}{2}(1-\sigma'+\sigma)}\},$$

and (5.7) is satisfied if $\tau' > A\tau^{\frac{1}{2}-\frac{1}{2}(\sigma-\sigma')}$;

(3) when $\sigma' \geq 1+\sigma$,

$$\zeta(s'-s) = O(\log \tau),$$

so that (5.7) is satisfied if $\tau' > A$. This completes the proof of the theorem as stated.

References.

1. G. H. Hardy and J. E. Littlewood, "The zeros of Riemann's zeta-function on the critical line", *Math. Zeitschrift*, 10 (1921), 283-317.
2. G. H. Hardy and J. E. Littlewood, "The approximate functional equations for $\zeta(s)$ and $\zeta^2(s)$ ", *Proc. London Math. Soc.* (2), 29 (1929), 81-97.
3. J. R. Wilton, "A note on the coefficients in the expansion of $\zeta(s, x)$ in powers of $s-1$ ", *Quart. J. of Math.*, 50 (1927), 329-332.

THE CONTINUITY OF A FUNCTION DEFINED BY A DEFINITE INTEGRAL

By R. L. JEFFERY.

[Received 3 April, 1929—Read 25 April, 1929.]

If a function of two variables $f(x, y)$ is bounded, continuous in y at y_0 for each x , and measurable in x for each y , then

$$F(y) = \int_a^b f(x, y) dx$$

is continuous at y_0 . This was first proved by W. H. Young*. Young obtained this result in conformity with the Heine sequence definition† of continuity. That this definition is equivalent to that of Cauchy‡ can be established only through the medium of the Zermelo axiom§. In this paper continuity of $F(y)$ according to Cauchy's definition is proved directly without recourse to the axiom of choice. The proof involves the consideration of non-measurable sets of points, and it is necessary to invent what may be described as the upper integral of a measurable function over a non-measurable set.

The problem of the continuity of $F(y)$ at y_0 has been considered by E. W. Hobson|. His results hold whenever, for a given $\epsilon > 0$ and any $\delta > 0$, it is possible to prove that the set of x -points G_δ is measurable for

* *Monatshefte für Math. und Phys.*, 21 (1910), 125–149.

† *Journal für Math.*, 74 (1872), 182. $F(y)$ is continuous at y_0 if, for every sequence $y_1, y_2, \dots$, approaching y_0 , $\lim_{n \rightarrow \infty} F(y_n) = F(y_0)$.

‡ Attributed to Cauchy by Sierpinski, *Comptes rendus*, 163 (1916), 688. $F(y)$ is continuous at y_0 if for a given ϵ there exists a δ such that $|F(y_0) - F(y)| < \epsilon$ when $|y_0 - y| < \delta$.

§ Sierpinski, *loc. cit.* Also see E. W. Hobson, *Theory of functions of a real variable*, 2nd ed., 1 (1921), § 211.

|| *Theory of functions of a real variable*, 2nd ed., 2 (1926), § 224. See also 3rd ed., 1 (1927), 727.

which $|f(x, y_0) - f(x, y)| < \epsilon$ when $|y_0 - y| < \delta$. For then we have

$$(1) \quad |F(y_0) - F(y)| \leq \int_a^b |f(x, y_0) - f(x, y)| dx \\ \leq \int_{G_\delta} |f(x, y_0) - f(x, y)| dx + \int_{CG_\delta} |f(x, y_0) - f(x, y)| dx.$$

If δ_i ($i = 1, 2, \dots$) is a monotone sequence of positive numbers tending to zero, and G_i is the set G_δ for $\delta = \delta_i$, it can be shown that $\lim mG_i = b - a$. Hence, if L and l are the least upper bound and greatest lower bound respectively of $f(x, y)$, then, for δ sufficiently small and $|y_0 - y| < \delta$, we have

$$|F(y_0) - F(y)| \leq \epsilon(b - a) + (L - l)\epsilon.$$

And, since ϵ is arbitrary, it follows that $F(y)$ is continuous at y_0 according to the definition of Cauchy. An example*, however, shows that, even when $f(x, y)$ satisfies the stated conditions, G_δ need not be measurable. Nevertheless, for the function $f(x, y)$ defined in this example, it can be shown that G_δ contains a measurable component whose measure approaches $b - a$ as δ approaches zero. Whenever such a measurable component exists, it is evident that it can replace G_δ in (1) above, and the argument there used is still applicable. We give an example† to show that such a measurable component does not always exist.

Let V be a Van Vleck set‡ on $(0, 1)$. The set V can be subdivided§ into two sets V'_1 and V''_1 , each of which is non-measurable with outer measure unity. The set V'_1 can be similarly subdivided into two sets V'_2 and V''_2 , and so on indefinitely. This leads to an infinite sequence of non-measurable sets, $V'_1, V''_1, V'_2, V''_2, \dots$, each of which is distinct from all the others and has outer measure unity. Let $f(x, y) = 0$ at all points of the square $0 \leq x \leq 1, 0 \leq y \leq 1$, except on the lines $y = x/n$ ($n = 1, 2, \dots$), and on these lines let $f(0, 0) = 0$. At other points on the line $y = x/n$ let $f(x, y) = 1$ if x belongs to V'_n . Otherwise on this line let $f(x, y) = 0$ ($n = 1, 2, \dots$). It can be shown that $f(x, y)$ is bounded, measurable in x for each y , and for each x continuous in y at y_0 . But for this function, if ϵ is less than unity, G_ϵ contains CV and consequently has outer measure

* Hobson, *loc. cit.*, 3rd ed., 1 (1927), 727.

† It has been pointed out to me by Professor E. W. Hobson that the axiom of choice enters into this example, since it is employed in the determination of a Van Vleck set. While this is true, both of this example and the one referred to above, yet these examples in no way affect the subsequent discussion.

‡ *Trans. Amer. Math. Soc.*, 9 (1908), 237-244.

§ Van Vleck, *loc. cit.*

unity. But CG_δ has outer measure unity, since it contains at least one of the sets V_n'' ($n = 1, 2, \dots$). Hence G_δ is non-measurable on $(0, 1)$ and contains no measurable components other than sets of zero measure.

As G_δ does not always contain a measurable component, we turn to a consideration of the relation existing between a measurable function and non-measurable sets. Let $f(x)$ be any function bounded and measurable on (a, b) . Let S be any set whatever on (a, b) . Divide the range of f by the points $A_n \equiv a_1, a_2, \dots, a_n$, and let s_i be the part of S for which $a_{i-1} \leq f < a_i$ ($i = 1, 2, \dots, n-1$), s_n the part for which $a_{n-1} \leq f \leq a_n$, and e_i the set of all points which satisfy the corresponding inequality. Let $\bar{m}s_i$ denote the outer measure of s_i . Form the sum

$$\sigma_n = \sum a_i \bar{m}s_i.$$

Since the sets s_i are contained in the sets e_i which are measurable and without points common to any two, the outer measures of the sets s_i are additive*. Thus we have

$$\sum \bar{m}s_i = \bar{m}(\sum s_i).$$

Hence, if $A_{n'}$ is a subdivision obtained by inserting additional points in the division A_n , then

$$(2) \quad \sigma_{n'} \leq \sigma_n.$$

Let I be the lower bound of σ_n for all possible subdivisions A_n . We show that, for any subdivision, $B_m \equiv b_1, b_2, \dots, b_n$, whatever, with norm sufficiently small,

$$\sigma_m - I < \epsilon.$$

Since I is the lower bound of σ_n for all subdivisions A_n , it is possible to find one A_n for which

$$\sigma_n - I < \frac{1}{2}\epsilon.$$

With A_n thus fixed, let B_m be any other subdivision such that the greatest length of any interval of B_m is less than the smallest interval of A_n , and at the same time less than $\frac{\epsilon(b-a)}{2n}$. Obtain a new subdivision C_l by superimposing B_m on A_n . Then, by (2) above,

$$(3) \quad \sigma_l - I < \frac{1}{2}\epsilon.$$

Writing σ_m and σ_l in a suitable notation, taking their difference, and applying the theorem of Carathéodory referred to above, it is seen that

$$\sigma_m - \sigma_l < \frac{1}{2}\epsilon.$$

* Carathéodory, *Reelle Funktionen* (Berlin, 1918), 273.

This, with (3), gives

$$\sigma_m - I < \epsilon.$$

This number I we designate by $\int_S f(x)dx$. Let A_n be any subdivision of the range of f . Since e_i contains s_i it follows that

$$me_i \geq \bar{m}s_i.$$

Consequently

$$\begin{aligned} \sum a_i me_i - \sum a_i \bar{m}s_i &= \sum a_i (me_i - \bar{m}s_i) \\ &\leq U(b-a-\bar{m}S), \end{aligned}$$

where U is the least upper bound of $|f(x)|$. As a result of this we have:

If $b-a-\bar{m}S < \epsilon$, then

$$(4) \quad \left| \int_a^b f(x)dx - \int_S f(x)dx \right| < U\epsilon,$$

and if $\bar{m}S = b-a$, then

$$(5) \quad \int_a^b f(x)dx = \int_S f(x)dx.$$

Returning now to the problem of the continuity of $F(y)$, let $\delta_i (i = 1, 2, \dots)$ be a monotonic sequence of positive numbers tending to zero, and G_i the set of x -points for which

$$|f(x, y_0) - f(x, y)| < \epsilon$$

when $|y_0 - y| < \delta_i$. We have seen that G_i may not be measurable, but it is clear that G_i contains G_{i-1} , and that $\lim G_i$ is all the points of (a, b) . It would then follow, from a theorem due to W. H. Young*, that

$$\lim \bar{m}G_i = b-a.$$

Hence we can fix $i = k$ great enough to ensure that

$$(6) \quad b-a-\bar{m}G_k < \epsilon.$$

Consider the inequality

$$(7) \quad |F(y_0) - F(y)| \leq \int_a^b |f(x, y_0) - f(x, y)| dx$$

* *Theory of sets of points* (Cambridge, 1906), Theorem 28, 104.

where $|y_0 - y| < \delta_k$. From (4) and (6) we have

$$(8) \quad \left| \int_b^b |f(x, y_0) - f(x, y)| dx - \int_{G_k} |f(x, y_0) - f(x, y)| dx \right| < U\epsilon,$$

where U is the least upper bound of $|f(x, y_0) - f(x, y)|$. Since over G_k $|f(x, y_0) - f(x, y)| < \epsilon$, we have

$$(9) \quad \int_{G_k} |f(x, y_0) - f(x, y)| dx < \epsilon(b-a).$$

It then follows, from (7), (8), and (9), that

$$|F(y_0) - F(y)| < U\epsilon + \epsilon(b-a),$$

and, since ϵ is arbitrary, $F(y)$ continuous at y_0 , in accordance with the definition of Cauchy.

A TYPE OF INFINITE MATRIX

By V. NAYLOR.

[Received 2 March, 1929.—Read 14 March 1929.]

I. "To sum a series" and "to solve an equation" may be regarded as inverse operations. Thus, in the equality

$$(1) \quad 1+x+x^2+\dots+x^{n-1}=K$$

we may give values to x and find the sum K ; or we may give values to K and find the solutions x . Furthermore, we may give the equality an interpretation in the Arithmetic of Matrices, for we may write it thus:

$$[1 \ 1 \ 1 \ \dots \ (n \text{ units})] \begin{Bmatrix} 1 \\ x \\ x^2 \\ \dots \\ x^{n-1} \end{Bmatrix} = [1+x+x^2+\dots+x^{n-1}] = [K].$$

It is a natural extension to inquire what happens when $n \rightarrow \infty$; we seek to apply the above ideas to the equality

$$(2) \quad 1+x+x^2+\dots=K,$$

the expression on the left being an infinite series.

When (1) is reduced to the form $(1-x^n)/(1-x)=K$, it can be shown that, if

$$a_{r+1} = \sqrt[n]{K \cdot a_r - K + 1}, \quad b_{r+1} = (b_r^n + K - 1)/K,$$

both of the sequences (a_r) and (b_r) tend to limits, $a(n)$ and $b(n)$ say, as $r \rightarrow \infty$, provided that certain conditions are satisfied with regard to the choice of a_1 and b_1 . Each of these limits is one of the roots of equation (1).

If either (or both) of $\lim_{n \rightarrow \infty} a(n)$ and $\lim_{n \rightarrow \infty} b(n)$ exists, we may use them as definitions of the roots of (2); and then, if

$$\lim_{n \rightarrow \infty} \lim_{r \rightarrow \infty} \sqrt[n]{K \cdot a_r - K + 1} = a, \quad \lim_{n \rightarrow \infty} \lim_{r \rightarrow \infty} (b_r^n + K - 1)/K = b,$$

we are confronted with the further problem of determining whether, or

under what conditions, the series

$$1+a+a^2+\dots, \quad 1+b+b^2+\dots$$

have the sum K .

It is not the object of this paper to examine this particular question; it is only desired to indicate its relation to the more general one. It should be noted, however, that if such conditions prevail that $\lim_{r \rightarrow \infty} b_r^n$ exists and is less than 1, then

$$\lim_{n \rightarrow \infty} \frac{b_r^n + K - 1}{K} = \frac{K-1}{K} = b, \text{ say,}$$

and, on substituting back, we get

$$1 + \left(\frac{K-1}{K}\right) + \left(\frac{K-1}{K}\right)^2 + \dots,$$

the sum of which is K , when $\left|\frac{K-1}{K}\right| < 1$, i.e. when $K > \frac{1}{2}$. In these circumstances the "substitution test" is satisfied. On the other hand, if $K = \frac{1}{2}$ then $b = -1$; and if we substitute back this value of b , the series obtained is $1-1+1-1+\dots$, of which the Eulerian sum is $\frac{1}{2}$. This type of summability is mentioned rather than, say, that of Cesàro, on account of the fact that some of the ensuing examples defy summation by the method of Cesàro.

II. The more general question involves the solution of the equations

$$\begin{cases} (1+a_{11})x_1 + a_{12}x_2 + \dots = c_1 \\ a_{21}x_1 + (1+a_{22})x_2 + \dots = c_2 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \end{cases}$$

i.e. an infinity of equations with an infinity of unknowns. The method of solution is the same as in § I, viz. to retain n equations and to solve for the n unknowns $x_1, x_2, \dots, x_n$.

The value of x_k so obtained is $\Delta_n^{(k)}/\Delta_n$, where

$$\Delta_n = \begin{vmatrix} 1+a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & 1+a_{nn} \end{vmatrix},$$

and $\Delta_n^{(k)}$ is the result of putting $c_1, c_2, \dots$, for $a_{1k}, a_{2k}, \dots$, in Δ_n .

Then $\lim_{n \rightarrow \infty} \Delta_n^{(k)}/\Delta_n$, if it exists, will give the root of the equations corresponding to suffix k . For the existence of this limit, (x_k) , certain restrictive conditions must be imposed.

Riesz*, with the assumption of absolute convergence,

$$\sum_{i,k} |a_{ik}| = \sum A_k = A,$$

has shown that when the c 's are bounded, $\Delta_n^{(k)}$ tends to a limit and Δ_n to a limit as $n \rightarrow \infty$, and that (x_k) is the ratio of these limits.

The question arises, do cases exist when, by assuming conditions less stringent than the convergence of $\sum_{i,k} |a_{ik}|$, a definite value is obtained

$$\text{for } \lim \frac{\Delta_n^{(k)}}{\Delta_n} \text{ but not for } \frac{\lim \Delta_n^{(k)}}{\lim \Delta_n} ?$$

The object of this paper is to give one or two examples in answer to this question.

III. Consider the set of n equations

$$(1) \quad \begin{cases} \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n = 1 \\ \lambda_1^3 x_1 + \lambda_2^3 x_2 + \dots + \lambda_n^3 x_n = 0 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \lambda_1^{2n-1} x_1 + \lambda_2^{2n-1} x_2 + \dots + \lambda_n^{2n-1} x_n = 0. \end{cases}$$

Provided that no two λ 's are the same, the solutions are unique, and are given by

$$(2) \quad x_k = \frac{1}{\lambda_k} \cdot \frac{\nabla_0}{\nabla} \quad (k = 1, 2, 3, \dots, n),$$

where ∇_0 denotes the result of putting $\lambda_k = 0$ in ∇ , and

$$(3) \quad \begin{aligned} \nabla &= \begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_n^2 \\ \dots & \dots & \dots & \dots \\ \lambda_1^{2n-2} & \lambda_2^{2n-2} & \dots & \lambda_n^{2n-2} \end{vmatrix} \\ &= \prod_{r=2}^n (\lambda_1^2 - \lambda_r^2) \cdot \prod_{r=3}^n (\lambda_2^2 - \lambda_r^2) \dots \prod_{r=n}^n (\lambda_{n-1}^2 - \lambda_r^2). \end{aligned}$$

This leads to

$$(4) \quad x_k = \frac{1}{\lambda_k} \frac{1}{\prod_{r=1}^n \left(1 - \frac{\lambda_k^2}{\lambda_r^2} \right)} \quad (k = 1, 2, 3, \dots, n),$$

* F. Riesz, *Les systèmes d'équations linéaires à une infinité d'inconnues* (Gauthier-Villars, 1913), ch. II.

as the solutions of the equations, where the dash means that the zero factor is to be omitted.

(5) *This product tends to a finite limit as $n \rightarrow \infty$ provided that $\sum_{r=1}^{\infty} \lambda_r^{-2}$ is convergent. The restriction that no two of $\lambda_1^2, \lambda_2^2, \dots$ are equal has already been assumed in § III (1).*

We denote this limit by (x_k) .

Clearly, by § III (5), a possible sequence of λ 's is $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3, \dots$, and (x_k) , in this example, is given by

$$(6) \quad (x_k) = \lim_{n \rightarrow \infty} \frac{1}{k} \cdot \frac{1}{1 - \frac{k^2}{1^2}} \cdot \frac{1}{1 - \frac{k^2}{2^2}} \cdots \frac{1}{1 - \frac{k^2}{(k-1)^2}} \cdot \frac{1}{1 - \frac{k^2}{(k+1)^2}} \cdots \frac{1}{1 - \frac{k^2}{n^2}}$$

$$= (-1)^{k+1} \cdot \frac{2}{k},$$

by using the expression for $\sin \theta$ in factors.

Thus, $(x_1) = 2, (x_2) = -\frac{2}{2}, (x_3) = \frac{2}{3}, \dots$, are the roots of the equations

$$(7) \quad \begin{cases} 1 \cdot x_1 + 2 \cdot x_2 + 3 \cdot x_3 + \dots = 1 \\ 1^3 \cdot x_1 + 2^3 \cdot x_2 + 3^3 \cdot x_3 + \dots = 0 \\ \dots \quad \dots \quad \dots \quad \dots \end{cases}$$

It is interesting to note that in this case $\nabla_0 \rightarrow \infty$ and $\nabla \rightarrow \infty$, but their quotient tends to a finite limit as $n \rightarrow \infty$.

In Riesz's tract the convergence of Δ_n , and hence of $\Delta_n^{(k)}$, and hence of $\Delta_n^{(k)}/\Delta_n$, is proved by using $\prod_{k=1}^n (1 + A_k)$, which contains more product terms of the a 's than does Δ_n . This is possible because conditions more restrictive are placed on the fundamental determinant at the outset.

We next ask the question, "Do these solutions (x_k) meet the substitution test of equations?"

The answer is supplied by a theorem, due to Landau, which states that the left-hand sides of the equations

$$\sum_{k=1}^{\infty} a_{ik} \cdot x_k = c_i \quad (i = 1, 2, 3, \dots)$$

converge absolutely for all systems (x_k) such that

$$\sum_{k=1}^{\infty} |x_k|^p \quad (p > 1) \text{ converges, provided that } \sum_{k=1}^{\infty} |a_{ik}|^{p/(p-1)}$$

converges for all i 's.

In our example, by taking $p = 2$, we get

$$\sum_{k=1}^{\infty} |x_k|^p = 2^2 \left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right], \text{ which converges,}$$

but
$$\sum_{k=1}^{\infty} |a_{ik}|^{p/(p-1)} = 1^2 + 2^2 + 3^2 + \dots, \text{ which diverges.}$$

Therefore the result of substituting the values of (x_k) from § III (6) in § III (7) will not give convergent series. This is easily seen to be so, for, on substituting back, we get the associated "sums"

$$(8) \quad \begin{cases} 1 - 1 + 1 - 1 \dots = \frac{1}{2} \\ 1^2 - 2^2 + 3^2 - 4^2 \dots = 0 \\ 1^4 - 2^4 + 3^4 - 4^4 \dots = 0 \\ \dots \quad \dots \quad \dots \quad \dots \end{cases}$$

It will be seen that these are sums in the true Eulerian sense, for the processes through which we have gone are equivalent to making the assumptions

$$\begin{cases} \lambda_1 \lim x_1^{(n)} + \lambda_2 \lim x_2^{(n)} + \dots = 1 = \lim \sum \lambda_n x_n, \\ \lambda_1^3 \lim x_1^{(n)} + \lambda_2^3 \lim x_2^{(n)} + \dots = 0 = \lim \sum \lambda_n^3 x_n, \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \end{cases}$$

where $x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)}$ denote the roots of § III (1) corresponding to the retention of n equations.

The reader will see that the foregoing analysis justifies the introduction of the infinite product matrix

$$(9) \quad \begin{bmatrix} 1 & 2 & 3 & \dots \\ 1^3 & 2^3 & 3^3 & \dots \\ 1^5 & 2^5 & 3^5 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{3} \\ \dots \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \dots \end{bmatrix}$$

in cases where product matrices are used to express transformations. The result may be written

$$[1, 2^2, 3^4, \dots] \{1, -1, 1, \dots\} = \{\frac{1}{2}, 0, 0, \dots\},$$

a suggestive way of expressing the associated sums § III (8).

By choosing other sequences of λ 's for the above sequence of c 's, or by using a different sequence of c 's, we may obtain other similar results.

(10) Thus, from the equations

$$\sum_{k=1}^{\infty} h^{2i-1} x_k = q^{2i-1} \quad (i = 1, 2, 3, \dots)$$

we get, by a process similar to the above, the infinite matrix product

$$[1, 2^3, 3^5, 4^7, \dots] \left\{ \frac{1}{1^2 - q^2}, -\frac{2}{2^2 - q^2}, \dots \right\} = \frac{\pi}{2 \sin q\pi} \{q, q^3, \dots\},$$

(11) From equations § III (1), if we use the sequence of λ 's expressed by $\lambda_m = r^m$, $r > 1$, we obtain the associated sums

$$\begin{cases} 1 + \frac{1}{1-r^2} + \frac{1}{(1-r^4)(1-r^2)} + \dots = \frac{1}{P}, \\ 1 + \frac{r^{2s}}{1-r^2} + \frac{r^{4s}}{(1-r^4)(1-r^2)} + \dots = 0, \end{cases}$$

where s is any positive integer, and $1/P = \{1 - (1/r^2)\} \{1 - (1/r^4)\} \dots$

These results also may be expressed in the form of a matrix product.

In conclusion, we desire to stress the threefold aspect of the problem.

First, we may regard the question as being one of finding the roots of an infinity of equations with an infinity of unknowns.

Secondly, we may use the analysis set out as a means of defining the sums of such series as have an Eulerian sum.

Thirdly, we may seek products of infinite matrices, whose determinants do not converge in the ordinary way.

These matrices may become more significant in certain branches of Physics.

CESÀRO SUMMATION OF THE DIFFERENTIATED SERIES OF FOURIER-LEBESGUE SERIES AND THEIR ALLIED SERIES

By K. I. SAYERS†.

[Received 27 November, 1928.—Read 13 December, 1928.—Revised 7 August, 1929.]

Summary. In this paper I discuss the summability of the differentiated series of Fourier-Lebesgue series and their allied series on lines similar to those adopted by Plessner‡, but remove the restriction that $F(x)$ is to be a function of bounded variation. The general case of summability (C, k) for $k > 0$ for the differentiated series and $k > 1$ for the allied series is discussed, and known results are deduced as special cases.

1. We assume that $F(x)$ is integrable in the sense of Lebesgue and periodic with period 2π ; also that

$$(1) \quad \int_{-\pi}^{\pi} F(t) dt = 0;$$

this can always be ensured by subtracting a suitable constant from $F(x)$.

The following cases of $F(x)$, satisfying special conditions, will also be considered:

A. $F(x)$ is of bounded variation;

B. Special points exist at which $F(x)$ has a symmetrical differential coefficient. At such a point the difference quotient

$$\frac{F(x+h) - F(x-h)}{2h}$$

tends to a unique finite limit as $h \rightarrow 0$. The limit will be denoted by $F'(x)$. It is known that it exists and coincides with the ordinary differ-

† M. T. Meyer Research Student, Girton College, 1926–1928.

‡ “Zur Theorie der konjugierten trigonometrischen Reichen”, *Mittheilungen des Math. Seminars der Univ. Giessen* (10), 1923.

ential coefficient $dF(x)/dx$ whenever this exists. The notation causes no confusion, since ordinary differentiability plays no part in the theory. It is convenient to use the modulus of symmetric differentiability $\xi(h)$ given by

$$(2) \quad \xi(h) = \left\| \frac{F(x+t) - F(x-t)}{2t} - F'(x) \right\| \quad (0 < t \leq h),$$

where $\|\phi(t)\|$ denotes the maximum value of $|\phi(t)|$ in the given interval. By definition, as $h \rightarrow 0$,

$$(3) \quad \xi(h) \rightarrow 0.$$

2. *The coefficients.* We define

$$(1) \quad a_n = \frac{n}{\pi} \int_{-\pi}^{\pi} F(t) \sin nt \, dt \quad (n = 0, 1, 2, \dots),$$

$$(2) \quad \beta_n = -\frac{n}{\pi} \int_{-\pi}^{\pi} F(t) \cos nt \, dt;$$

we have

$$(3) \quad a_0 = 0, \text{ obviously; } \beta_0 = 0, \text{ by (1, 1);}$$

further, in all cases,

$$(4) \quad a_n = o(n); \quad \beta_n = o(n);$$

and, unless $F(x)$ satisfies special conditions, no more than this can be proved.

If $F(x)$ satisfies A, then

$$(5) \quad a_n = O(1); \quad \beta_n = O(1).$$

3. *The series*

$$(1) \quad \sum_{n=1}^{\infty} (a_n \cos nx + \beta_n \sin nx)$$

is the differentiated series of the Fourier-Lebesgue series of $F(x)$, and the series

$$(2) \quad \sum_{n=1}^{\infty} (-\beta_n \cos nx + a_n \sin nx)$$

is the series allied with (1).

From (2, 3) it is seen that the series (1) possesses no constant term, and, further, that if a_n , b_n are typical coefficients of the ordinary Fourier series associated with $F(x)$, then

$$a_n = nb_n; \quad \beta_n = -na_n.$$

4. The k -th Rieszian mean of the Fourier series of $F(t)$ at x is known to be

$$\frac{\omega}{\pi} \int_0^\infty \{F(x+t) + F(x-t)\} \gamma_{k+1}(\omega t) dt,$$

where
$$\gamma_{k+1}(u) = \int_0^1 (1-t)^k \cos ut dt,$$

and this in turn may be written

$$\frac{\Pi(k) C_{k+1}(u)}{u^{k+1}},$$

where $C_{k+1}(u)$ is the case $p = k+1$ of the function

$$C_p(u) = \sum_{r=0}^{\infty} (-)^r \frac{u^{p+2r}}{\Pi(p+2r)} \quad (p > -1).$$

The principal properties of the functions $C_p(u)$ and $\gamma_k(u)$ have been given by W. H. Young† and S. Pollard‡.

Let the k -th Rieszian means of the differentiated series and its allied series be $\sigma_k(\omega)$ and $\tau_k(\omega)$ respectively. Then

$$\sigma_k(\omega) = \frac{\omega}{\pi} \frac{d}{dx} \int_0^\infty \{F(x+t) + F(x-t)\} \gamma_{k+1}(\omega t) dt.$$

But

$$\begin{aligned} \frac{d}{dx} \int_0^\infty F(x+t) \gamma_{k+1}(\omega t) dt &= \frac{d}{dx} \int_x^\infty F(t) \gamma_{k+1}(\omega(t-x)) dt, \\ &= -\omega \int_x^\infty F(t) \gamma'_{k+1}(\omega(t-x)) dt - F(x) \gamma_{k+1}(0), \\ &= -\omega \int_0^\infty F(x+t) \gamma'_{k+1}(\omega t) dt - F(x) \gamma_{k+1}(0). \end{aligned}$$

† W. H. Young, *Quart. J. of Math.*, 43 (1912), 161-177.

‡ *Proc. Camb. Phil. Soc.*, 23 (1926), 373-382 (375).

Similarly,

$$\frac{d}{dx} \int_0^\infty F(x-t) \gamma_{k+1}(\omega t) dt = \omega \int_0^\infty F(x-t) \gamma'_{k+1}(\omega t) dt + F(x) \gamma_{k+1}(0).$$

Therefore

$$(1) \quad \sigma_k(\omega) = -\frac{\omega^2}{\pi} \int_0^\infty \{F(x+t) - F(x-t)\} \gamma'_{k+1}(\omega t) dt.$$

Further, the k -th Rieszian mean of the series allied with the Fourier series of $F(t)$ at x is

$$(2) \quad \frac{\omega}{\pi} \int_0^\infty \{F(x+t) - F(x-t)\} \frac{\omega t \gamma_{k+2}(\omega t)}{k+1} dt,$$

and a precisely similar argument shows that

$$(3) \quad \tau_k(\omega) = \frac{\pi}{(k+1)\pi} \int_0^\infty \{F(x+t) + F(x-t)\} \frac{d}{dt} [\omega t \gamma_{k+2}(\omega t)] dt.$$

To prove formula (2) we observe that, if

$$F(x) \sim \frac{1}{2} a_0 + \sum_{m=1}^\infty (a_m \cos mx + b_m \sin mx),$$

then

$$\frac{1}{2} \{F(x+t) - F(x-t)\} \sim \sum_{m=1}^\infty (b_m \cos mx - a_m \sin mx) \sin mt.$$

If we write

$$\begin{aligned} g_{k+1}(t) &= \int_0^1 (1-u)^k \sin ut \, du, \\ &= \frac{t \gamma_{k+2}(t)}{k+1} = \frac{1-k \gamma_k(t)}{t}, \end{aligned}$$

then $g_{k+1}(\omega t)$ is of bounded variation in $(0, \infty)$ for $k \geq 0$, and it tends to 0 as $t \rightarrow \infty$. Hence, by a theorem of Hardy†,

$$\begin{aligned} \int_0^\infty \frac{F(x+t) - F(x-t)}{2} g_{k+1}(\omega t) dt \\ = \sum_{m=1}^\infty (b_m \cos mx - a_m \sin mx) \int_0^\infty g_{k+1}(\omega t) \sin mt \, dt. \end{aligned}$$

Now

$$g_{k+1}(\omega t) = \frac{1}{\omega} \int_0^\omega \left(1 - \frac{u}{\omega}\right)^k \sin tu \, du,$$

† E. W. Hobson, *The theory of functions of a real variable*, 2 (1926), 583.

so that, by Fourier's integral theorem,

$$\int_0^\infty g_{k+1}(\omega t) \sin mt \, dt = \frac{\pi}{2\omega} \left(1 - \frac{m}{\omega}\right)^k \quad (m < \omega),$$

$$= 0 \quad (m \geq \omega);$$

which proves (2).

5. Write

$$L_{k, \omega}(t) = \frac{d}{dt} \gamma_{k+1}(\omega t),$$

$$M_{k, \omega}(t) = \frac{d}{dt} [\omega t \gamma_{k+2}(\omega t)];$$

then the asymptotic behaviour of the integrals (4, 1) and (4, 3) may be deduced from the known properties of the function $\gamma_{k+1}(\omega t)$. In particular,

$$|L_{k, \omega}(t)| \leq \frac{A}{\omega^\alpha |t|^{1+\alpha}} \quad (k \geq 0),$$

where $\alpha = \min(k, 2)$ and A is a constant relative to the function considered; also

$$|M_{k, \omega}(t)| \leq \frac{A}{\omega^\alpha |t|^{1+\alpha}} \quad (k \geq -1),$$

where $\alpha = \min(k, 1)$ and A is a constant as before.

6. It follows from (4, 1) and the properties of the γ function that, if S is a constant,

$$\sigma_k(\omega) - S = -\frac{\omega^2}{\pi} \int_0^\infty \left\{ \frac{F(x+t) - F(x-t)}{t} - 2S \right\} t \gamma'_{k+1}(\omega t) \, dt.$$

If we write

$$\Phi_1(t) = F(x+t) - F(x-t) - 2St,$$

the necessary and sufficient condition for summability (C, k) to S at x is, for $k > 0$,

$$(1) \quad \lim_{\omega \rightarrow \infty} \omega^2 \int_0^\infty \Phi_1(t) \gamma'_{k+1}(\omega t) \, dt = 0.$$

7. In certain circumstances the condition (6, 1) can be simplified by a Riemann reduction. Let η be a positive number, and divide the range

into $(0, \eta)$, (η, ∞) . Then

$$\begin{aligned} \omega^2 \int_{\eta}^{\infty} \Phi_1(t) \gamma'_{k+1}(\omega t) dt \\ = \omega^2 \int_{\eta}^{\infty} \{F(x+t) - F(x-t)\} \gamma'_{k+1}(\omega t) dt - 2S\omega^2 \int_{\eta}^{\infty} t \{ \gamma'_{k+1}(\omega t) \} dt, \end{aligned}$$

and for $k > 1$ this tends to zero, provided only that $F(x)$ is integrable, since

$$\gamma'_{k+1}(t) = O\left(\frac{1}{t^{k+1}}\right), \quad \text{or} \quad O\left(\frac{1}{t^3}\right), \quad \text{for } k > 1.$$

For $k > 0$, the second term tends to zero, as may be seen by integrating by parts; moreover, if $F(x)$ is of bounded variation in $(0, 2\pi)$ say, then it is of bounded variation in $(\eta, 2r\pi)$ where r is a positive integer, and so

$$(1) \quad \omega^2 \int_{\eta}^{2r\pi} \{F(x+t) - F(x-t)\} \gamma'_{k+1}(\omega t) dt \rightarrow 0$$

as $\omega \rightarrow \infty$, as may be seen by expressing $\{F(x+t) - F(x-t)\}$ as the difference of two bounded monotone functions and applying the second mean value theorem. Further

$$(2) \quad \left| \omega \int_{s\pi}^{(s+2)\pi} \{F(x+t) - F(x-t)\} \gamma'_{k+1}(\omega t) dt \right| \\ < \left| \gamma'_{k+1}(\omega s\pi) \omega \int_{s\pi}^{(s+2)\pi} \{F(x+t) - F(x-t)\} dt \right| + MV,$$

where

$$M = \max \left| \int_a^{\beta} \omega \gamma'_{k+1}(\omega t) dt \right|, \quad s\pi \leq a < \beta \leq (s+2)\pi,$$

and V is the total variation of $\{F(x+t) - F(x-t)\}$ in $\{s\pi, (s+2)\pi\}^\dagger$.

The first term on the right-hand side of (2) is zero; V is constant, and M is $O\left(\frac{1}{(\omega s\pi)^{k+1}}\right)$, so that

$$(3) \quad \omega^2 \int_{2r\pi}^{\infty} \{F(x+t) - F(x-t)\} \gamma'_{k+1}(\omega t) dt = O\left(\frac{1}{\omega^k} \sum_{s=r}^{\infty} \frac{1}{(s\pi)^{k+1}}\right) \\ \rightarrow 0$$

as $\omega \rightarrow \infty$.

[†] Cf. E. W. Hobson, *The theory of functions of a real variable*, 1 (1927), 623.

Hence the necessary and sufficient condition for summability (C, k) to S at x of the differentiated series may be expressed in the form

$$(4) \quad \lim_{\eta \rightarrow 0} \overline{\lim}_{\omega \rightarrow \infty} \left| \omega^2 \int_0^\eta \Phi_1(t) \gamma'_{k+1}(\omega t) dt \right| = 0,$$

where it is supposed that $k > 0$ if $F(x)$ is a function of bounded variation, but $k > 1$ if $F(x)$ is merely integrable in the Lebesgue sense.

8. Write

$$\begin{aligned} \Phi_2(t) &= \int_0^t \Phi_1(u) du, \\ &\dots \quad \dots \quad \dots \\ \Phi_r(t) &= \int_0^t \Phi_{r-1}(u) du. \end{aligned}$$

Take $r \leq k-1$, and integrate the integral (7, 4) r times by parts. We obtain

$$\begin{aligned} \omega^2 \int_0^\eta \Phi_1(t) \gamma'_{k+1}(\omega t) dt &= [\omega^2 \gamma'_{k+1}(\omega t) \Phi_2(t) - \dots + (-1)^{r+1} \gamma^{(r)}_{k+1}(\omega t) \Phi_{r+1}(t)]_0^\eta \\ &\quad + (-1)^{r+2} \omega^{r+2} \int_0^\eta \gamma^{(r+1)}_{k+1}(\omega t) \Phi_{r+1}(t) dt. \end{aligned}$$

Since
$$\omega^{r+1} \gamma^{(r)}_{k+1}(\omega t) = O\left(\frac{1}{\omega}\right) \quad (r \leq k-1),$$

the condition (7, 4) is equivalent to

$$(1) \quad \lim_{\eta \rightarrow 0} \overline{\lim}_{\omega \rightarrow \infty} \left| \omega^{r+2} \int_0^\eta \gamma^{(r+1)}_{k+1}(\omega t) \Phi_{r+1}(t) dt \right| = 0 \quad (r \leq k-1).$$

9. A sufficient condition for summability (C, k) to S at x is

$$(1) \quad \phi_r^*(t) = \frac{1}{t} \int_0^t \frac{|\Phi_r(u)|}{u^r} du \rightarrow 0$$

as $t \rightarrow 0$, where r is a positive integer less than k .

For $r-1 \leq k-1$, and so

$$(2) \quad \lim_{\eta \rightarrow 0} \overline{\lim}_{\omega \rightarrow \infty} \left| \omega^{r+1} \int_0^\eta \gamma^{(r)}_{k+1}(\omega t) \Phi_r(t) dt \right| = 0$$

is equivalent to (8, 1).

In particular, when k is an integer,

$$(3) \quad \phi_{k-1}^*(t) \rightarrow 0$$

is sufficient; a special case of (3) is

$$(4) \quad \frac{\Phi_{k-1}(t)}{t^{k-1}} \rightarrow 0,$$

which ensures (3). Further, if k is integral and

$$\phi_k(t) = \frac{\Phi_k(t)}{t^k} \rightarrow 0,$$

the necessary and sufficient condition becomes

$$(5) \quad \lim_{\eta \rightarrow 0} \overline{\lim}_{\omega \rightarrow \infty} \left| \omega^{k+1} \int_{1/\omega}^{\eta} \Phi_k(t) \gamma_{k+1}^{(k)}(\omega t) dt \right| = 0,$$

and it can be shown that in this case summability may be reduced to convergence.

Using Young's criterion for the convergence of the Fourier series of $\phi_k(t)$ we find further that the conditions

$$(6) \quad \phi_k(t) = o(1), \quad \phi_{k-1}^*(t) = O(1),$$

are sufficient for summability (C, k) to $S(k > 1)$.

These results follow by precisely the same reasoning as in Pollard's paper†.

10. If $F(x)$ satisfies B of § 1, consider the condition (7, 4). Divide the range of integration into $(0, 1/\omega)$, $(1/\omega, \eta)$. Then

$$\left| \omega^2 \int_{1/\omega}^{\eta} \Phi_1(t) \gamma'_{k+1}(\omega t) dt \right| \leq \omega^2 \int_{1/\omega}^{\eta} |\Phi_1(t)| \frac{A}{|\omega t|^{1+\alpha}} dt,$$

where $\alpha = \min(k, 2)$. Integrating by parts, we have

$$\begin{aligned} \left| \omega^2 \int_{1/\omega}^{\eta} \Phi_1(t) \gamma'_{k+1}(\omega t) dt \right| &\leq \frac{A}{\omega^{\alpha-1}} \int_{1/\omega}^{\eta} \frac{|\Phi_1(t)|}{|t|^{\alpha}} dt \\ &= \frac{A}{\omega^{\alpha-1}} \left[\frac{\phi_1^*(t)}{t^{\alpha-1}} \right]_{1/\omega}^{\eta} + \frac{A\alpha}{\omega^{\alpha-1}} \int_{1/\omega}^{\eta} \frac{\phi_1^*(t)}{t^{\alpha}} dt \\ &= M_1 + M_2, \end{aligned}$$

† *Loc. cit.*, 379 *et seq.*

say. Let $S = F'(x)$; then, by (1, 2), $\xi(h)$ exists and

$$\xi(h) = \frac{1}{2} \left\| \frac{\Phi_1(t)}{t} \right\| \quad (0 < t \leq h);$$

consequently

$$2 \|\phi_1(t)\| = 2\xi(h),$$

and

$$\|\phi_1^*(t)\| = 2\xi(h).$$

Now

$$\xi(h) \rightarrow 0,$$

$$\text{as } h \rightarrow 0, \text{ and } |M_1| \leq \frac{A}{\omega^{a-1}} \left| \frac{\phi_1^*(\eta)}{\eta^{a-1}} \right| + A \left| \phi_1^* \left(\frac{1}{\omega} \right) \right|;$$

therefore

$$\lim_{\eta \rightarrow 0} \lim_{\omega \rightarrow \infty} |M_1| = 0 \quad (k > 0),$$

Further,

$$|M_2| \leq \frac{A'}{\omega^{a-1}} \omega^a \int_{1/\omega}^{\eta} |\phi_1^*(t)| dt \leq 2\omega A' \xi(\eta) \left[\eta - \frac{1}{\omega} \right],$$

which tends to 0 as $\eta \rightarrow 0$.

For the remainder of the integral

$$\left| \omega^2 \int_0^{1/\omega} \Phi(t) \gamma'_{k+1}(\omega t) dt \right| \leq \omega \int_0^{1/\omega} |\gamma_{k+1}(\omega t)| |d\Phi_1(t)| \leq A\omega \int_0^{1/\omega} |d\Phi_1(t)|.$$

Thus, if $F(x)$ satisfies B , the necessary and sufficient condition for summability (C, k) to $S = F'(x)$ at x is

$$(1) \quad \lim_{\omega \rightarrow \infty} \omega \int_0^{1/\omega} |d\Phi_1(t)| = 0 \quad (k > 1).$$

Further, in the special case A , $F'(x)$ exists almost everywhere, and by a theorem of Lebesgue, extended by Plessner, (1) is satisfied for almost all x ($k > 0$); hence the differentiated series of the Fourier-Lebesgue series of $F(x)$ is summable (C, k) , for $k > 0$, almost everywhere to $F'(x)$, when $F(x)$ is a function of bounded variation. It is, in particular, summable $(C, 1)$ almost everywhere, which is Plessner's result.

11. The summability of the allied series. Write

$$\Psi(t) = F(x+t) + F(x-t) - 2F(x).$$

Using (4, 3) we have

$$\tau_k(\omega) = \frac{\omega}{(k+1)\pi} \int_0^\infty \Psi(t) \frac{d}{dt} [\omega t \gamma_{k+2}(\omega t)] dt,$$

since

$$\int_0^\infty \frac{d}{dt} [\omega t \gamma_{k+2}(\omega t)] dt = 0.$$

Since

$$\frac{\omega t \gamma_{k+2}(\omega t)}{1+k} = \frac{1}{\omega t} - \frac{k \gamma_k(\omega t)}{\omega t},$$

we have

$$\frac{\omega}{(k+1)\pi} \int_\eta^\infty \Psi(t) \frac{d}{dt} [\omega t \gamma_{k+2}(\omega t)] dt = \frac{\omega}{\pi} \int_\eta^\infty \Psi(t) \left[-\frac{1}{\omega t^2} - \frac{d}{dt} \left\{ \frac{k \gamma_k(\omega t)}{\omega t} \right\} \right] dt.$$

Also

$$\frac{d}{dt} \left[\frac{k \gamma_k(\omega t)}{\omega t} \right] = -\frac{k \gamma_k(\omega t)}{\omega t^2} + \frac{k \gamma'_k(\omega t)}{t},$$

and it follows that, for $k > 1$,

$$(1) \quad \lim_{\omega \rightarrow \infty} \frac{\omega}{(k+1)\pi} \int_\eta^\infty \Psi(t) \frac{d}{dt} [\omega t \gamma_{k+2}(\omega t)] dt = -\frac{1}{\pi} \int_\eta^\infty \frac{\Psi(t)}{t^2} dt.$$

Now consider

$$\lim_{\eta \rightarrow 0} \lim_{\omega \rightarrow \infty} \frac{\omega}{(k+1)\pi} \int_0^\eta \Psi(t) \frac{d}{dt} [\omega t \gamma_{k+2}(\omega t)] dt.$$

Suppose that

$$(2) \quad \frac{\Psi(t)}{t} \rightarrow 0$$

as $t \rightarrow 0$. Then

$$(3) \quad \begin{aligned} & \frac{\omega}{(k+1)\pi} \int_{1/\omega}^\eta \Psi(t) \frac{d}{dt} [\omega t \gamma_{k+2}(\omega t)] dt \\ &= -\frac{1}{\pi} \int_{1/\omega}^\eta \frac{\Psi(t)}{t^2} dt + \frac{\omega}{\pi} \int_{1/\omega}^\eta \Psi(t) \left\{ \frac{k \gamma_k(\omega t)}{\omega t^2} - \frac{k \gamma'_k(\omega t)}{t} \right\} dt, \end{aligned}$$

and since $\Psi(t) = o(t)$, the second term in (3) tends to zero after $\lim_{\eta \rightarrow 0} \lim_{\omega \rightarrow \infty}$ since $k > 1$.

Finally,

$$\begin{aligned} & \lim_{\omega \rightarrow \infty} \frac{\omega}{(k+1)\pi} \int_0^{1/\omega} \Psi(t) \frac{d}{dt} [\omega t \gamma_{k+2}(\omega t)] dt \\ &= \lim_{\omega \rightarrow \infty} \frac{\omega}{(k+1)\pi} \int_0^{1/\omega} \Psi(t) [\omega \gamma_{k+2}(\omega t) + \omega^2 t \gamma'_{k+2}(\omega t)] dt, \\ &= o \left(\omega \int_0^{1/\omega} (\omega t + \omega^2 t^2) dt \right), \\ &= o(1). \end{aligned}$$

Hence the series allied with the differentiated series is summable (C, k) for $k > 1$ to T , where

$$(4) \quad T = \lim_{\eta \rightarrow 0} \frac{-1}{\pi} \int_{\eta}^{\infty} \frac{\Psi(t)}{t^2} dt,$$

provided that (2) is satisfied.

I have pleasure in taking this opportunity of expressing my thanks to Mr. Pollard for many valuable suggestions, and also to the referee for his careful criticism and help (particularly in §§ 7 and 11), and for his suggestions as to the form in which the paper should be presented.

THE QUADRATIC EQUATION IN QUATERNIONS

By D. E. LITTLEWOOD.

[Received 1 March, 1929.—Read 14 March, 1929.—Revised 20 June, 1929].

1. *Introduction.*

Little is known of equations other than linear* in non-commutative algebras. It has been shown†, however, that an equation of the form

$$ax^2 + bx + c = 0,$$

or

$$x^2a + xb + c = 0,$$

where a, b, c are constant quaternions, has in general six roots of which not more than two are real.

The general quadratic equation in quaternions takes the form

$$\Sigma a_0 x a_1 x a_2 + \Sigma b_0 x b_1 + c = 0.$$

It has been shown‡ that quaternion quadratics with an infinite number of roots exist, and that those with only a finite number of roots may have as many as sixteen. The question is further complicated by the existence of the unique equation satisfied by all quaternions§,

$$(xi)^2 + (xj)^2 + (xk)^2 - (ix)^2 - (jx)^2 - (kx)^2 = 0.$$

* A. R. Richardson, "Simultaneous linear equations over a division algebra", *Proc. London Math. Soc.* (2), 28 (1928), 395-420.

† Sir W. R. Hamilton, "Continued fractions in quaternions", *Phil. Mag.*, 3 (1852), 371; 4 (1852), 305; 5 (1853), 117, 236, 321. Sir W. R. Hamilton, *Lectures on quaternions* (1853), § 649. J. J. Sylvester, *Quart. J. of Math.*, 20 (1885), 305-312.

‡ J. J. Sylvester, *Comptes rendus*, 48 (1884), 651-652.

§ A. R. Richardson, "Equations over a division algebra", *Messenger of Math.*, 58 (1928), 1-6.

In this paper the solution of the quaternion quadratic is exhibited as the intersection of four quadrics in four-space. For this reason we consider only the quadratic with *real* quaternion coefficients, although the method is applicable to the quadratic with complex quaternion coefficients also. When the coefficients are unrestricted, there may be as many as sixteen roots, of which some, all, or none may be real. At most eleven only of the sixteen are independent, and these determine the other five.

Henceforward, for clarity, F will be used to denote a quaternion function and other capital letters to represent quaternion constants, while small letters will be used for scalars, and f will denote a scalar function. The letters i, j, k have their usual significance.

2. Reduction of quaternion quadratic.

The general quaternion quadratic equation is

$$\Sigma A_0 Q A_1 Q A_2 + \Sigma B_0 Q B_1 + C_0 = 0. \quad (1)$$

We proceed first to simplify the terms of the first degree $\Sigma B_0 Q B_1$.

If $B_1 = p + qi + rj + sk,$

then $B_0 Q B_1 = B_0 p Q + B_0 q Q i + B_0 r Q j + B_0 s Q k.$

Suppose that

$$\Sigma B_0 p = E, \quad \Sigma B_0 q = F, \quad \Sigma B_0 r = G, \quad \Sigma B_0 s = H,$$

then $\Sigma B_0 Q B_1 = EQ + FQi + GQj + HQk, \quad (2)$

which may be written

$$EQ - F' i Q i - G' j Q j - H' k Q k.$$

This form is chosen since it leads to greater simplicity.

Now, if $Q = x_0 + ix_1 + jx_2 + kx_3, \quad (3)$

then $4x_0 = Q - iQi - jQj - kQk,$

$$4ix_1 = Q - iQi + jQj + kQk,$$

$$4jx_2 = Q + iQi - jQj + kQk,$$

$$4kx_3 = Q + iQi + jQj - kQk,$$

and we have

$$\Sigma B_0 Q B_1 = U_0 x_0 + U_1 ix_1 + U_2 jx_2 + U_3 kx_3, \quad (4)$$

where

$$U_0 = E + F' + G' + H',$$

$$U_1 = E + F' - G' - H',$$

$$U_2 = E - F' + G' - H',$$

$$U_3 = E - F' - G' + H'.$$

We have thus expressed the linear terms as the sum of quaternion multiples of x_0, x_1, x_2, x_3 respectively. If now we separate the real part, and the coefficients of i, j, k respectively, in $U_0, U_1 i, U_2 j, U_3 k$, we arrive at

$$\Sigma B_0 Q B_1 = l_1 + i l_2 + j l_3 + k l_4, \quad (5)$$

where l_1, l_2, l_3, l_4 are real homogeneous linear forms in the four variables x_0, x_1, x_2, x_3 .

We next show that the second degree terms, $\Sigma A_0 Q A_1 Q A_2$, can be expressed in the form $f_1 + i f_2 + j f_3 + k f_4$, where f_1, f_2, f_3, f_4 are homogeneous quadratic forms in the same four variables.

If the quaternion constants A_1 and A_2 in $A_0 Q A_1 Q A_2$ be written in terms of scalars and the units i, j , and k , the corresponding terms may be separated as in the case of the linear expressions.

Then, clearly, $\Sigma A_0 Q A_1 Q A_2$ may be written in terms of

$$\begin{array}{llll} Q^2, & -iQ^2i, & -jQ^2j, & -kQ^2k \\ iQiQ, & QiQi, & +kQjQj, & -jQiQk, \\ jQjQ, & -kQjQi, & QjQj, & iQjQk, \\ kQkQ, & +jQkQi, & -iQkQj, & QkQk, \end{array}$$

with left-hand quaternion coefficients as in (2).

$$\begin{aligned} \text{Now let} \quad \alpha &= Q^2 - iQ^2i - jQ^2j - kQ^2k, \\ \beta &= Q^2 - iQ^2i + jQ^2j + kQ^2k, \\ \gamma &= Q^2 + iQ^2i - jQ^2j + kQ^2k, \\ \delta &= Q^2 + iQ^2i + jQ^2j - kQ^2k, \\ \alpha_1 &= iQiQ + QiQi - kQjQj + jQiQk, \\ &\dots \quad \dots \quad \dots \quad \dots \\ \delta_3 &= kQkQ - jQkQi + iQkQj + QkQk. \end{aligned}$$

As with the linear terms, the second degree terms may be expressed in the form

$$A\alpha + B\beta + C\gamma + D\delta + A_1\alpha_1 + B_1\beta_1 + \dots + D_3\delta_3, \quad (6)$$

the quaternion coefficients A, B, C , etc., being found in the same way as the linear terms (4).

It is clear that α is the real part of Q^2 ; β, γ, δ are the parts of Q^2 involving i, j, k respectively; α_1 is the real part of $(iQ)^2$, etc.

Since, from (3),

$$Q^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 + 2ix_0x_1 + 2jx_0x_2 + 2kx_0x_3,$$

it follows from (3) that

$$\begin{aligned} \alpha &= x_0^2 - x_1^2 - x_2^2 - x_3^2, & \beta &= 2ix_0x_1, & \gamma &= 2jx_0x_2, & \delta &= 2kx_0x_3, \\ \alpha_1 &= x_1^2 - x_0^2 - x_2^2 - x_3^2, & \beta_1 &= -2ix_0x_1, & \gamma_1 &= 2jx_1x_3, & \delta_1 &= -2kx_1x_2, \\ \alpha_2 &= x_2^2 - x_0^2 - x_1^2 - x_3^2, & \beta_2 &= -2ix_2x_3, & \gamma_2 &= -2jx_0x_2, & \delta_2 &= 2kx_1x_2, \\ \alpha_3 &= x_3^2 - x_0^2 - x_1^2 - x_2^2, & \beta_3 &= 2ix_2x_3, & \gamma_3 &= -2jx_1x_3, & \delta_3 &= -2kx_0x_3. \end{aligned}$$

We can thus express our quadratic terms as the sum of quaternion multiples of the terms of the general homogeneous quadratic in four variables.

On separating the scalar part, and those parts involving i, j, k respectively, we obtain

$$\Sigma A_0 Q A_1 Q A_2 = f_1 + if_2 + jf_3 + kf_4,$$

where f_1, f_2, f_3, f_4 are real homogeneous quadratic expressions in the four variables x_0, x_1, x_2, x_3 .

If we combine with (5) and the constant in (1), the equation (1) may be written

$$\phi_1 + i\phi_2 + j\phi_3 + k\phi_4 = 0,$$

where $\phi_1, \phi_2, \phi_3, \phi_4$ are four non-homogeneous scalar polynomials in the four scalar variables x_0, x_1, x_2, x_3 .

If x_0, x_1, x_2, x_3 are Cartesian coordinates of four-space, then $\phi_1 = 0, \phi_2 = 0, \phi_3 = 0, \phi_4 = 0$ represent four quadrics in four-space.

Our equation (1) will be satisfied if, and only if, the four expressions $\phi_1, \phi_2, \phi_3, \phi_4$ vanish together. That is to say, the roots of our equation are those quaternions represented by the points of intersection of the four quadrics.

The terms $\alpha, \beta, \dots, \alpha_1, \dots, \delta_3$ are independent and the coefficients are arbitrary. Though the terms $x_0^2, x_1^2, x_2^2, x_3^2$ are not given independently, clearly a simple linear transformation on the α 's will give them independently, so that our quadrics are in no way restricted.

3. *The quadratic identity.*

Of the terms $\alpha, \beta, \dots, \alpha_1, \dots, \delta_3$, it will be seen that some differ only in sign, *e.g.*, $\beta = 2ix_0x_1$, $\beta_1 = -2ix_0x_1$. In fact,

$$\begin{aligned}\beta + \beta_1 &= 0, & \gamma + \gamma_2 &= 0, & \delta + \delta_3 &= 0, \\ \beta_2 + \beta_3 &= 0, & \gamma_1 + \gamma_3 &= 0, & \delta_1 + \delta_2 &= 0.\end{aligned}$$

These represent equations satisfied by all quaternions, and consequently they are equivalent to the quadratic identity in all its forms

$$(iQ)^2 + (jQ)^2 + (kQ)^2 - (Qi)^2 - (Qj)^2 - (Qk)^2 = 0.$$

4. *The roots of the quadratic.*

As noted in § 1, we consider only the cases in which the coefficients in the quadratic are real quaternions. The resulting quadrics, then, have real coefficients. Corresponding to real points of intersection will be real quaternion roots of (1), and to unreal points, complex quaternion roots of (1).

In general, coefficients will be unrelated, and the equation (1) will then have sixteen quaternion roots. These roots are not independent, as will be shown. Fourteen points determine a quadric in four-space; two quadrics can be drawn through thirteen points; three linearly independent quadrics through twelve points; and four through eleven unconnected points.

Given eleven unconnected roots, these will determine but four quadrics through the corresponding points, the other five roots being given uniquely by the residual intersection.

Given twelve unconnected roots, only three quadrics will pass through the corresponding points. The four quadrics $\phi_1, \phi_2, \phi_3, \phi_4$ must be linearly dependent on these, and the whole eighth degree curve of intersection represents an infinity of roots of the equation.

Given thirteen roots, there will be but two independent quadrics intersecting in a quartic surface representing a double infinity of roots.

Given fourteen roots, the unique quadric through the corresponding points must represent a triple infinity of roots.

Given fifteen unconnected roots, in general no quadric will pass through all the corresponding points, and the quadratic satisfied by all the roots will be the quadratic identity in its various forms.

If some of the points of intersection of the quadrics are at infinity, the quadratic will be deficient in roots. Thus, if one of the equations, say

$\phi_4 = 0$, is linear only, there will be but eight roots. Again, if one of the four variables x_0, x_1, x_2, x_3 is absent from all four equations, these will represent four cylinders with parallel generators which in general have no common intersection. Thus the quaternion quadratic will have no roots, all having, in fact, become infinite.

5. Quadratics with the same roots.

If two quadratics have the same sixteen roots, they are linearly dependent over the quaternions, save for a quadratic identity. Suppose that the quadratics are

$$F = \phi_1 + i\phi_2 + j\phi_3 + k\phi_4 = 0,$$

$$F' = \phi'_1 + i\phi'_2 + j\phi'_3 + k\phi'_4 = 0.$$

The quadrics ϕ' must be linearly dependent on the quadrics ϕ . And

$$4\phi_1 = F - iFi - jFj - kFk, \text{ etc.}$$

Hence there will be a relation

$$F' = \Sigma A_1 F A_2 + R,$$

A_1, A_2 being constant quaternions and R a quadratic identity.

Similarly, if a quadratic F_1 is not deficient in roots, and a second quadratic F_2 is satisfied by all the roots of F_1 , then F_2 is linearly dependent on F_1 save for a quadratic identity.

6. Some examples.

Example 1. An instance of a quaternion quadratic with sixteen real quaternion roots.

$a = a_1 = a_2 = a_3 = -2$ is satisfied by the sixteen roots $\pm 1 \pm i \pm j \pm k$.

The equation is

$$\begin{aligned} Q^2 - iQ^2i - jQ^2j - kQ^2k - QiQ + iQiQi - jQiQj - kQiQk - QjQ - iQjQi \\ + jQjQj - kQjQk - QkQ - iQkQi - jQkQj + kQkQk + 8(1 + i + j + k) = 0. \end{aligned}$$

Example 2. Infinity of roots.

$$\begin{aligned} Q^2 - iQ^2i - jQ^2j - kQ^2k - QiQ + iQiQi - jQiQj - kQiQk - QjQ - iQjQi \\ + jQjQj - kQjQk + 8(1 + i + j) = 0. \end{aligned}$$

This has an eighth degree curve of roots $x_0 + ix_1 + jx_2 + kx_3$, where the x_s satisfy $x_0^2 = x_1^2 = x_2^2 = 2 - x_3^2$.

Example 3. A double infinity of roots.

$$Q^2 - iQ^2i - jQ^2j - kQ^2k - QiQ + iQiQi - jQiQj - kQiQk + 8(1+i) = 0.$$

The roots are represented by points on the quartic surface

$$x_0^2 = x_1^2, \quad x_2^2 = 2 - x_3^2.$$

Example 4. A triple infinity of roots.

$$Q^2 - iQ^2i - jQ^2j - kQ^2k + 8 = 0.$$

All points on the quadric $x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0$ represent roots.

Example 5.

The following equation has no roots in real or complex quaternions:—

$$Q^2 + kQ^2k - iQiQ - kQiQk + Q - iQi - jQj - kQk = 2i + j + k.$$

I wish to thank Prof. A. R. Richardson for introducing the problem to me.

DIVISOR FUNCTIONS: THEIR DIFFERENTIAL EQUATIONS AND RECURRENCE FORMULAE

By S. W. P. STEEN.

[Received 22 April, 1929.—Read 25 April, 1929.]

1. Introduction.

This paper introduces a class of functions which arises in connection with the expression of an integer n as the product of ν positive integers, that is in connection with the positive integral solutions of the Diophantine equation $n = n_1 n_2 \dots n_\nu$. The functions have, in an altered form, many of the properties of Bessel functions; in fact, the simpler varieties of these functions reduce to slight modifications of Bessel functions of the third kind. For instance, they satisfy a differential equation and recurrence formulae, they appear as coefficients in a certain expansion, exhibit a similar asymptotic behaviour, give rise to a Fourier integral theorem, provide discontinuous integrals, yield an integral representation for $(x+y)^{-1}$, furnish a formula similar to the Euler sum formula,* and admit of Fourier transforms. They may be obtained the one from the other by certain operations, one of which is fractional integration.†

* For the extension of this formula in connection with the solutions of the Diophantine equation $n = n_1^2 + n_2^2$, see Landau, *Vorlesungen über Zahlentheorie*, 2 (1927), 274, Satz 559. This formula, which is given for Bessel functions of order zero, may be extended to Bessel functions of any order.

† Bessel functions may be obtained the one from the other by fractional integration. We have

$$(2\sqrt{x})^\nu J_\nu(\sqrt{x}) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1} (2\sqrt{u})^\beta J_\beta(\sqrt{u}) du,$$

where $\alpha + \beta = \nu$, $R(\alpha) > 0$, $R(\beta) \geq -\frac{1}{2}$. When $\beta = -\frac{1}{2}$ the above equation is a slight modification of a well known integral for $J_\nu(x)$. In words the above equation becomes $(2\sqrt{x})^\nu J_\nu(\sqrt{x})$ is the α -th integral of $(2\sqrt{x})^\beta J_\beta(\sqrt{x})$ if $\alpha + \beta = \nu$.

Another formula will be given later.

The functions under consideration arise when we find alternative expressions for the meromorphic function

$$(1.10) \quad D_\nu(z) = \sum_{n=1}^{\infty} \frac{2z\sigma_\nu(n)}{z^2 - n^2},$$

where $\sigma_\nu(n)$ is the coefficient of n^{-s} in $\{\zeta(s)\}^\nu$. When ν is an integer, $\sigma_\nu(n)$ is the number of positive integral solutions of the equation $n = n_1 n_2 \dots n_\nu$, the order in which the n_i 's occur being material. In the present paper only the case when ν is an integer will be dealt with. The alternative expressions for the function $D_\nu(z)$ are given by the equations

$$(1.11) \quad D_\nu(z) = \begin{cases} -2\pi i \phi_\nu(z) + 2(-\frac{1}{2})^\nu z^{-1} - \sum_{n=1}^{\infty} \sigma_\nu(n) d_\nu(nz) & (Iz > 0), \\ 2\pi i \bar{\phi}_\nu(z) + 2(-\frac{1}{2})^\nu z^{-1} + \sum_{n=1}^{\infty} \sigma_\nu(n) \bar{d}_\nu(nz) & (Iz < 0), \end{cases}$$

where $\phi_\nu(z)$ is the residue of $\frac{z^s \{\zeta(s+1)\}^\nu}{1+e^{\pi i s}}$ at $s=0$, and where

$$(1.12) \quad d_\nu(z) = 2^{\nu-1} \int_{-c-i\infty}^{-c+i\infty} \{(2\pi)^\nu z\}^s \{\Gamma(-s)\}^\nu \{\cos \tfrac{1}{2}s\pi\}^{\nu-1} e^{-\frac{1}{2}\pi i s} ds \quad (c > 0),$$

$$(1.13) \quad \bar{d}_\nu(z) = 2^{\nu-1} \int_{-c-i\infty}^{-c+i\infty} \{(2\pi)^\nu z\}^s \{\Gamma(-s)\}^\nu \{\cos \tfrac{1}{2}s\pi\}^{\nu-1} e^{\frac{1}{2}\pi i s} ds \quad (c > 0).$$

The second expression in the equations (1.11) is obtained from the first by changing the sign of i throughout. These equations exhibit the asymptotic behaviour of the function $D_\nu(z)$ as z tends to infinity. The term $-2\pi i \phi_\nu(z)$ is the principal term in the upper half-plane, and the term $2\pi i \bar{\phi}_\nu(z)$ is the principal term in the lower half-plane*.

* The asymptotic behaviour of functions of the type (1.10) may also be found by the method illustrated by the following two examples:

$$(a) \quad \lim_{z \rightarrow \infty} \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = 2 \lim_{z \rightarrow \infty} \sum_{n=1}^{\infty} \frac{1}{1 - (n/z)^2} \frac{1}{z} = 2 \int_0^{\infty} \frac{\exp(-i\theta)}{1 - t^2} \frac{dt}{t} = \mp \pi i,$$

according as $Iz \gtrless 0$, where $z = \rho \exp(i\theta)$;

$$\begin{aligned} (b) \quad \lim_{z \rightarrow \infty} \sum_{n=1}^{\infty} \frac{2z r_2(n)}{z^2 - n^2} &= \lim_{z \rightarrow \infty} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{2z}{z^2 - (n^2 + m^2)^2} \\ &= 2 \lim_{z \rightarrow \infty} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{1 - \{(n/\sqrt{z})^2 + (m/\sqrt{z})^2\}^2} \frac{1}{\sqrt{z}} \frac{1}{\sqrt{z}} \\ &= 2 \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \frac{e^{-i\theta}}{1 - (u^2 + v^2)^2 e^{-i\theta}} = 2\pi \int_0^{\infty} \frac{\exp(-i\theta)}{1 - t^2} \frac{dt}{t} = \mp \pi^2 i, \end{aligned}$$

according as $Iz \gtrless 0$. In the case of $D_\nu(z)$ this method is more troublesome.

The divisor functions are defined by the equation

$$(1.2) \quad Y(a_1, a_2, \dots, a_\nu; \mu; z) = - \int_C z^s e^{\mu \pi i s} \Gamma(a_1 - s) \Gamma(a_2 - s) \dots \Gamma(a_\nu - s) ds,$$

where μ is an integer, and where the path of integration starts at $+\infty$ and, after encircling all the poles of the integrand counter-clockwise, returns to $+\infty$. We shall always take

$$(1.21) \quad -\pi < \arg z < \pi$$

unless otherwise stated, so that $z^s = \exp(s \log z)$, where $-\pi < I \log z < \pi$. The divisor functions are easily seen to be defined for all values of $\mu, a_1, a_2, \dots, a_\nu$ by the equation (1.2). For fixed values of $a_1, a_2, \dots, a_\nu$ we might consider $Y(a_1, a_2, \dots, a_\nu; \mu; z)$ as a many-valued function of z by absorbing the term $e^{\mu \pi i s}$ into the term z^s and removing the restriction (1.21). But it will be found more convenient (as in the case of Bessel functions of the third kind) to retain the restriction (1.21), and to consider the equation (1.2) as defining an infinity of one-valued functions. It will be shown in the course of this paper that only ν of these functions are independent.

Let $a_h = a_h + ib_h$ and $s = \sigma + it$, then the modulus of the subject of integration on the right-hand side of equation (1.2) is equal in absolute value to

$$\begin{aligned} \exp \left\{ \rho \sigma - \theta t - \pi \mu t + \sum_{h=1}^{\nu} \left(\frac{1}{2} (a_h - \sigma - \frac{1}{2}) \log [(a_h - \sigma)^2 + (b_h - t)^2] \right. \right. \\ \left. \left. - a_h + \sigma - (b_h \mp t)(\mp \frac{1}{2} \pi) \right) + O(1) \right\} \\ = \exp \left\{ -\theta t - \pi \mu t - \frac{1}{2} \nu \pi |t| + O(\log t) \right\} \quad \text{as } t \rightarrow \infty, \end{aligned}$$

where the $\mp$ sign is taken according as $t \gtrless 0$, and where $z = e^{\rho + i\theta}$. Thus

$$(1.22) \quad Y(a_1, a_2, \dots, a_\nu; \mu; z) = \int_{-c-i\infty}^{-c+i\infty} z^s e^{\mu \pi i s} \Gamma(a_1 - s) \Gamma(a_2 - s) \dots \Gamma(a_\nu - s) ds$$

if μ and z are restricted by

$$(1.23) \quad \left\{ \begin{array}{lll} \nu \text{ even,} & \mu = \frac{1}{2} \nu, & -\pi < \theta < 0, \\ & \mu = -\frac{1}{2} \nu, & 0 < \theta < \pi, \\ \nu \text{ odd,} & \mu = \frac{1}{2} (\nu - 1), & -\pi < \theta < \frac{1}{2} \pi, \\ & \mu = -\frac{1}{2} (\nu - 1), & -\frac{1}{2} \pi < \theta < \pi, \\ \text{for other values of } \mu \text{ such that } & -\frac{1}{2} \nu < \mu < \frac{1}{2} \nu, & -\pi < \theta < \pi, \end{array} \right.$$

while, if $\frac{1}{2}\nu\pi = |\theta + \mu\pi|$, c must be chosen so that

$$(1.231) \quad \sum_{\nu=1}^h (a_h + c - \tfrac{1}{2}) < -1.$$

The numbers $a_1, a_2, \dots, a_\nu$ will be called the orders of the divisor function $Y(a_1, a_2, \dots, a_\nu; \mu; z)$. Clearly the order in which the orders occur is immaterial.

The connection between the divisor functions and Bessel functions of the third kind is given by the equations

$$(1.24) \quad -2\pi^2 H_\nu^{(1)}(2\sqrt{z}) = e^{-\pi i \nu} z^{\frac{1}{2}\nu} Y(-\nu, 0; -1; z), \quad -\pi < \arg z < \pi,$$

$$2\pi^2 H_\nu^{(2)}(2\sqrt{z}) = e^{\pi i \nu} z^{\frac{1}{2}\nu} Y(-\nu, 0; 1; z), \quad -\pi < \arg z < \pi,$$

so that

$$(1.241) \quad 4\pi^2 J_\nu(2\sqrt{z}) = \{-e^{-\pi i \nu} Y(-\nu, 0; -1; z) + e^{\pi i \nu} Y(-\nu, 0; 1; z)\} z^{\frac{1}{2}\nu}.$$

Also

$$(1.25) \quad Y(a; \mu; z) = 2\pi i z^a e^{\mu\pi i a} \exp\{(-)^{\mu+1} z\}.$$

In the present paper § 2 provides the equations (1.11), introduces the divisor functions and gives their expansion as power series, notes certain cases of reduction consequent on the multiplication theorem of the Γ -function, and discusses the divisor functions with different values of μ . The proof of the equations (1.11) depends on the functional equation of the zeta-function. § 3 shows how the divisor functions may be obtained the one from the other by certain operations, one of which is fractional integration. It also gives various integral representations of these functions. § 4 gives the differential equation, which is satisfied by the divisor functions; it is a differential equation of order ν , the ν independent solutions are given. § 5 gives the recurrence formulae. The results of these two paragraphs follow easily from the integral defining the divisor function. § 6 exhibits a manner in which a certain combination of divisor functions appear as coefficients in a Laurent expansion. This is done by the use of the equation

$$e^{\frac{1}{2}z(t-t^{-1})} = \sum_{n=-\infty}^{\infty} t^n J_n(z)$$

and its fractional integrals, together with the results of § 3. § 7 gives a compact representation of $D_\nu(z)$. § 8 gives some functions allied to the function $D_\nu(z)$.

The remaining results may be obtained in the following ways. The Fourier integral theorem may be proved by a transformation of variables in the equation*

$$(1.3) \quad (2\pi)^{\nu} F(x_1 x_2 \dots x_{\nu}) \\ = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} du_1 \dots du_{\nu} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} F(v_1 v_2 \dots v_{\nu}) e^{-i \sum_{h=1}^{\nu} u_h (v_h - x_h)} dv_1 \dots dv_{\nu};$$

discontinuous integrals arise by taking

$$(1.31) \quad F(x) = F(x_1 x_2 \dots x_{\nu}) = \begin{cases} f(x) & (x < X) \\ 0 & (x > X). \end{cases}$$

The Fourier transforms follow from the Fourier integral, which is of the form

$$(1.32) \quad F(x) = \int_0^{\infty} dt \int_0^{\infty} F(s) K(st) K_1(tx) ds.$$

The integral representation of $(x+y)^{-1}$ arises from the consideration of integrals of the form†

$$(1.33) \quad \int_0^{\infty} \dots \int_0^{\infty} du_1 \dots du_{\nu} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} F(v_1 v_2 \dots v_{\nu}) e^{-i \sum_{h=1}^{\nu} u_h (v_h - x_h)} dv_1 \dots dv_{\nu}.$$

* The transformation (3.321) is applied to the v 's and then to the u 's; change the v 's by means of the scheme

$$v_1 u_1 = y_1, \quad v_2 u_2 = y_2/y_1, \quad \dots, \quad v_{\nu-1} u_{\nu-1} = y_{\nu-1}/y_{\nu-2}, \quad v_{\nu} = y_{\nu} u_1 u_2 \dots u_{\nu-1}/y_{\nu-1},$$

then $v_1 v_2 \dots v_{\nu} = y_{\nu}$ and the Jacobian is $(y_1 y_2 \dots y_{\nu-1})^{-1}$; then change the u 's by means of the scheme

$$u_1 x_1 = z_1, \quad u_2 x_2 = z_2/z_1, \quad \dots, \quad u_{\nu-1} x_{\nu-1} = z_{\nu-1}/z_{\nu-2}, \quad u_{\nu} = z_{\nu} x_1 x_2 \dots x_{\nu-1}/z_{\nu-1},$$

then $u_1 u_2 \dots u_{\nu} = z_{\nu}$ and the Jacobian is $(z_1 z_2 \dots z_{\nu-1})^{-1}$.

Write
$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{dz_1}{z_1} \dots \frac{dz_{\nu-1}}{z_{\nu-1}} \exp i \left\{ z_1 + \frac{z_2}{z_1} + \dots + \frac{z_{\nu-1}}{z_{\nu-2}} + \frac{z_{\nu}}{z_{\nu-1}} \right\} = H(z_{\nu} x),$$

then
$$(2\pi)^{\nu} F(x) = \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dy F(y) H(-zy) H(zx),$$

if $F(x_1, x_2, \dots, x_{\nu}) = F(x_1 x_2 \dots x_{\nu}) = F(x)$; $x_1 x_2 \dots x_{\nu} = x$.

† We have

$$\frac{1}{2\pi i} \frac{d}{dt} \int_0^{\infty} du \int_{-\infty}^{\infty} \{ \log_u(v-t) - \log_u v - \log_t(v-t) + \log_t v \} e^{-iu(v-x)} dv = \frac{i}{x-t},$$

where $\log_u x$ is to be continued by variation of x in the upper half-plane and $\log_t x$ is to be continued by variation in the lower half-plane.

Integrals of this type also yield some discontinuous integrals. Using one of these discontinuous integrals, or Cauchy's theorem, a formula analogous to the Euler sum formula may be proved, namely:

$$(1.34) \quad \sum_{n=1}^{\infty} \sigma_{\nu}(n) f(n) = \int_0^{\infty} f(x) \{ \bar{\phi}_{\nu}(x) + \phi_{\nu}(x) \} dx \\ + \sum_{n=1}^{\infty} \frac{\sigma_{\nu}(n)}{2\pi i} \int_0^{\infty} f(x) \{ \bar{d}_{\nu}(nx) + d_{\nu}(nx) \} dx.$$

In cases where either or both sides are divergent, but are summable by some method, we mean summability by this method.

If equation (1.3) is transformed by polar coordinates it yields the Fourier-Bessel integral theorem*, similarly the integral (1.33) transformed into polar coordinates yields discontinuous integrals involving Bessel functions of the third kind.

Much assistance has been obtained from the use of Prof. Watson's invaluable book on Bessel functions.

2. The function $D_{\nu}(z)$.

Let $z = \exp(\rho + i\theta)$ and $s = \sigma + it$, so that $z^s = \exp\{(\rho + i\theta)(\sigma + it)\}$, and let

$$(2.0) \quad I = \int_{a-i\infty}^{a+i\infty} \frac{z^s ds}{1+e^{\pi is}}, \quad \text{where } 0 < a < 1.$$

Then the following results are easily proved by Cauchy's theorem:—

If $0 < \theta < \pi$, $\rho < 0$, then

$$I = 2z(1-z^2)^{-1},$$

if $0 < \theta < \pi$, $\rho > 0$, then

$$I = -2z^{-1}(1-z^{-2})^{-1} = 2z(1-z^2)^{-1},$$

if $0 < \theta < \pi$, $\rho = 0$, then

$$I = \int_{a-i\infty}^{a+i\infty} \frac{e^{i\theta s}}{1+e^{\pi is}} ds = \lim_{\rho \rightarrow 0} \int_{a-i\infty}^{a+i\infty} \frac{e^{(\rho+i\theta)s}}{1+e^{\pi is}} = \frac{2z}{1-z^2}.$$

* See Ermakoff, *Math. Annalen*, 5 (1872), 639-40, for the case of two variables.

By changing the sign of i throughout in the above results we obtain the equation

$$I' = \int_{a-i\infty}^{a+i\infty} \frac{z^s}{1+e^{-\pi is}} ds = -\frac{2z}{1-z^2}$$

when $-\pi < \theta < 0$ and for all values of ρ .

Now let $0 < \theta < \pi$, then

$$(2.10) \quad -\int_{a-i\infty}^{a+i\infty} \frac{z^s \{\zeta(s+1)\}^\nu}{1+e^{\pi is}} ds = -\sum_{n=1}^{\infty} \frac{\sigma_\nu(n)}{n} \int_{a-i\infty}^{a+i\infty} \frac{(z/n)^s}{1+e^{\pi is}} ds \\ = \sum_{n=1}^{\infty} \frac{2z\sigma_\nu(n)}{z^2-n^2} = D_\nu(z);$$

similarly, when $-\pi < \theta < 0$,

$$\int_{a-i\infty}^{a+i\infty} \frac{z^s \{\zeta(s+1)\}^\nu}{1+e^{-\pi is}} ds = D_\nu(z).$$

Now take ν to be a positive integer.

Remembering that $|\zeta(s)| = O(t^k)$ for $\sigma_1 \leq \sigma \leq \sigma_2$, where k is a constant, we see that we may apply Cauchy's theorem to the integral on the left-hand side of equation (2.10), and so obtain the equation

$$\int_{a-i\infty}^{a+i\infty} \frac{z^s \{\zeta(s+1)\}^\nu}{1+e^{\pi is}} ds = 2\pi i \phi_\nu(z) - 2z^{-1}(-\tfrac{1}{2})^\nu + \int_{-c-i\infty}^{-c+i\infty} \frac{z^s \{\zeta(s+1)\}^\nu}{1+e^{\pi is}} ds,$$

where $1 < c < 3$, and $\phi_\nu(z)$ is the residue at $s=0$ of

$$\frac{z^s \{\zeta(s+1)\}^\nu}{1+e^{\pi is}}.$$

Clearly $\phi_\nu(z)$ is a polynomial in $\log z$ of degree $\nu-1$.

$$\text{Now} \quad 2\zeta(s) \Gamma(s) \cos(\tfrac{1}{2}s\pi) = (2\pi)^s \zeta(1-s),$$

$$\text{hence} \quad \zeta(s+1) = 2(2\pi)^s \zeta(-s) \Gamma(-s) \cos(\tfrac{1}{2}s\pi).$$

Thus, when $0 < \theta < \pi$,

$$(2.11) \quad D_\nu(z) = -2\pi i \phi_\nu(z) + 2z^{-1}(-\tfrac{1}{2})^\nu \\ - 2^{\nu-1} \int_{-c-i\infty}^{-c+i\infty} \{(2\pi)^\nu z\}^s \{\zeta(-s)\}^\nu \{\Gamma(-s)\}^\nu \{\cos(\tfrac{1}{2}s\pi)\}^{\nu-1} e^{-\frac{1}{2}\pi is} ds \\ = -2\pi i \phi_\nu(z) + 2z^{-1}(-\tfrac{1}{2})^\nu \\ - 2^{\nu-1} \sum_{n=1}^{\infty} \sigma_\nu(n) \int_{-c-i\infty}^{-c+i\infty} \{(2\pi)^\nu nz\}^s \{\Gamma(-s)\}^\nu \{\cos(\tfrac{1}{2}s\pi)\}^{\nu-1} e^{-\frac{1}{2}\pi is} ds,$$

since the series for $\zeta(s)$ is uniformly convergent, and the infinite integrals are absolutely convergent.

Now write

$$(2.12) \quad d_\nu(z) = 2^{\nu-1} \int_{-c-i\infty}^{-c+i\infty} \{(2\pi)^\nu z\}^s \{\Gamma(-s)\}^\nu \{\cos(\tfrac{1}{2}s\pi)\}^{\nu-1} e^{-\frac{1}{2}\pi is} ds.$$

The subject of integration in (2.12) is

$$\exp\{-\theta t - \tfrac{1}{2}\pi|t| + \tfrac{1}{2}\pi t + O(\log t)\},$$

so that, in the integral on the right-hand side of equation (2.12), we may take $c > 0$. Equation (2.11) now becomes

$$(2.21) \quad D_\nu(z) = -2\pi i \phi_\nu(z) + 2z^{-1}(-\tfrac{1}{2})^\nu - \sum_{n=1}^{\infty} \sigma_\nu(n) d_\nu(nz) \quad (0 < \theta < \pi).$$

By changing the sign of i throughout we obtain the equation

$$(2.22) \quad D_\nu(z) = 2\pi i \bar{\phi}_\nu(z) + 2z^{-1}(-\tfrac{1}{2})^\nu + \sum_{n=1}^{\infty} \sigma_\nu(n) \bar{d}_\nu(nz) \quad (-\pi < \theta < 0),$$

where

$$(2.23) \quad \bar{d}_\nu(nz) = 2^{\nu-1} \int_{-c-i\infty}^{-c+i\infty} \{(2\pi)^\nu nz\}^s \{\Gamma(-s)\}^\nu \{\cos(\tfrac{1}{2}s\pi)\}^{\nu-1} e^{\frac{1}{2}\pi is} ds.$$

(2.21) and (2.22) give (1.11). The integrals on the right-hand sides of (2.12) and (2.23) consist of a number of integrals of the form

$$(2.30) \quad \int_{-c-i\infty}^{-c+i\infty} z^s e^{\frac{1}{2}\mu\pi is} \{\Gamma(-s)\}^\nu ds,$$

where μ is an integer such that $-\nu \leq \mu \leq \nu$. Instead of considering the integral (2.30) we shall consider the more general integral

$$(1.2) \quad Y(a_1, a_2, \dots, a_\nu; \mu; z) = - \int_C z^s e^{\mu\pi is} \Gamma(a_1-s) \Gamma(a_2-s) \dots \Gamma(a_\nu-s) ds,$$

where μ is an integer, and where the path of integration starts at $+\infty$ and, after encircling all the poles of the integrand counterclockwise, returns to $+\infty$. Also

$$(1.21) \quad -\pi < \arg z < \pi.$$

The table (1.23) gives conditions under which the alternative integral

(1.22) may be used. For shortness we shall write

$$(2.40) \quad Y_\nu(\mu; z) = Y(0, 0, \dots, 0; \mu; z) = - \int_C z^s e^{\mu \pi i s} \{\Gamma(-s)\}^\nu ds \\ = \lim_{a_1 \rightarrow 0, \dots, a_\nu \rightarrow 0} Y(a_1, a_2, \dots, a_\nu; \mu; z).$$

The functions $Y(a_1, a_2, \dots, a_\nu; \mu; z)$ arise when we find a second expression for the function

$$(2.41) \quad \sum_{n=1}^{\infty} \frac{2z\tau_\nu(n)}{z^2 - n^2},$$

where $\tau_\nu(n) = \sum n_1^{a_1} n_2^{a_2} \dots n_\nu^{a_\nu}$ summed over all positive integral solutions of $n = n_1 n_2 \dots n_\nu$. [Clearly the series (2.41) is convergent when all the a 's are negative.]

We may obtain a power series for the divisor functions from the equation (1.2). The subject of integration has poles at $s = a_j + n$, $n = 0, 1, 2, \dots$, and the residue there is

$$-(-)^n z^{a_j+n} e^{\pi i \mu (a_j+n)} \Gamma(a_1 - a_j - n) \\ \dots \Gamma(a_{j-1} - a_j - n) \Gamma(a_{j+1} - a_j - n) \dots \Gamma(a_\nu - a_j - n) (n!)^{-1} \\ = \frac{(-)^{v(n+1)} z^{a_j+n} e^{\pi i \mu (a_j+n)} \pi^{v-1}}{n! \Gamma(n+1+a_j-a_1) \dots \Gamma(n+1+a_j-a_\nu) \sin \pi(a_j-a_1) \dots \sin \pi(a_j-a_\nu)},$$

where the asterisks denote that the terms $\Gamma(n+1+a_j-a_j)$ and $\sin \pi(a_j-a_j)$ have been omitted from the products in the denominators. Thus, if we suppose that $a_i - a_j$ is not an integer, we obtain the expansion

$$(2.50) \quad Y(a_1, a_2, \dots, a_\nu; \mu; z) = -2\pi i \frac{(-)^v \pi^{v-1} z^{a_1} e^{\pi i \mu a_1}}{\sin \pi(a_1-a_2) \dots \sin \pi(a_1-a_\nu)} \\ \times \sum_{n=0}^{\infty} \frac{(-)^{nv} z^n e^{\pi i \mu n}}{n! \Gamma(n+1+a_1-a_2) \dots \Gamma(n+1+a_1-a_\nu)} \\ + \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ -2\pi i \frac{(-)^v \pi^{v-1} z^{a_\nu} e^{\pi i \mu a_\nu}}{\sin \pi(a_\nu-a_1) \dots \sin \pi(a_\nu-a_{\nu-1})} \\ \times \sum_{n=0}^{\infty} \frac{(-)^{nv} z^n e^{\pi i \mu n}}{n! \Gamma(n+1+a_\nu-a_1) \dots \Gamma(n+1+a_\nu-a_{\nu-1})}.$$

This gives the behaviour of the divisor functions at the origin. If the

difference between two a 's, say a_1 and a_2 , is an integer, we may write $a_2 = a_1 + m + \xi$ in the equation (2.50), and proceed to the limit $\xi \rightarrow 0$, and so obtain the expansion corresponding to (2.50) in this case. Only two of the series on the right-hand side of (2.50) require consideration, the first and the second; they become:

$$\begin{aligned}
 & -2\pi i \frac{(-)^{\nu} \pi^{\nu-1} z^{a_1} e^{\pi i \mu a_1}}{\sin \pi(-m-\xi) \sin \pi(a_1-a_3) \dots \sin \pi(a_1-a_{\nu})} \\
 & \quad \times \sum_{n=0}^{\infty} \frac{(-)^{n\nu} z^n e^{\pi i \mu n}}{n! \Gamma(n+1-m-\xi) \Gamma(n+1+a_1-a_3) \dots \Gamma(n+1+a_1-a_{\nu})} \\
 & -2\pi i \frac{(-)^{\nu} \pi^{\nu-1} z^{a_1+m+\xi} e^{\pi i \mu(a_1+m+\xi)}}{\sin \pi(m+\xi) \sin \pi(a_1+m+\xi-a_3) \dots \sin \pi(a_1+m+\xi-a_{\nu})} \\
 & \quad \times \sum_{n=0}^{\infty} \frac{(-)^{n\nu} z^n e^{\pi i \mu n}}{n! \Gamma(n+1+m+\xi) \Gamma(n+1+m+\xi+a_1-a_3) \dots \Gamma(n+1+m+\xi+a_1-a_{\nu})} \\
 & = +2\pi i \frac{(-)^{\nu} \pi^{\nu-2} z^{a_1} e^{\pi i \mu a_1}}{\sin \pi(a_1-a_3) \dots \sin \pi(a_1-a_{\nu})} \\
 & \quad \times \sum_{n=0}^{m-1} \frac{(-)^{n(\nu+1)} z^n e^{\pi i \mu n} \Gamma(m-n)}{n! \Gamma(n+1+a_1-a_3) \dots \Gamma(n+1+a_1-a_{\nu})} \\
 & -2\pi i \frac{(-)^{\nu+m+1} \pi^{\nu-2} z^{a_1} e^{\pi i \mu a_1}}{\sin \pi(a_1-a_3) \dots \sin \pi(a_1-a_{\nu})} \\
 & \quad \times \sum_{n=m}^{\infty} \frac{(-)^{n\nu} z^n e^{\pi i \mu n} \Gamma'(n+1-m)}{n! \{\Gamma(n+1-m)\}^2 \Gamma(n+1+a_1-a_3) \dots \Gamma(n+1+a_1-a_{\nu})} \\
 & -2\pi i (-)^{\nu m} \pi^{\nu-2} z^{a_1} e^{\pi i \mu a_1} \left(\frac{d}{d\xi} \right)_{\xi=0} \\
 & \quad \times \sum_{n=0}^{\infty} \frac{(-)^{n\nu} z^{n+m+\xi} e^{\pi i \mu n}}{n! \Gamma(n+1+m+\xi) \Gamma(n+1+m+\xi+a_1-a_3) \dots \Gamma(n+1+m+\xi+a_1-a_{\nu}) \sin \pi(a_1+\xi-a_3) \dots \sin \pi(a_1+\xi-a_{\nu})};
 \end{aligned}$$

we proceed similarly in other cases.

Thus, if $a_1 = a_2$ and $R(a_1) < R(a_3) < R(a_j)$ for $j \geq 4$, then

$$Y(a_1, a_2, \dots, a_{\nu}; \mu; z) = z^{a_1}(a \log z + b) + O(z^{R(a_3)})$$

for small $|z|$, where a and b are constants.

We also have, if $a_i - a_j$ is not an integer,

$$(2.51) \quad Y(a_1, a_2, \dots, a_\nu; \mu; z)$$

$$= -\frac{2\pi i (-)^{\nu} \pi^{\nu-1} z^{a_1} e^{\pi i \mu a_1}}{\sin \pi(a_1 - a_2) \dots \sin \pi(a_1 - a_\nu)} \Gamma(1 + a_1 - a_2) \dots \Gamma(1 + a_1 - a_\nu) \\ \times {}_0F_\nu(; 1 + a_1 - a_2, \dots, 1 + a_1 - a_\nu; -e^{\pi i \mu} z) \\ + \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ -\frac{2\pi i (-)^{\nu} \pi^{\nu-1} z^{a_\nu} e^{\pi i \mu a_\nu}}{\sin \pi(a_\nu - a_1) \dots \sin \pi(a_\nu - a_{\nu-1})} \Gamma(1 + a_\nu - a_1) \dots \Gamma(1 + a_\nu - a_{\nu-1}) \\ \times {}_0F_\nu(; 1 + a_\nu - a_1, \dots, 1 + a_\nu - a_{\nu-1}; -e^{\pi i \mu} z)$$

in the notation of the hypergeometric functions. Writing

$$(2.60) \quad Y(a_1, a_2, \dots, a_\nu; \mu; z) = \sum_{h=1}^{\nu} z^{a_h} e^{\pi i \mu a_h} P_h(-e^{\pi i \mu} z)$$

we see that

$$Y(a_1, a_2, \dots, a_\nu; 2\mu; z) = \sum_{h=1}^{\nu} z^{a_h} e^{\pi i 2\mu a_h} P_h(-z)$$

and that

$$Y(a_1, a_2, \dots, a_\nu; 2\mu + 1; -z) = \sum_{h=1}^{\nu} z^{a_h} e^{\pi i (2\mu + \frac{1}{2}) a_h} P_h(-z),$$

where we take 0 or 2, according as z is in the upper or in the lower half-plane; thus the divisor functions with μ odd can be expressed in terms of divisor functions with μ even. Now consider the divisor functions corresponding to $\mu = 0, 2, \dots, 2\nu - 2$. Then any other divisor function with even μ is expressible linearly in terms of these, for we can find $c_0, \dots, c_{\nu-1}$ so that

$$(2.61) \quad e^{\pi i 2\mu a_h} = \sum_{k=0}^{\nu-1} c_k e^{\pi i 2k a_h} \quad \text{for } h = 1, 2, \dots, \nu$$

(the c 's depend on μ); whence we obtain the equation

$$(2.62) \quad Y(a_1, a_2, \dots, a_\nu; 2\mu; z) = \sum_{k=0}^{\nu-1} c_k Y(a_1, a_2, \dots, a_\nu; 2k; z).$$

Thus there are only ν independent divisor functions with even μ ; similarly there are ν independent divisor functions with odd μ .

If two or more α 's differ by an integer the divisor functions corresponding to $\mu = 0, 2, \dots, 2\nu-2$ are still independent.

The connection between the divisor functions and the function $d_\nu(z)$ is given by the equation

$$(2.70) \quad d_\nu(z) = - \int_C \left\{ (2\pi)^\nu z^{1/s} \right\}^\nu \Gamma(-s) \sum_{h=0}^{\nu-1} C_h e^{\frac{1}{2}\pi(\nu-2h-2)s} ds \\ = \sum_{h=0}^{\nu-1} C_h Y_\nu \left\{ \left[\frac{1}{2}\nu \right] - h - 1; (2\pi)^\nu z i^{\frac{1}{2}\nu - \left[\frac{1}{2}\nu \right]} \right\},$$

where $-\pi < \arg z i^{\frac{1}{2}\nu - \left[\frac{1}{2}\nu \right]} < \pi$; similarly

$$(2.71) \quad \bar{d}_\nu(z) = \sum_{h=0}^{\nu-1} C_h Y_\nu \left\{ \left[\frac{1}{2}\nu \right] - h; (2\pi)^\nu z i^{\frac{1}{2}\nu - \left[\frac{1}{2}\nu \right]} \right\},$$

where $-\pi < \arg z i^{\frac{1}{2}\nu - \left[\frac{1}{2}\nu \right]} < \pi$ and ${}_{\nu-1}C_h$ is a binomial coefficient.

There are various cases in which a divisor function with ν orders reduces to one with fewer orders. Consider

$$(2.72) \quad Y\left(\alpha_1, \alpha_1 + \frac{1}{n}, \dots, \alpha_1 + \frac{n-1}{n}, \alpha_2, \alpha_2 + \frac{1}{n}, \dots, \alpha_h + \frac{n-1}{n}; \mu; z\right) \\ = - \int_C z^s e^{\mu\pi is} \Gamma(\alpha_1 - s) \dots \Gamma\left(\alpha_1 + \frac{n-1}{n} - s\right) \Gamma(\alpha_2 - s) \dots \Gamma\left(\alpha_h + \frac{n-1}{n} - s\right) ds \\ = - \int_C z^s e^{\mu\pi is} n^{\frac{1}{2}h - n(\alpha_1 + \alpha_2 + \dots + \alpha_h) + hns} (2\pi)^{\frac{1}{2}h(n-1)} \Gamma(\alpha_1 n - ns) \dots \Gamma(\alpha_h n - ns) ds \\ = - n^{\frac{1}{2}h - 1 - n(\alpha_1 + \dots + \alpha_h)} (2\pi)^{\frac{1}{2}h(n-1)} \int_C z^{s/n} e^{\mu\pi is/n} n^{hns} \Gamma(\alpha_1 n - s) \dots \Gamma(\alpha_h n - s) ds \\ = - n^{\frac{1}{2}h - 1 - n(\alpha_1 + \dots + \alpha_h)} (2\pi)^{\frac{1}{2}h(n-1)} Y(\alpha_1 n, \dots, \alpha_h n; [\mu/n]; z^{1/n} n^{hn} e^{\pi i(\mu/n)}),$$

where $[\mu/n]$ denotes the integral part of μ/n , and (μ/n) denotes the fractional part of μ/n .

3. Methods of generating divisor functions.

It is easily seen that, for all values of z ,

$$(3.0) \quad \int_{C'} t^{z-1} e^t dt = 2i\Gamma(z) \sin \pi z,$$

where C' is a path beginning at $-\infty$, encircling the origin counter-clock-

wise and returning to $-\infty$. The many-valued function t^{z-1} is defined by the equation $t^{z-1} = \exp \{ (z-1) \log t \}$, where $-\pi < \arg t < \pi$.

Now consider $Y(a_1, a_2, \dots, a_\nu; \mu; z/t)$, where t is real and positive, it has a definite value by virtue of the convention (1.21). Let t vary along the contour C' , then the corresponding values of $Y(a_1, a_2, \dots, a_\nu; \mu; z/t)$ will vary continuously until $\arg(z/t) = \pm\pi$, when the convention (1.21) will cause a discontinuity; we can overcome this by taking

$$(3.01) \quad -2\pi < \arg(z/t) < 2\pi.$$

Now consider

$$(3.10) \quad \int_{C'} e^t Y(a_1, a_2, \dots, a_\nu; \mu; z/t) t^{a-1} dt,$$

where the inequalities (3.01) are satisfied and where, when $\arg t = 0$, the Y function is to have its previous meaning.

The integral (3.10) is equal to

$$\begin{aligned} & - \int_{C'} e^t \int_C z^s t^{-s+a-1} e^{\mu\pi is} \Gamma(a_1-s) \dots \Gamma(a_\nu-s) ds dt \\ &= -2i \int_C z^s e^{\mu\pi is} \sin \pi(a-s) \Gamma(a-s) \Gamma(a_1-s) \dots \Gamma(a_\nu-s) ds \\ &= - \int_C z^s \{ e^{\pi ia} e^{(\mu-1)\pi is} - e^{-\pi ia} e^{(\mu+1)\pi is} \} \Gamma(a-s) \Gamma(a_1-s) \dots \Gamma(a_\nu-s) ds \\ (3.11) \quad &= e^{\pi ia} Y(a, a_1, a_2, \dots, a_\nu; \mu-1; z) \\ & \quad - e^{-\pi ia} Y(a, a_1, a_2, \dots, a_\nu; \mu+1; z). \end{aligned}$$

Note that the convention (3.01) could be avoided by the use of a contour beginning at $\infty \exp(\theta-\pi)i$, encircling the origin counter-clockwise, and returning to $\infty \exp(\theta+\pi)i$, instead of the contour C' .

Next consider the integral

$$(3.20) \quad \int_0^\infty e^{-t} Y(a_1, a_2, \dots, a_\nu; \mu; z/t) t^{a-1} dt,$$

where the divisor function is expressed by the integral (1.22); this implies that we are restricted to the table (1.23). We shall also suppose

that $c + R(a) > 1^*$, all the integrals that occur are then absolutely convergent, and the integrands are bounded. Hence the integral (3.20) is equal to

$$(3.21) \quad \int_{-c-i\infty}^{-c+i\infty} z^s e^{\mu\pi is} \Gamma(a-s) \Gamma(a_1-s) \dots \Gamma(a_\nu-s) ds \\ = Y(a, a_1, a_2, \dots, a_\nu; \mu; z).$$

Thus we obtain successively, if $R(z) > 0$,

$$(3.22) \quad Y(a_1, a_2, \dots, a_\nu; 0; z) = \int_0^\infty e^{-t_1} t_1^{a_1-1} Y(a_2, a_3, \dots, a_\nu; 0; z/t_1) dt_1 \\ = \int_0^\infty e^{-t_1} t_1^{a_1-1} \int_0^\infty e^{-t_2} t_2^{a_2-1} \dots \int_0^\infty e^{-t_{\nu-1}} t_{\nu-1}^{a_{\nu-1}-1} \\ \times Y(a_\nu; 0; z/t_1 t_2 \dots t_{\nu-1}) dt_{\nu-1} \dots dt_2 dt_1 \\ = 2\pi i z^{a_\nu} \int_0^\infty e^{-t_1} t_1^{a_1-a_\nu-1} \int_0^\infty e^{-t_2} t_2^{a_2-a_\nu-1} \\ \dots \int_0^\infty e^{-t_{\nu-1}} t_{\nu-1}^{a_{\nu-1}-a_\nu-1} \exp \{ -z/(t_1 t_2 \dots t_{\nu-1}) \} dt_{\nu-1} \dots dt_1.$$

Again we have, if $Ra > 0$, and if the divisor function is expressed by the integral (1.22),

$$(3.23) \quad \int_0^\infty e^{-at} Y(a_1, a_2, \dots, a_\nu; \mu; z/t) t^{a-1} dt \\ = a^{-a} \int_{-c-i\infty}^{-c+i\infty} z^s e^{\mu\pi is} a^s \Gamma(a-s) \Gamma(a_1-s) \dots \Gamma(a_\nu-s) ds \\ = a^{-a} Y(a, a_1, a_2, \dots, a_\nu; \mu; za)$$

if $-\pi < \arg az < \pi$ and (1.21),

these last two restrictions are certainly satisfied if $-\frac{1}{2}\pi < \arg z < \frac{1}{2}\pi$. If $\arg az > \pi$, then μ is to be replaced by $\mu+1$, and if $\arg az < -\pi$, then μ is to be replaced by $\mu-1$.

* $\text{Max} \{ R(-a_1), \dots, R(-a_\nu) \} < c$.

Next, again using the integral (1.22) for the divisor function, consider

$$(3.24) \quad \int_0^{\infty} e^{it} Y(a_1, a_2, \dots, a_\nu; \mu; z/t) t^{a-1} dt,$$

where we suppose that it is possible to choose c so that $0 < c + R(a) < 1$; this is possible if $\text{Max} \{R(-a_1), \dots, R(-a_\nu)\} < 1 - R(a)$. Divide the range of integration in (3.24) into two parts 0 to 1 and 1 to ∞ . In the first part, let the divisor function be expressed by the integral (1.22) with $c > 1 - R(a)$, and then we may interchange the order of the integrations: that is, we obtain the equation

$$(3.241) \quad \int_0^1 e^{it} t^{a-1} Y(a_1, a_2, \dots, a_\nu; \mu; z/t) dt \\ = \int_{-c-i\infty}^{-c+i\infty} \int_0^1 e^{it} t^{a-s-1} z^s e^{\mu\pi is} \Gamma(a_1-s) \dots \Gamma(a_\nu-s) dt ds.$$

In the second part, let the divisor function be represented by the integral (1.2), then we may interchange the orders of integration, and so obtain the equation

$$(3.242) \quad \int_1^{\infty} e^{it} t^{a-1} Y(a_1, a_2, \dots, a_\nu; \mu; z/t) dt \\ = - \int_c^{\infty} \int_1^{\infty} e^{it} t^{a-s-1} z^s e^{\mu\pi is} \Gamma(a_1-s) \dots \Gamma(a_\nu-s) dt ds,$$

where $R(a-s) < 1$ on the contour.

Now $\int_0^1 e^{it} t^{a-s-1} dt$ is a regular function of s for all values of s such that $R(a-s) > 0$; also for such values of s its absolute value is less than 1; hence the contour on the right-hand side of equation (3.241) may be replaced by one for which $0 < R(a+c) < 1$. Again, $\int_1^{\infty} e^{it} t^{a-s-1} dt$ is a regular function of s for all values of s such that $R(a-s) < 1$, for

$$\int_1^{\infty} e^{it} t^{a-s-1} dt = i + i \int_1^{\infty} (a-s-1) e^{it} t^{a-s-2} dt,$$

also its absolute value is $O(|s|)$ when $R(a-s) < 1$. Thus the contour on the right-hand side of equation (3.242) may be replaced by one from $-c-i\infty$ to $-c+i\infty$ where $0 < R(a+c) < 1$. Thus, finally, if it

is possible to take $0 < R(a+c) < 1$,

$$\begin{aligned}
 (3.25) \quad & \int_0^\infty e^{it} Y(a_1, a_2, \dots, a_\nu; \mu; z/t) t^{a-1} dt \\
 &= e^{\frac{1}{2}\pi ia} \int_{-c-i\infty}^{-c+i\infty} z^s e^{\mu\pi is} e^{-\frac{1}{2}\pi is} \Gamma(a-s) \Gamma(a_1-s) \dots \Gamma(a_\nu-s) ds \\
 &= e^{\frac{1}{2}\pi ia} Y(a, a_1, a_2, \dots, a_\nu; \mu; -iz) \quad \text{if } -\frac{1}{2}\pi < \arg z < \pi;
 \end{aligned}$$

if $-\pi < \arg z < -\frac{1}{2}\pi$, then μ must be replaced by $\mu-1$. From the equation (3.25) we obtain the equation

$$\begin{aligned}
 (3.26) \quad & \int_0^\infty e^{it} Y(a_1, a_2, \dots, a_\nu; \mu; iz/t) t^{a-1} dt \\
 &= e^{\frac{1}{2}\pi ia} Y(a, a_1, a_2, \dots, a_\nu; \mu; z) \quad \text{if } -\pi < \arg z < \frac{1}{2}\pi;
 \end{aligned}$$

and if $\frac{1}{2}\pi < \arg z < \pi$, μ is to be replaced by $\mu+1$.

From the equation (3.26) we obtain successively, if $\mu = -\frac{1}{2}\nu\pi$, z is real, and (1.231) is valid,

$$\begin{aligned}
 Y(a_1, a_2, \dots, a_\nu; \mu; z) &= e^{-\frac{1}{2}\pi i(a_1+\dots+a_{\nu-1})} \int_0^\infty e^{it_1} t_1^{a_1-1} \dots \int_0^\infty e^{it_{\nu-1}} t_{\nu-1}^{a_{\nu-1}-1} \\
 &\quad \times Y(a_\nu; \mu; i^{\nu-1} z/t_1 \dots t_{\nu-1}) dt_{\nu-1} \dots dt_1,
 \end{aligned}$$

where

$$\begin{aligned}
 Y(a_\nu; \mu; i^{\nu-1} z/t_1 \dots t_{\nu-1}) &= - \int_C z^s e^{\mu\pi is} e^{\frac{1}{2}\pi i(\nu-1)s} (t_1 \dots t_{\nu-1})^{-s} \Gamma(a_\nu-s) ds \\
 &= 2\pi i z^{\sigma_\nu} (t_1 \dots t_{\nu-1})^{-a_\nu} e^{a_\nu \pi i \{\mu + \frac{1}{2}(\nu-1)\}} \\
 &\quad \times \exp \{ e^{\pi i \{\mu + \frac{1}{2}(\nu+1)\}} z/t_1 \dots t_{\nu-1} \},
 \end{aligned}$$

so that

$$\begin{aligned}
 (3.30) \quad & Y(a_1, a_2, \dots, a_\nu; -\frac{1}{2}\nu\pi; z) \\
 &= 2\pi i z^{a_\nu} e^{-\frac{1}{2}\pi i(a_1+\dots+a_\nu)} \\
 &\quad \times \int_0^\infty e^{it_1} t_1^{a_1-a_\nu-1} \dots \int_0^\infty e^{it_{\nu-1}} t_{\nu-1}^{a_{\nu-1}-a_\nu-1} \exp \{ iz/t_1 \dots t_{\nu-1} \} dt_{\nu-1} \dots dt_1,
 \end{aligned}$$

where $0 < R(a_i - a_\nu) < 1^*$, and $Ra_1 \geq Ra_2 \geq \dots \geq Ra_\nu$.

A similar formula is obtained by changing the sign of i throughout.

* For the first step we require $0 \leq R(a_1 - a_i) < 1$, for $i = 2, 3, \dots, \nu$;
 for the second step we require $0 \leq R(a_2 - a_i) < 1$, for $i = 3, 4, \dots, \nu$;

 for the last step we require $0 \leq R(a_{\nu-1} - a_\nu) < 1$.

If all the α 's are zero we obtain the equation

$$(3.31) \quad Y_\nu(-\tfrac{1}{2}\nu\pi; z) = 2\pi i \int_0^\infty e^{it_1} \frac{dt_1}{t_1} \dots \int_0^\infty e^{it_{\nu-1}} \frac{dt_{\nu-1}}{t_{\nu-1}} e^{iz/t_1 \dots t_{\nu-1}}.$$

If A denotes the domain $y_1 y_2 \dots y_\nu \leq x$, $y_i \geq 0$, we have

$$(3.32) \quad 2\pi i \int_A e^{i(y_1 + y_2 + \dots + y_\nu)} dv \\ = 2\pi i \int_0^x dt_\nu \int_0^\infty \frac{dt_1}{t_1} \dots \int_0^\infty \frac{dt_{\nu-1}}{t_{\nu-1}} e^{it_1} e^{it_2/t_1} \dots e^{it_\nu/t_{\nu-1}};$$

where we have used the transformation

$$y_1 = t_1, \quad y_2 = \frac{t_2}{t_1}, \quad \dots, \quad y_\nu = \frac{t_\nu}{t_{\nu-1}},$$

the Jacobian is $(t_1 t_2 \dots t_{\nu-1})^{-1}$.

The multiple integral on the right-hand side of (3.32) is equal to

$$(3.33) \quad 2\pi i \int_0^x dt_\nu \int_0^\infty \frac{dt_1}{t_1} \dots \int_0^\infty \frac{dt_{\nu-1}}{t_{\nu-1}} e^{i(t_1 + t_2 + \dots + t_{\nu-1}) + it_\nu/(t_1 t_2 \dots t_{\nu-1})}.$$

Just as we obtained the equation (3.31) so we may obtain the equation

$$(3.311) \quad - \int_C z^s e^{i\mu\pi is} \{\Gamma(-s)\}^\nu ds \\ = 2\pi i \int_0^\infty e^{\pm it_1} \frac{dt_1}{t_1} \dots \int_0^\infty e^{\pm it_{\nu-1}} \frac{dt_{\nu-1}}{t_{\nu-1}} e^{\pm iz/(t_1 t_2 \dots t_{\nu-1})}$$

where μ is the difference between the number of minus signs and the number of plus signs.

Hence we obtain the equation

$$(3.34) \quad \frac{1}{2\pi i} \int_C \{(2\pi)^\nu nx\}^s \{\Gamma(-s)\}^\nu \{\cos \tfrac{1}{2}\pi s\}^\nu ds \\ = \int_0^\infty \frac{dt_1}{t_1} \dots \int_0^\infty \frac{dt_{\nu-1}}{t_{\nu-1}} \cos t_1 \cos t_2/t_1 \dots \cos t_{\nu-1}/t_{\nu-2} \cos \{(2\pi)^\nu xn/t_{\nu-1}\},$$

and thus finally, if $n_1 n_2 \dots n_\nu = n$, we have

$$(3.35) \quad 2^\nu \int_A \cos 2n_1 \pi x_1 \dots \cos 2n_\nu \pi x_\nu dv = \frac{1}{2\pi i} \int_0^x \{\bar{d}_\nu(nx) + d_\nu(nx)\} dx^*.$$

* The same function integrated over a quadrant of a hypersphere $x_1^2 + x_2^2 + \dots + x_\nu^2 \leq x$ can be expressed in terms of Bessel functions.

This equation gives the connection with the real variable counterpart of the present paper, *i.e.* Pfeiffer's method. In a similar manner we have the equation

$$(3.36) \quad 2^{\nu} \int_A f(x_1 x_2 \dots x_{\nu}) \cos 2n_1 \pi x_1 \dots \cos 2n_{\nu} \pi x_{\nu} dv \\ = \frac{1}{2\pi i} \int_0^x f(x) \{ \bar{d}_{\nu}(nx) + d_{\nu}(nx) \} dx.$$

Divisor functions may also be obtained the one from the other by fractional integration. The α -th fractional integral of a function $f(x)$ is given by the equation

$$(3.40) \quad f_{\alpha}(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad R(\alpha) > 0.$$

Consider the function

$$(3.41) \quad f(z) = - \int_C \sin \pi(s-a_1) z^{s-a_1} e^{\mu \pi i s} \Gamma(a_1-s) \dots \Gamma(a_{\nu}-s) ds \\ = -\frac{1}{2} i z^{-a_1} e^{-\pi i a_1} Y(a_1, a_2, \dots, a_{\nu}; \mu+1; z) \\ + \frac{1}{2} i z^{-a_1} e^{\pi i a_1} Y(a_1, a_2, \dots, a_{\nu}; \mu-1; z).$$

The α -th fractional integral of this function is

$$(3.42) \quad f_{\alpha}(z) = - \frac{1}{\Gamma(\alpha)} \int_C \sin \pi(s-a_1) e^{\mu \pi i s} \Gamma(a_1-s) \dots \Gamma(a_{\nu}-s) \\ \times \int_0^z (z-t)^{\alpha-1} t^{s-a_1} dt,$$

where we shall suppose that $\text{Min} \{R(a_1), \dots, R(a_{\nu})\} > R(a_1)-1$, so that it is possible to have $R(s-a_1) > -1$ on the contour C ; the integral on the right-hand side of equation (3.42) is equal to

$$- \frac{1}{\Gamma(\alpha)} \int_C \sin \pi(s-a_1) e^{\mu \pi i s} \Gamma(a_1-s) \dots \Gamma(a_{\nu}-s) \frac{z^{s+a-\alpha_1} \Gamma(\alpha) \Gamma(s-a_1+1)}{\Gamma(s-a_1+\alpha+1)} ds \\ = - \int_C \sin \pi(s-a_1+\alpha) e^{\mu \pi i s} z^{s+a-\alpha_1} \Gamma(a_1-\alpha-s) \Gamma(a_2-s) \dots \Gamma(a_{\nu}-s) ds \\ (3.43) = -\frac{1}{2} i z^{-(a_1-\alpha)} e^{-\pi i (a_1-\alpha)} Y(a_1-\alpha, a_2, \dots, a_{\nu}; \mu+1; z) \\ + \frac{1}{2} i z^{-(a_1-\alpha)} e^{\pi i (a_1-\alpha)} Y(a_1-\alpha, a_2, \dots, a_{\nu}; \mu-1; z).$$

This is the same as the function $f(z)$ defined by equation (3.41), except

that a_1 has been replaced by $a_1 - \alpha$. For the case $\nu = 2$ this becomes a result concerning Bessel functions.

The above result may be extended by removal of the restriction $\text{Min}\{R(a_1), \dots, R(a_\nu)\} > R(a_1) - 1$ if we replace equation (3.40) by the equation

$$(3.44) \quad 2i \sin \pi \alpha f(a) = -\frac{1}{\Gamma(a)} \int_x^{(0+)} (t-x)^{a-1} f(t) dt.$$

Note that, if a divisor function reduces to the form

$$(3.50) \quad -\int_C z^s e^{\mu\pi i s} \Gamma(a_1 + \dots + a_n - ns) ds,$$

by employing Dirichlet's integral

$$(3.51) \quad \begin{aligned} & \int_A x_1^{a_1-s-1} \dots x_{n-1}^{a_{n-1}-s-1} (1-x_1-\dots-x_{n-1})^{a_n-s-1} dv \\ &= \frac{\Gamma(a_1-s) \dots \Gamma(a_n-s)}{\Gamma(a_1+\dots+a_n-ns)}, \end{aligned}$$

we may obtain divisor functions of orders $a_1, a_2, \dots, a_n$.

In the integral (3.51) integration is over the domain

$$x_1 + x_2 + \dots + x_n \leq 1; \quad x_i \geq 0 \quad (i = 1, 2, \dots, n); \quad \text{and} \quad R(a_i - s) > 1.$$

4. The differential equation.

First, we shall find the differential equation satisfied by $Y_\nu(\mu; z)$. We have

$$\frac{d}{dz} Y_\nu(\mu; z) = \int_C z^{s-1} e^{\mu\pi i s} \{\Gamma(-s)\}^{\nu-1} \Gamma(-s+1) ds,$$

$$\frac{d^2}{dz^2} Y_\nu(\mu; z) = -\int_C z^{s-2} e^{\mu\pi i s} \{\Gamma(-s)\}^{\nu-1} \Gamma(-s+2) ds,$$

and, generally,

$$(4.1) \quad \frac{d^k}{dz^k} Y_\nu(\mu; z) = (-)^{k+1} \int_C z^{s-k} e^{\mu\pi i s} \{\Gamma(-s)\}^{\nu-1} \Gamma(-s+k) ds.$$

Now

$$(4.2) \quad (-s)^n = \sum_{h=0}^{n-1} (-)^{n+1} C_{n-h+1}^h (-s)(-s+1) \dots (-s+h),$$

where the coefficients are Stirling's numbers of the second kind.* Thus

$$\begin{aligned}
 (-)^{\nu+1} \sum_{h=0}^{\nu-1} C_{\nu-h+1}^h (-)^{h+1} z^{h+1} \frac{d^{h+1}}{dz^{h+1}} Y_{\nu}(\mu; z) \\
 = (-)^{\nu+2} \int_C z^s e^{\mu\pi is} \{\Gamma(-s)\}^{\nu} \sum_{h=0}^{\nu-1} C_{\nu-h+1}^h (-s)(-s+1) \dots (-s+h) ds \\
 = - \int_C z^s e^{\mu\pi is} \{\Gamma(-s)\}^{\nu} (-s)^{\nu} ds \\
 = - \int_C z^s e^{\mu\pi is} \{\Gamma(-s+1)\}^{\nu} ds = z e^{\mu\pi i} Y_{\nu}(\mu; z).
 \end{aligned}$$

Hence the differential equation

$$(4.3) \quad zy = (-)^{\nu+1} \sum_{h=0}^{\nu-1} C_{\nu-h+1}^h (-z)^{h+1} y^{(h+1)}$$

is satisfied by $Y_{\nu}(2\mu; z)$ for all values of μ , and the differential equation

$$(4.31) \quad -zy = (-)^{\nu+1} \sum_{h=0}^{\nu-1} C_{\nu-h+1}^h (-z)^{h+1} y^{(h+1)}$$

is satisfied by $Y_{\nu}(2\mu+1; z)$ for all values of μ .

The differential equation for $Y(a_1, a_2, \dots, a_{\nu}; \mu; z)$ is found in a similar manner. We have

$$\begin{aligned}
 (4.4) \quad z^k \frac{d^k}{dz^k} Y(a_1, a_2, \dots, a_{\nu}; \mu; z) \\
 = - \int_C z^s e^{\mu\pi is} s(s-1) \dots (s-k+1) \Gamma(a_1-s) \dots \Gamma(a_{\nu}-s) ds,
 \end{aligned}$$

also

$$\begin{aligned}
 z e^{\mu\pi i} Y(a_1, a_2, \dots, a_{\nu}; \mu; z) \\
 = - \int_C z^s e^{\mu\pi is} (a_1-s) \dots (a_{\nu}-s) \Gamma(a_1-s) \dots \Gamma(a_{\nu}-s) ds \\
 = - \int_C z^s e^{\mu\pi is} \{A_{\nu} - A_{\nu-1}s + \dots + (-)^{\nu} A_0 s^{\nu}\} \Gamma(a_1-s) \dots \Gamma(a_{\nu}-s) ds,
 \end{aligned}$$

* Nielson, *Handbuch der Theorie der Gammafunktion* (1906), 70.

where the A 's are the symmetric functions of the α 's. Using (4.2) we obtain the equation

$$(4.5) \quad \sum_{n=0}^{\nu} A_{\nu-n}(-s)^n = \sum_{n=0}^{\nu} A_{\nu-n}(-)^{n+1} \sum_{h=0}^{n-1} C_{n-h+1}^h(-s)(-s+1) \dots (-s+h).$$

Hence the differential equation

$$(4.6) \quad zy = \sum_{n=0}^{\nu} A_{\nu-n}(-)^{n+1} \sum_{h=0}^{n-1} C_{n-h+1}^h(-z)^{h+1} y^{(h+1)}$$

is satisfied by $Y(\alpha_1, \alpha_2, \dots, \alpha_{\nu}; 2\mu; z)$.

It has been shown [see (2.62)] that, if $\alpha_i - \alpha_j$ is not an integer, only ν of these functions are independent; the ν functions given by $\mu = 0, 1, \dots, \nu-1$ then give the ν independent solutions of the differential equation (4.6). The ν independent solutions of the differential equation (4.3) may be obtained from the ν independent solutions of the differential equation (4.6) by letting the α 's tend to zero. They are thus given by $Y_{\nu}(2\mu; z)$ where $\mu = 0, 1, \dots, \nu-1$.

We notice that, if $z \neq 0$, the divisor function $Y(\alpha_1, \alpha_2, \dots, \alpha_{\nu}; \mu; z)$ cannot have a zero of multiplicity ν , for if it did, then the first $\nu-1$ differential coefficients would vanish at this zero, and hence from the differential equation the ν -th differential coefficient would also vanish at the same place, and hence, by repeated differentiation of the differential equation, we conclude that all the differential coefficients would vanish at this place. This implies that the solution vanishes identically.

In a similar manner we can find the differential equation satisfied by the functions $d_{\nu}(z)$, $\bar{d}_{\nu}(z)$, and $d_{\nu}(z) + \bar{d}_{\nu}(z)$.

5. The recurrence formulae.

The recurrence formulae are numerous. We have

$$Y(\alpha_1+1, \alpha_2, \dots, \alpha_{\nu}; \mu; z) = - \int_C z^s e^{u\pi is} (a_1-s) \Gamma(a_1-s) \dots \Gamma(a_{\nu}-s) ds$$

$$(5.1) \quad = a_1 Y(\alpha_1, \alpha_2, \dots, \alpha_{\nu}; \mu; z) - z \frac{d}{dz} Y(\alpha_1, \alpha_2, \dots, \alpha_{\nu}; \mu; z)$$

$$(5.2) \quad = (\alpha_1 - \alpha_2) Y(\alpha_1, \alpha_2, \dots, \alpha_{\nu}; \mu; z) + Y(\alpha_1, \alpha_2+1, \alpha_3, \dots, \alpha_{\nu}; \mu; z).$$

Again we have

$$\begin{aligned}
 & Y(a_1+1, a_2+1, a_3, \dots, a_\nu; \mu; z) \\
 &= - \int_C z^s \{ a_1 a_2 - s(a_1 + a_2 - 1) + s(s-1) \} e^{\mu\pi i s} \Gamma(a_1-s) \dots \Gamma(a_\nu-s) ds \\
 (5.11) \quad &= a_1 a_2 Y(a_1, a_2, \dots, a_\nu; \mu; z) + (1 - a_1 - a_2) z \frac{d}{dz} Y(a_1, a_2, \dots, a_\nu; \mu; z) \\
 &\quad + z^2 \frac{d^2}{dz^2} Y(a_1, a_2, \dots, a_\nu; \mu; z),
 \end{aligned}$$

and, generally,

$$\begin{aligned}
 & Y(a_1+1, \dots, a_h+1, a_{h+1}, \dots, a_\nu; \mu; z) \\
 &= - \int_C z^s \{ A_h^h - s A_{h-1}^h + \dots + (-)^h A_0^h s^h \} e^{\mu\pi i s} \Gamma(a_1-s) \dots \Gamma(a_\nu-s) ds,
 \end{aligned}$$

where the numbers A_j^h , $0 \leq j \leq h$, are the symmetric functions of $a_1, a_2, \dots, a_h$. Using the identity (4.5) we obtain the formula

$$\begin{aligned}
 (5.12) \quad & Y(a_1+1, \dots, a_h+1, a_{h+1}, \dots, a_\nu; \mu; z) \\
 &= \sum_{n=0}^h A_{h-n}^h (-)^{n+1} \sum_{k=0}^{n-1} C_{n-k+1}^k (-z)^{k+1} \frac{d^{k+1}}{dz^{k+1}} Y(a_1, a_2, \dots, a_\nu; \mu; z).
 \end{aligned}$$

Clearly the formula (5.2) may be repeated a number of times, giving a variety of formulae.

Again we clearly have

$$\begin{aligned}
 (5.3) \quad & \frac{d}{dz} \{ z^{-a_1} Y(a_1, a_2, \dots, a_\nu; \mu; z) \} \\
 &= -z^{-(a_1+1)} Y(a_1+1, a_2, \dots, a_\nu; \mu; z),
 \end{aligned}$$

and, generally,

$$\begin{aligned}
 (5.31) \quad & \frac{d^n}{dz^n} \{ z^{-a_1} Y(a_1, a_2, \dots, a_\nu; \mu; z) \} \\
 &= (-)^n z^{-(a_1+n)} Y(a_1+n, a_2, \dots, a_\nu; \mu; z).
 \end{aligned}$$

Thus*

$$\begin{aligned}
 & \frac{d^{n_2}}{dz^{n_2}} \left\{ z^{a_1-a_2+n_1} \frac{d^{n_1}}{dz^{n_1}} \{ z^{-a_1} Y(a_1, a_2, \dots, a_\nu; \mu; z) \} \right\} \\
 &= (-)^{n_1+n_2} z^{-(a_2+n_2)} Y(a_1+n_1, a_2+n_2, a_3, \dots, a_\nu; \mu; z),
 \end{aligned}$$

etc.

* This provides a new way of obtaining the differential equation; it will be dealt with in a later paper.

Also it is obvious that

$$(5.4) \quad Y(a_1+a, a_2+a, \dots, a_\nu+a; \mu; z) = z^a e^{\mu\pi i a} Y(a_1, a_2, \dots, a_\nu; \mu; z).$$

Hence, using equation (5.12),

$$\begin{aligned} (5.5) \quad Y(a_1-1, a_2-1, \dots, a_h-1, a_{h+1}, \dots, a_\nu; \mu; z) \\ = z^{-1} e^{-\mu\pi i} Y(a_1, a_2, \dots, a_h, a_{h+1}+1, \dots, a_\nu+1; \mu; z) \\ = z^{-1} e^{-\mu\pi i} \sum_{n=0}^{v-h} B_{v-h-n}^{v-h} (-)^{n+1} \sum_{k=0}^{n-1} C_{n-k-1}^k (-z)^{k+1} \frac{d^{k+1}}{dz^{k+1}} \\ \times Y(a_1, a_2, \dots, a_\nu; \mu; z), \end{aligned}$$

where the numbers B_j^h , $0 \leq j \leq h$ are the symmetric functions of $a_{h+1}, \dots, a_\nu$.

In a similar manner recurrence formulae may be obtained for the functions $d_\nu(z)$, $\bar{d}_\nu(z)$, and $d_\nu(z) + \bar{d}_\nu(z)$. Note that when $\mu = 0$ and the a 's are real, so that the divisor function $(2\pi i)^{-1} Y(a_1, a_2, \dots, a_\nu; \mu; z)$ is real when z is real, then the equation (5.3) shows that between consecutive zeros of $Y(a_1, a_2, \dots, a_\nu; 0; z)$ there is at least one zero of $Y(a_1+1, a_2, \dots, a_\nu; 0; z)$.

6. The divisor coefficients.

Bessel functions of integral order are the coefficients of powers of t in the Laurent expansion of

$$(6.1) \quad e^{\frac{1}{2}z(t-t^{-1})} = \sum_{n=-\infty}^{\infty} t^n J_n(z).$$

From this equation we easily obtain the equation

$$(6.11) \quad e^{t-zt^{-1}} = \pi^{-2} \sum_{n=-\infty}^{\infty} (-t)^n \{Y(-n, 0; 1; z) - Y(-n, 0; -1; z)\}.$$

Now write zt for t , and integrate fractionally; on referring to equation (3.42), we obtain the equation

$$\begin{aligned} (6.12) \quad \frac{1}{\Gamma(\alpha)} \int_0^z (z-u)^{\alpha-1} e^{ut-t^{-1}} du \\ = \pi^{-2} \sum_{n=-\infty}^{\infty} (-zt)^n z^\alpha \{e^{\pi i \alpha} Y(-n-\alpha, 0; 1; z) \\ - e^{-\pi i \alpha} Y(-n-\alpha, 0; -1; z)\}. \end{aligned}$$

By means of the equation (1.241) this result may be written

$$(6.121) \quad \frac{1}{\Gamma(a)} \int_0^z (z-u)^{a-1} e^{ut-t^{-1}} du = \sum_{n=-\infty}^{\infty} t^n z^{\frac{1}{2}(n+a)} J_{n+a}(2\sqrt{z}).$$

Now consider

$$(6.2) \quad \int_{C'} e^y e^{t-z(ty)^{-1}} y^{a-1} dy = 2\pi i e^t (z/t)^{\frac{1}{2}a} J_{-a}\{2\sqrt{z/t}\},$$

where the contour C' is the same as in the integral (3.10). The function on the right-hand side of equation (6.2) is regular and one-valued for all values of $t \neq 0$. Hence it has a Laurent expansion, which is therefore given by the equation

$$\begin{aligned} (6.21) \quad & 2\pi i e^t (z/t)^{\frac{1}{2}a} J_{-a}\{2\sqrt{z/t}\} \\ &= \pi^{-2} \sum_{n=-\infty}^{\infty} (-t)^n \{e^{\pi i a} Y(a, -n, 0; 0; z) - e^{-\pi i a} Y(a, -n, 0; 2; z) \\ &\quad - e^{\pi i a} Y(a, -n, 0; -2; z) + e^{-\pi i a} Y(a, -n, 0; 0; z)\} \\ &= -\pi^{-2} \sum_{n=-\infty}^{\infty} (2i)^2 (-t)^n \int_C z^s \sin \pi s \sin \pi(a-s) \\ &\quad \times \Gamma(a-s) \Gamma(-n-s) \Gamma(-s) ds. \end{aligned}$$

Generally we have

$$\begin{aligned} (6.3) \quad & \int_{C'} dy_1 \dots \int_{C'} dy_v e^{y_1 + \dots + y_v} e^{t-z(ty_1 \dots y_v)^{-1}} y_1^{a_1-1} \dots y_v^{a_v-1} \\ &= -\pi^{-2} (2i)^{v+1} \sum_{n=-\infty}^{\infty} (-t)^n \int_C z^s \sin \pi s \sin \pi(a_1-s) \dots \sin \pi(a_v-s) \\ &\quad \times \Gamma(a_1-s) \dots \Gamma(a_v-s) \Gamma(-n-s) \Gamma(-s) ds. \end{aligned}$$

In this equation the coefficient of t^n may be expressed as a sum of divisor functions.

The equation (6.12) may be similarly treated. Referring to the integral (3.10) we see that the integral on the left-hand side of equation (6.3) is equal to

$$\begin{aligned} & -e^t (2i)^{v+1} \pi^{-1} \int_C (z/t)^s \sin \pi(a_1-s) \dots \sin \pi(a_v-s) \Gamma(a_1-s) \dots \Gamma(a_v-s) \Gamma(-s) ds \\ &= -e^t (2\pi i)^{v+1} \pi^{-2} \int_C (z/t)^s \Gamma(-s) \{\Gamma(s-a_1+1) \dots \Gamma(s-a_v+1)\}^{-1} ds \\ &= \pi^{-2} e^t (2\pi i)^{v+2} \sum_{n=0}^{\infty} (-z/t)^n \frac{1}{n! \Gamma(n+1-a_1) \dots \Gamma(n+1-a_v)} \\ (6.31) \quad &= \pi^{-2} e^t (2\pi i)^{v+2} {}_0F_v(; 1-a_1, \dots, 1-a_v; -z/t) \\ &\quad \times \{\Gamma(1-a_1) \dots \Gamma(1-a_v)\}^{-1}. \end{aligned}$$

The equation (6.3) can thus be written as the equation*

$$(6.32) \quad e'_0 F_\nu(; 1-a_1, \dots, 1-a_\nu; z/t) \\ = \sum_{n=-\infty}^{\infty} t^n \{\Gamma(n+1)\}^{-1} {}_0F_{\nu+1} (; n+1, 1-a_1, \dots, 1-a_\nu; z),$$

where $\{\Gamma(n+1) \cdot (n+1)(n+2) \dots (n+m)\}^{-1} = \{\Gamma(n+m+1)\}^{-1}$ for all values of n . Differential equations and recurrence formulae may be obtained for these hypergeometric functions as for the divisor functions.

From the equation (6.32) we obtain integral representations of these hypergeometric functions on dividing by t^{n+1} and integrating for t round the unit circle; these integrals, as in the case of Bessel functions, admit of a number of varieties.

7. A compact integral for $D_\nu(z)$.

Consider the integral

$$(7.1) \quad \int_{-\infty}^{\infty} \frac{dy}{(zy-n^2)(z-m^2y)} = \frac{\pm 1}{z^2-n^2m^2} \quad \text{according as } Iz \gtrless 0,$$

and the integral

$$(7.11) \quad \frac{1}{2\pi i} \int_{\gamma} \frac{dy}{(zy-n^2)(z-m^2y)} = \frac{-1}{z^2-n^2m^2} \quad \text{where } |z| < 1,$$

γ is the unit circle, and so includes all points of the form z/m^2 , and excludes all of the form n^2/z if $|z| < 1$. Since the series

$$\sum_{n=1, m=1}^{\infty} (zy+n^2)^{-1} (z+m^2y)^{-1}$$

is absolutely and uniformly convergent we may interchange the order of summation and integration in the equation

$$(7.2) \quad D_2(z) = -\frac{1}{2\pi i} \sum_{n=1, m=1}^{\infty} \int_{\gamma} \frac{dy}{(zy-n^2)(z-m^2y)},$$

and so obtain the equation

$$(7.3) \quad D_2(z) = -\frac{1}{2\pi i} \int_{\gamma} \{ \pi \cot \pi(zy)^{\frac{1}{2}} - (zy)^{-\frac{1}{2}} \} \{ \pi \cot \pi(z/y)^{\frac{1}{2}} - (y/z)^{\frac{1}{2}} \} \frac{dy}{y}.$$

* Cf. Kummer's relation. Watson, *Bessel functions* (1922), 191.

Again we have

$$(7.4) \quad \frac{1}{(2\pi i)^2} \int_{\gamma} dy_2 \int_{\gamma} dy_1 \frac{1}{(z^{\frac{1}{2}} y_2^{\frac{1}{2}} y_1 - n^2)(z^{\frac{1}{2}} y_2^{\frac{1}{2}} - m^2 y_1)(z - \ell^2 y_2)} \\ = \frac{-1}{2\pi i} \int_{\gamma} \frac{1}{(zy_2 - n^2 m^2)(z - \ell^2 y_2)} dy_2 = \frac{1}{z^2 - n^2 m^2 \ell^2} \quad (|z| < 1),$$

whence we obtain the equation

$$(7.5) \quad D_3(z) = \frac{1}{(2\pi i)^2} \int_{\gamma} \frac{dy_2}{y_2} \int_{\gamma} \frac{dy_1}{y_1} \{ \pi \cot [\pi y_1^{\frac{1}{2}} (zy_2)^{\frac{1}{2}}] + y_1^{-\frac{1}{2}} (zy_2)^{-\frac{1}{2}} \} \\ \times \{ \pi \cot [\pi y_1^{-\frac{1}{2}} (zy_2)^{\frac{1}{2}}] - y_1^{\frac{1}{2}} (zy_2)^{-\frac{1}{2}} \} + \{ \pi \cot [\pi z^{\frac{1}{2}} y_2^{-\frac{1}{2}}] - z^{-\frac{1}{2}} y_2^{\frac{1}{2}} \}.$$

Proceeding in this manner we obtain a compact integral representation for the function $D_\nu(z)$ which may be expressed briefly in the equation

$$(7.6) \quad D_{\nu+1}(z) = \frac{-1}{2\pi i} \int_{\gamma} D_\nu \{ \sqrt{(zy)} \} \{ \pi \cot [n\sqrt{(z/y)}] - \sqrt{(y/z)} \} \frac{dy}{y} \quad (|z| < 1),$$

or in the equation*

$$(7.7) \quad D_\nu(z) = \frac{(-1)^{\nu-1}}{2\pi i} \int_{\gamma} \{ \cot(z^{2/\nu}/y_1) - y_1/z^{2/\nu} \} \frac{dy_1}{y_1} \\ \dots \int_{\gamma} \{ \cot(z^{2/\nu}/y_{\nu-1}) - y_{\nu-1}/z^{2/\nu} \} \\ \times \{ \cot(z^{2/\nu} y_1 \dots y_{\nu-1}) - (z^{2/\nu} y_1 \dots y_{\nu-1})^{-1} \} \frac{dy_{\nu-1}}{y_{\nu-1}},$$

together with

$$(7.71) \quad D_1(z) = \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \pi \cot \pi z - z^{-1}.$$

In equation (7.7) notice that the integrand *qua* function of z has a pole of order ν at $z = n = n_1 \dots n_\nu$, if $y_1, \dots, y_{\nu-1}$ be suitably chosen, the effect of the elimination of the numbers $y_1, \dots, y_{\nu-1}$ by means of integrations reduces this pole to a simple one. For we have poles when

$$z^{2/\nu}/y_1 = n_1, \dots, z^{2/\nu}/y_{\nu-1} = n_{\nu-1}, z^{2/\nu} y_1 \dots y_{\nu-1} = n_\nu,$$

hence, if

$$y_1 = n^{2/\nu}/n_1, \dots, y_{\nu-1} = n^{2/\nu}/n_{\nu-1},$$

* Cf. equation (8.90) below. For infinite integrals with a modified integrand see equation (8.2) below.

all these poles coincide at the point $z = n = n_1 \dots n_\nu$, which thus becomes a pole of order ν .

The function $D_\nu(z)$ may be expressed as a power series convergent for $|z| < 1$;

$$(7.8) \quad D_\nu(z) = -2z \sum_{m=0}^{\infty} z^{2m} \sum_{n=1}^{\infty} \frac{\sigma_\nu(n)}{n^{2m}} = -2z \sum_{m=0}^{\infty} \left\{ (-)^{m-1} \frac{B_m(2\pi)^{2m}}{2(2m)!} \right\}^\nu z^{2m},$$

where B_m is the m -th Bernoullian number, so that the coefficients in the power series for $D_\nu(z)$ are the ν -th powers of the coefficients in the power series for $\pi \cot \pi z - 1/z$. Hence we obtain

$$(7.81) \quad D_2(z) = \frac{-1}{2\pi} \int_0^{2\pi} \{ \pi \cot(\pi z^{\frac{1}{2}} e^{i\theta}) - z^{-\frac{1}{2}} e^{-i\theta} \} \{ \pi \cot(\pi z^{\frac{1}{2}} e^{-i\theta}) - z^{-\frac{1}{2}} e^{i\theta} \} d\theta,$$

and generally

$$(7.82) \quad D_\nu(z) = \frac{-1}{2\pi} \int_0^{2\pi} D_\lambda(z^{\frac{1}{2}} e^{i\theta}) D_\mu(z^{\frac{1}{2}} e^{-i\theta}) d\theta \quad (\lambda + \mu = \nu),$$

where we take

$$(7.71) \quad D_1(z) = \pi \cot \pi z - z^{-1}.$$

These integrals may be obtained from the equation (7.7). The equation (7.7) may be used to obtain the equations (1.11), but the method actually used is simpler. If equation (7.7) is used for this purpose the cotangents are replaced by their expansions as a series of exponentials, the resulting series is then integrated term by term.

8. Some functions allied to $D_\nu(z)$.

Functions allied to the functions $D_\nu(z)$ arise in finding integral representations for powers of the zeta-function. Thus, if $\sigma > 1$,

$$(8.0) \quad \{\Gamma(s) \xi(s)\}^\nu = \int_0^\infty dx_1 \dots \int_0^\infty dx_\nu (x_1 \dots x_\nu)^{s-1} \frac{e^{-x_1}}{1-e^{-x_1}} \dots \frac{e^{-x_\nu}}{1-e^{-x_\nu}};$$

by means of the substitution (3.321) this becomes

$$(8.01) \quad \begin{aligned} &= \int_0^\infty t_\nu^{s-1} dt_\nu \int_0^\infty \frac{dt_1}{t_1} \dots \int_0^\infty \frac{dt_{\nu-1}}{t_{\nu-1}} \frac{e^{-t_1}}{1-e^{-t_1}} \frac{e^{-t_2/t_1}}{1-e^{-t_2/t_1}} \dots \frac{e^{-t_\nu/t_{\nu-1}}}{1-e^{-t_\nu/t_{\nu-1}}} \\ &= \int_0^\infty t_\nu^{s-1} dt_\nu \int_0^\infty \frac{dt_1}{t_1} \dots \int_0^\infty \frac{dt_{\nu-1}}{t_{\nu-1}} \frac{e^{-t_1}}{1-e^{-t_1}} \frac{e^{-t_2}}{1-e^{-t_2}} \\ &\quad \dots \frac{e^{-t_{\nu-1}}}{1-e^{-t_{\nu-1}}} \frac{e^{-t_\nu(t_1 \dots t_{\nu-1})^{-1}}}{1-e^{-t_\nu(t_1 \dots t_{\nu-1})^{-1}}}. \end{aligned}$$

If $R(x) > 0$, we shall write

$$\begin{aligned}
 (8.2) \quad \Delta_\nu(x) &= \int_0^\infty \frac{dt_1}{t_1} \cdots \int_0^\infty \frac{dt_{\nu-1}}{t_{\nu-1}} \frac{e^{-t_1}}{1-e^{-t_1}} \cdots \frac{e^{-t_{\nu-1}}}{1-e^{-t_{\nu-1}}} \frac{e^{-x(t_1 \dots t_{\nu-1})^{-1}}}{1-e^{-x(t_1 \dots t_{\nu-1})^{-1}}} \\
 &= \int_0^\infty \frac{dt_1}{t_1} \cdots \int_0^\infty \frac{dt_{\nu-1}}{t_{\nu-1}} \sum_{n_1=1, \dots, n_{\nu-1}=1}^\infty e^{-[n_1 t_1 + \dots + n_{\nu-1} t_{\nu-1} + n_\nu x(t_1 \dots t_{\nu-1})^{-1}]} \\
 &= \sum_{n_1=1, \dots, n_{\nu-1}=1}^\infty \int_0^\infty \frac{dt_1}{t_1} \cdots \int_0^\infty \frac{dt_{\nu-1}}{t_{\nu-1}} \\
 &\quad \times \exp \{ -[t_1 + \dots + t_{\nu-1} + (n_1 \dots n_\nu) x(t_1 \dots t_{\nu-1})^{-1}] \} \\
 (8.21) \quad &= \sum_{n=1}^\infty \frac{\sigma_\nu(n)}{2\pi i} Y_\nu(0; nx)^* \quad \text{from equation (3.22).}
 \end{aligned}$$

This series gives the continuation of the function $\Delta_\nu(x)$.

Referring to equations (2.70) and (2.71), we see that

$$(8.3) \quad D_\nu(z) = \begin{cases} -2\pi i \phi_\nu(z) + 2(-\frac{1}{2})^\nu z^{-1} - 2\pi i \sum_{h=0}^{\nu-1} C_h \Delta_\nu \{ (2\pi)^\nu n z e^{\frac{1}{2}\pi i(\nu-2h-2)} \} \\ 2\pi i \phi_\nu(z) + 2(-\frac{1}{2})^\nu z^{-1} + 2\pi i \sum_{h=0}^{\nu-1} C_h \Delta_\nu \{ (2\pi)^\nu n z e^{\frac{1}{2}\pi i(\nu-2h)} \} \end{cases}$$

according as $Iz \gtrless 0$, where

$$(8.31) \quad \Delta_\nu(z e^{\frac{1}{2}\pi i}) = \sum_{n=1}^\infty \frac{\sigma_\nu(n)}{2\pi i} Y_\nu([\tfrac{1}{2}\mu]; z e^{\frac{1}{2}\pi i(\mu-2[\frac{1}{2}\mu])}.$$

We now obtain alternative expressions for the functions $\Delta_\nu(z)$.

If $\sigma < 1$, we have

$$(8.4) \quad \frac{1}{2}\pi \sec(\tfrac{1}{2}s\pi)(2\pi\nu)^{s-1} = \frac{1}{2i \sin \pi s} \int_C \frac{(-z)^s}{z^2 + 4\pi^2 \nu^2} dz;$$

thus, if $-1 < \sigma < 1$, we obtain the equation

$$(8.41) \quad \frac{1}{2}\pi \sec(\tfrac{1}{2}s\pi)(2\pi\nu)^{s-1} = \int_0^\infty \frac{x^s}{x^2 + 4\pi^2 \nu^2} dx.$$

Now

$$\begin{aligned}
 (8.5) \quad \Delta_\nu(z) &= \frac{1}{2\pi i} \sum_{n=1}^\infty \sigma_\nu(n) \int_{-1-c-i\infty}^{-1-c+i\infty} (nz)^s \{ \Gamma(-s) \}^\nu ds \\
 &= \frac{1}{2\pi i} \int_{-1-c-i\infty}^{-1-c+i\infty} \{ \xi(-s) \Gamma(-s) \}^\nu z^s ds \\
 &= -\psi_{\nu 0}(z) - \psi_{\nu 1}(z) + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} z^s \{ \xi(-s) \Gamma(-s) \}^\nu ds,
 \end{aligned}$$

* See Oppenheim, *Proc. London Math. Soc.* (2), 26 (1927), 303, for the case $\nu = 2$.

where $\psi_{\nu 0}(z)$ and $\psi_{\nu 1}(z)$ are the residues of $z^s \{\zeta(-s) \Gamma(-s)\}^\nu$ at the points $s=0$ and $s=-1$ respectively, and where $0 < c < 1$. Using the functional equation of the zeta-function, namely

$$\zeta(-s) \Gamma(-s) = \frac{1}{2} (2\pi)^{-s} \sec \left(\frac{1}{2} s \pi \right) \zeta(s+1),$$

we obtain the equation

$$\begin{aligned} (8.51) \quad \Delta_\nu(z) &= -\psi_{\nu 0}(z) - \psi_{\nu 1}(z) + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} z^s \left\{ \frac{1}{2} (2\pi)^{-s} \sec \left(\frac{1}{2} s \pi \right) \zeta(s+1) \right\}^\nu ds \\ &= -\psi_{\nu 0}(z) - \psi_{\nu 1}(z) \\ &\quad + \frac{1}{2\pi i} \left(\frac{1}{2} \right)^\nu \sum_{n=1}^{\infty} \sigma_\nu(n) \int_{c-i\infty}^{c+i\infty} z^s (2\pi)^{-\nu s} n^{-s-1} \left\{ \sec \left(\frac{1}{2} s \pi \right) \right\}^\nu ds. \end{aligned}$$

Now

$$\begin{aligned} (8.52) \quad & \left(\frac{1}{2} \right)^\nu \int_{c-i\infty}^{c+i\infty} z^s (2\pi)^{-\nu s} (n_1 \dots n_\nu)^{-s-1} \left\{ \sec \left(\frac{1}{2} s \pi \right) \right\}^\nu ds \\ &= z 2^{-\nu} \pi^{-2\nu} (n_1 \dots n_\nu)^{-2} \int_{c-i\infty}^{c+i\infty} \left\{ \frac{1}{2} \pi [z^{1/\nu} (2\pi n_1)^{-1}]^{s-1} \sec \left(\frac{1}{2} s \pi \right) \right\} \\ &\quad \dots \left\{ \frac{1}{2} \pi [z^{1/\nu} (2\pi n_\nu)^{-1}]^{s-1} \sec \left(\frac{1}{2} s \pi \right) \right\} ds \\ &= z 2^{-\nu} \pi^{-2\nu} (n_1 \dots n_\nu)^{-2} \int_{c-i\infty}^{c+i\infty} ds \int_0^\infty \frac{x_1^s dx_1}{x_1^2 + z^{2/\nu} (2\pi n_1)^{-2}} \dots \int_0^\infty \frac{x_\nu^s dx_\nu}{x_\nu^2 + z^{2/\nu} (2\pi n_\nu)^{-2}} * \end{aligned}$$

if we apply the transformation (3.321) to the integration for $x_1, \dots, x_\nu$, the integral (8.52) is seen to be equal to the integral

$$\begin{aligned} (8.53) \quad & z 2^{-\nu} \pi^{-2\nu} (n_1 \dots n_\nu)^{-2} \int_{c-i\infty}^{c+i\infty} t_\nu^s ds \int_0^\infty \frac{t_1^{-1} dt_1}{t_1^2 + z^{2/\nu} (2\pi n_1)^{-2}} \\ & \times \int_0^\infty \frac{t_2^{-1} dt_2}{t_2^2 t_1^{-1} + z^{2/\nu} (2\pi n_2)^{-2}} \dots \int_0^\infty \frac{t_{\nu-1}^{-1} dt_{\nu-1}}{t_{\nu-1}^2 t_{\nu-2}^{-2} + z^{2/\nu} (2\pi n_{\nu-1})^{-2}} \int_0^\infty \frac{dt_\nu}{t_\nu^2 t_{\nu-1}^{-2} + z^{2/\nu} (2\pi n_\nu)^{-2}} \\ &= z 2^{-\nu} \pi^{-2\nu} (n_1 \dots n_\nu)^{-2} \int_0^\infty \frac{t_1^{-1} dt_1}{t_1^2 + z^{2/\nu} (2\pi n_1)^{-2}} \\ &\quad \dots \int_0^\infty \frac{t_{\nu-1}^{-1} dt_{\nu-1}}{t_{\nu-1}^2 t_{\nu-2}^{-2} + z^{2/\nu} (2\pi n_{\nu-1})^{-2}} \int_{c-i\infty}^{c+i\infty} ds \int_0^\infty \frac{t_\nu^s dt_\nu}{t_\nu^2 t_{\nu-1}^{-2} + z^{2/\nu} (2\pi n_\nu)^{-2}}. \end{aligned}$$

* The alternative expansion may be obtained by replacing $(1-e^{-ti})^{-1}$ in equation (8.2) by its partial fraction expression, but the method actually used is preferable.

Now

$$\begin{aligned}
 t_{\nu-1}^2 \int_{e-i\infty}^{e+i\infty} ds \int_0^\infty \frac{t_\nu^s dt_\nu}{t_\nu^2 + z^{2/\nu} t_{\nu-1}^2 (2\pi n_\nu)^{-2}} \\
 = t_{\nu-1}^2 \int_{e-i\infty}^{e+i\infty} \frac{1}{2} \pi \sec\left(\frac{1}{2}\pi s\right) \{z^{1/\nu} t_{\nu-1} (2\pi n_\nu)^{-1}\}^{s-1} ds \\
 = t_{\nu-1}^2 \pi z^{-1/\nu} t_{\nu-1}^{-1} 2\pi n_\nu i 2z^{1/\nu} t_{\nu-1} (2\pi n_\nu)^{-1} \{1 + z^{2/\nu} t_{\nu-1}^2 (2\pi n_\nu)^{-2}\}^{-1} \\
 = 2\pi i t_{\nu-1}^2 \{1 + z^{2/\nu} t_{\nu-1}^2 (2\pi n_\nu)^{-2}\}^{-1},
 \end{aligned}$$

where we have used the results of evaluating the integral (2.0). Hence the integral (8.53) is equal to the integral

$$\begin{aligned}
 (8.54) \quad 2\pi i z 2^{-\nu} \pi^{-2\nu} (n_1 \dots n_\nu)^{-2} \int_0^\infty \frac{t_1^{-1} dt_1}{t_1^2 + z^{2/\nu} (2\pi n_1)^{-2}} \int_0^\infty \frac{t_2^{-1} dt_2}{t_2^2 t_1^{-2} + z^{2/\nu} (2\pi n_2)^{-2}} \\
 \dots \int_0^\infty \frac{t_{\nu-1}^{-1} dt_{\nu-1}}{\{t_{\nu-1}^2 t_{\nu-2}^{-2} + z^{2/\nu} (2\pi n_{\nu-1})^{-2}\} \{t_{\nu-1}^{-2} + z^{2/\nu} (2\pi n_\nu)^{-2}\}}.
 \end{aligned}$$

Now employ the transformation

$$(8.55) \quad \begin{cases} t_1 = \sqrt{y_1} (2\pi n_1)^{-1}, \\ t_2 = \sqrt{(y_1 y_2)} (2\pi)^{-2} (n_1 n_2)^{-1}, \\ \dots \quad \dots \quad \dots \quad \dots \\ t_{\nu-1} = \sqrt{(y_1 \dots y_{\nu-1})} (2\pi)^{-\nu+1} (n_1 \dots n_{\nu-1})^{-1}, \end{cases}$$

and the integral (8.54) is seen to be equal to the integral

$$\begin{aligned}
 (8.56) \quad 4\pi i z \int_0^\infty \frac{y_1^{-1} dy_1}{y_1 + z^{2/\nu}} \dots \int_0^\infty \frac{y_{\nu-2}^{-1} dy_{\nu-2}}{y_{\nu-2} + z^{2/\nu}} \\
 \times \int_0^\infty \frac{y_{\nu-1}^{-1} dy_{\nu-1}}{\{y_{\nu-1} + z^{2/\nu}\} \left\{ \frac{(2\pi)^{2\nu} (n_1 \dots n_\nu)^2}{(y_1 \dots y_{\nu-1})} + z^{2/\nu} \right\}} = 2\pi i \delta_\nu(n, z) \quad (\text{say}).
 \end{aligned}$$

The equation (8.51) now becomes the equation

$$(8.57) \quad \Delta_\nu(z) = -\psi_{\nu 0}(z) - \psi_{\nu 1}(z) + \sum_{n=1}^\infty \sigma_\nu(n) \delta_\nu(n, z).$$

Since

$$(8.6) \quad \int_0^\infty \frac{dy}{(\xi y + a)(\xi + by)} = \frac{\log(\xi^2/ab)}{\xi^2 - ab},$$

the integration for y_{v-1} may be replaced by

$$(8.61) \quad \frac{(y_1 \dots y_{v-2})}{(2\pi)^{2v} (n_1 \dots n_v)^2 - z^{4/v} (y_1 \dots y_{v-2})} \log \left(\frac{(2\pi)^{2v} (n_1 \dots n_v)^2}{z^{4/v} (y_1 \dots y_{v-2})} \right).$$

In the case $v = 2^*$ we then obtain the result

$$(8.62) \quad \Delta_2(z) = \pi i \sum_{n=1}^{\infty} \sigma_2(n) H_0^{(1)}(2i\sqrt{z}) \\ = -\frac{1}{2}(\gamma + \frac{1}{2} \log z) - 2(2\pi\sqrt{z})^{-2} \log(2\pi\sqrt{z}) \\ + 2z \sum_{n=1}^{\infty} \frac{\sigma_2(n) \log[(2\pi)^4 n^2 z^{-2}]}{(2\pi)^4 n^2 - z^2}.$$

Let

$$(8.7) \quad I_h = I_h(a, b, A) = \int_0^{\infty} \frac{1}{(y-a)(b-Ay)} \left(\log \frac{b}{Ay} \right)^h dy,$$

where h is an integer, a and A are complex numbers, while b is real and positive, and where

$$(8.701) \quad \log \left(\frac{b}{Ay} \right) = \log b - \log |A| - i \arg A - \log y,$$

the logarithms on the right-hand side being real.

Now consider

$$(8.71) \quad J_{h+1} = J_{h+1}(a, b, A) = \frac{1}{2\pi i} \int_C \frac{1}{(y-a)(b-Ay)} \left(\log \frac{b}{Ay} \right)^{h+1} dy,$$

where the contour C is as before, and where the logarithm is determined by (8.701), together with $0 \leq \arg y \leq 2\pi$, so that $\log y$ is real on the first part of C , namely from ∞ to 0, and is increased by $2\pi i$ on the second part of C , namely from 0 to ∞ . Then, by Cauchy's theorem,

$$(8.72) \quad J_{h+1} = -\frac{1}{b-Aa} [\{\log(b/Aa)\}^{h+1} - \{-2\pi i\}^{h+1}],$$

$\log(b/Aa)$ is to be interpreted by (8.701), and we have taken $0 < \arg a < 2\pi$, $0 < \arg A < 2\pi$. Also

$$(8.73) \quad J_{h+1} = \frac{1}{2\pi i} \int_{\infty}^0 \frac{1}{(y-a)(b-Ay)} \left(\log \frac{b}{Ay} \right)^{h+1} dy \\ + \frac{1}{2\pi i} \int_0^{\infty} \frac{1}{(y-a)(b-Ay)} \left(\log \frac{b}{Ay} - 2\pi i \right)^{h+1} dy,$$

* This agrees with Oppenheim, *Proc. London Math. Soc.* (2), 26 (1927), 306, equation (3.31).

where the logarithm is given by equation (8.701) with $\log y$ real. Hence

$$(8.731) \quad J_{h+1} = \frac{1}{2\pi i} \left\{ (-2\pi i)(h+1) I_h + (-2\pi i)^2 \frac{(h+1)h}{2!} I_{h-1} \right. \\ \left. + \dots + (-2\pi i)^{h+1} I_0 \right\},$$

eliminating $I_{h-1}, I_{h-2}, \dots, I_0$ from equation (8.731) and similar equations with $h-1, h-2, \dots, 1$ instead of h , we obtain I_h as a linear function of $J_1, \dots, J_{h+1}$. That is to say, I_h is a polynomial of degree $h+1$ in $\log(b/Aa)$ divided by $b-Aa$. Now consider the integral

$$(8.80) \quad \int_0^\infty \frac{dy_n}{y_n - a_n} \dots \int_0^\infty \frac{dy_1}{y_1 - a_1} \frac{1}{b - a_0 y_1 \dots y_n};$$

we shall show that it is equal to

$$(8.801) \quad \frac{P_n[\log(b/a_0 a_1 \dots a_n)]}{b - (a_0 a_1 \dots a_n)},$$

where $P_n(x)$ is a polynomial of degree n in x . The assertion is true when $n=1$, for

$$(8.802) \quad \int_0^\infty \frac{dy}{(y-a_1)(b-a_0 y)} = \frac{\log(b/a_0 a_1)}{b-a_0 a_1},$$

the logarithm having its imaginary part between 0 and 2π if $0 < \arg a_0 + \arg a_1 < 2\pi$. Suppose that it is true for $n=2, 3, \dots, n$; then for $n+1$ we have

$$(8.81) \quad \int_0^\infty \frac{dy_{n+1}}{y_{n+1} - a_{n+1}} \dots \int_0^\infty \frac{dy_1}{y_1 - a_1} \frac{1}{b - a_0 y_1 \dots y_{n+1}} \\ = \int_0^\infty \frac{dy}{y - a_{n+1}} \frac{P_n[\log(b/a_0 a_1 \dots a_n y)]}{b - (a_0 a_1 \dots a_n y)} \\ = \frac{P_{n+1}[\log(b/a_0 a_1 \dots a_{n+1})]}{b - (a_0 a_1 \dots a_{n+1})},$$

using the results for the integral I_h . Thus the assertion is true generally. Hence, if $Ia_j > 0$,

$$(8.82) \quad \int_0^\infty \frac{dy_n}{y_n - a_n} \dots \int_0^\infty \frac{dy_1}{y_1 - a_1} \frac{1}{b - (a_0 y_1 \dots y_n)} = \frac{P_n[\log(b/a_0 \dots a_n)]}{b - a_0 \dots a_n}.$$

The polynomial P_n is different if $Ia_j < 0$. Equation (8.82) is only proved if $0 < \arg(a_0 \dots a_j) < 2\pi^*$ for $j = 0, 1, \dots, n$, its value for other values of $a_0, \dots, a_n$ such that $0 < \arg a_j < 2\pi$ is given by analytic continuation. Thus the integral (8.56) is equal to

$$(8.821) \quad 4\pi iz \frac{P_{\nu-1}[\log \{(2\pi)^{2\nu}(n_1 \dots n_\nu)^2/z^2\} - \nu\pi i]}{(2\pi)^{2\nu}(n_1 \dots n_\nu)^2 - z^2(-)^\nu} \quad (Iz > 0).$$

The second expression [equation (8.57)] for $\Delta_\nu(z)$ is now given by

$$(8.83) \quad \Delta_\nu(z) = -\psi_{\nu 0}(z) - \psi_{\nu 1}(z) + 2z \sum_{n=1}^{\infty} \sigma_\nu(n) \frac{P_{\nu-1}[\log \{(2\pi)^{2\nu} n^2/z^2\} - \nu\pi i]}{(2\pi)^{2\nu} n^2 - z^2(-)^\nu}.$$

The coefficients in the polynomials $P_\nu(x)$ may be obtained as follows: from (8.731) we see that, if $h \geq 1$, the numerator of I_h is given by the equation

$$(8.90) \quad I_h(x) = \frac{x^{h+1}}{h+1} + \pi i x^h - \sum_{1 \leq n \leq \frac{1}{2}(h+1)} \frac{(2\pi)^{2n} B_n}{(2n)!} x^{h-2n+1} h(h-1) \dots (h-2n+2)$$

together with

$$(8.901) \quad I_0(x) = x + 2\pi i.$$

If $Q(x) = \sum_{j=1}^n q_j x^j$, then we shall write

$$(8.91) \quad I\{Q(x)\} = \sum_{j=1}^n q_j I_j(x)$$

and

$$(8.911) \quad I\{Q(x)\}_{x-\pi i} = \sum_{j=1}^n q_j I_j(x-\pi i);$$

then from (8.81) and (8.911) we obtain

$$(8.92) \quad P_n\{x-(n+1)\pi i\} = I\{P_{n-1}(x-n\pi i)\}_{x-\pi i},$$

and, since $P_1(x-2\pi i) = x$, this enables us to calculate the polynomials

* I.e. $0 < \arg a_0 + \dots + \arg a_j < 2\pi$.

$P_n(x)$ successively; we have, in fact,

$$P_1(x-2\pi i) = x,$$

$$P_2(x-3\pi i) = \frac{1}{2!} \{x^2 - (\pi i)^2\},$$

$$P_3(x-4\pi i) = \frac{1}{3!} \{x^3 - 4(\pi i)^2 x - 3(\pi i)^3\},$$

$$P_4(x-5\pi i) = \frac{1}{4!} \{x^4 - 10(\pi i)^2 x^2 - 12(\pi i)^3 x - 3(\pi i)^4\},$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

Also we have

$$(8.93) \quad \sum_{h=0}^{v-1} {}_{v-1}C_h (-)^h P_{v-1}[x-2\pi h i] \equiv (2\pi i)^{v-1},$$

whence we obtain (8.3) from (8.83) and (8.93).

ON THE RELATION BETWEEN THE DIFFERENT TYPES
OF ABEL SUMMATION

By MARY L. CARTWRIGHT.

[Received 31 January, 1929.—Read 14 February, 1929.

Received in revised form 20 June, 1929.]

1.1. Suppose that the series $\sum_1^{\infty} a_n e^{-yn^{\nu}}$, where $\nu > 0$, is convergent for $R(y) > 0$, so that it defines an analytic function $\phi_{\nu}(y) = \phi_{\nu}(re^{i\theta})$ regular in the right half-plane; and suppose further that

$$\phi_{\nu}(y) \rightarrow l$$

when $y \rightarrow 0$ so that $-a \leq \theta \leq a$, where $0 \leq a < \frac{1}{2}\pi$; then the series Σa_n is said to be *summable* (A_a, ν) to sum l . It is sufficient to assume that $\phi_{\nu}(y) \rightarrow l$ when $y \rightarrow 0$ on $\theta = a$ and $\theta = -a$, and that $\phi_{\nu}(y)$ is bounded in the angle $-a \leq \theta \leq a$, for then, by a theorem due to Lindelöf*, $\phi_{\nu}(y) \rightarrow l$ uniformly, when $y \rightarrow 0$ so that $-a \leq \theta \leq a$.

1.2. I have proved in a previous paper† that, if $\Sigma a_n n^{-s} = \Sigma a_n n^{-\sigma-it}$ is summable (A_a, n) for $\sigma > 0$, and if $a_n = O(n^K)$ for some K , then $\Sigma a_n n^{-s}$ is summable (A_{β}, μ) to the same sum, provided that

$$(1.21) \quad \beta \leq \frac{1}{2}\pi - (\frac{1}{2}\pi - a)\mu/\nu.$$

The chief object of this paper is to show that if Σa_n is summable (A_a, ν) , and if $\Sigma a_n e^{-yn^{\mu}}$ is convergent for $R(y) > 0$, then Σa_n is summable (A_{β}, μ) , provided that

$$(1.22) \quad \beta < \frac{1}{2}\pi - (\frac{1}{2}\pi - a)\mu/\nu,$$

but I shall also prove several other theorems connecting the series $\Sigma a_n e^{-yn^{\nu}}$ and $\Sigma a_n e^{-yn^{\mu}}$.

* E. Lindelöf, *Acta Soc. Sci. Fennicae*, 35 (7) (1909), 28.† M. L. Cartwright, *Journal London Math. Soc.*, 3 (1928), 262-267 (267).

1.3. I assume throughout that $\sum a_n e^{-yn^\nu}$ is convergent for $R(y) > 0$, and I prove first

THEOREM I. *If*

$$(1.31) \quad \phi_\nu(y) = O(y^{-c}),$$

where $c \geq 0$, when $y \rightarrow 0$ so that $-a \leq \theta \leq a$, then $\sum a_n e^{-yn^\mu}$ is summable (A_a, ν) to sum $\phi_\mu(y)$, where

$$(1.32) \quad \mu \phi_\mu(y^\mu) = \frac{\nu}{4\pi i} \int_0^\infty \{ \phi_\nu(u^\nu e^{ia}) + \phi_\nu(u^\nu e^{-ia}) \} \\ \times \frac{du}{u} \int_{\kappa-i\infty}^{\kappa+i\infty} \left(\frac{y}{u} \right)^{-s} \frac{\Gamma(s/\mu) ds}{\Gamma(s/\nu) \cos(as/\nu)},$$

provided that $r > 0$ and $-\beta/\mu \leq \theta \leq \beta/\mu$, where $\beta < \frac{1}{2}\pi - (\frac{1}{2}\pi - a)\mu/\nu$, and $\kappa > cv$.

When $a = 0$, the formula is written

$$(1.33) \quad \mu \phi_\mu(y^\mu) = \frac{\nu}{2\pi i} \int_0^\infty \phi_\nu(u^\nu) \frac{du}{u} \int_{\kappa-i\infty}^{\kappa+i\infty} \left(\frac{y}{u} \right)^{-s} \frac{\Gamma(s/\mu) ds}{\Gamma(s/\nu)}.$$

In this case μ must be less than ν . For $a > 0$, there is an alternative formula involving $\sin(as/\nu)$. I shall show later that (1.32) actually holds for $\kappa > \text{Max}(cv - \frac{1}{2}\pi\nu/a; -\mu)$, and (1.33) for $\kappa > -\mu$. If $\sum a_n e^{-yn^\mu}$ is convergent, then $\phi_\mu(y)$ is equal to the sum in the ordinary sense, but the only hypotheses required for the existence of $\phi_\mu(y)$ are (1.31) and the convergence of $\sum a_n e^{-yn^\nu}$ for $R(y) > 0$. This theorem involves the heaviest work, and is perhaps less interesting than the others, but the importance of the result lies in the formula (1.32) on which the rest of the work is based.

My principal result is

THEOREM II. *If $\sum a_n$ is summable (A_a, ν) to sum l , and if $\phi_\mu(y)$ is the sum of the series $\sum a_n e^{-yn^\mu}$ in the (A_a, ν) sense, then*

$$\phi_\mu(y) \rightarrow l,$$

when $y \rightarrow 0$ so that $-\beta \leq \theta \leq \beta < \frac{1}{2}\pi - (\frac{1}{2}\pi - a)\mu/\nu$, and in particular if $\sum a_n e^{-yn^\mu}$ is convergent, then $\sum a_n$ is summable (A_a, ν) .

If $\mu > \nu$, the series $\sum a_n e^{-yn^\mu}$ is necessarily convergent, so that if $\sum a_n$ is summable (A_a, ν) , where $a > 0$, then it is also summable (A_β, μ) , provided that $\beta < \frac{1}{2}\pi - (\frac{1}{2}\pi - a)\mu/\nu$ and $\nu < \mu < \pi\nu/(\pi - 2a)$.

I generalize Theorem II as follows :—

THEOREM III. If $\phi_\nu(y) \sim ly^{-c}$,

where $c \geq 0$, when $y \rightarrow 0$ so that $-a \leq \theta \leq a$, then

$$\phi_\mu(y) \sim \frac{l\Gamma(1+c\nu/\mu)}{\Gamma(1+c)} y^{-c\nu/\mu},$$

when $y \rightarrow 0$ so that $-\beta \leq \theta \leq \beta$, where $\beta < \frac{1}{2}\pi - (\frac{1}{2}\pi - a)\mu/\nu$.

In Theorems IV and V, I show that, if $\phi_\nu(y)$ is regular or has a pole or algebraic branch point at the origin, then $\phi_\mu(y)$ is regular all over the plane except, in the last two cases, for a corresponding singularity at the origin.

1.4. It may be observed that near the origin it is always easier to argue from $\phi_\nu(y)$ to $\phi_\mu(y)$ if $\mu < \nu$, although for $R(y) > 0$ $\sum a_n e^{-\gamma n^\mu}$ does not necessarily even converge. If the hypotheses of the first two theorems hold for every $a < \frac{1}{2}\pi$, then the conclusion holds for every $\mu > 0$ and for every $\beta < \frac{1}{2}\pi$. In the last two theorems I use the sum in the (A_a, ν) sense of the series $\sum a_n n^{-s}$, and though it does not appear explicitly in the main proofs of the other theorems, the ideas involved depend on its properties.

1.5. The methods of this paper may be applied to the series $\sum_1^\infty a_n e^{-y\lambda_n^\nu}$, where $0 < \lambda_n < \lambda_{n+1}$ and $\lambda_{n+1} \rightarrow \infty$. Using Hardy's definition*, we say that $\sum a_n$ is summable (A_a, λ, ν) if $\sum a_n e^{-y\lambda_n^\nu}$ is convergent for $R(y) > 0$ and if

$$(1.51) \quad \Phi_\nu(y) = \sum_1^\infty a_n e^{-y\lambda_n^\nu} \rightarrow l,$$

when $y \rightarrow 0$ so that $-a < \theta < a$. If we assume that $\sum a_n e^{-y\lambda_n^\nu}$ has a region of absolute convergence, we can obtain the corresponding theorems about $\sum a_n e^{-y\lambda_n^\mu}$ without much difficulty.

2.1. Lemma 2.1. Let $\phi_\nu(w, s) = \sum a_n n^{-s} e^{-wn^\nu}$, then

$$(2.11) \quad 2\Gamma\left(\frac{s}{\nu}\right) \cos \frac{\gamma s}{\nu} \phi_\nu(w^\nu, s) \\ = \nu \int_0^\infty u^{s-1} \{ \phi_\nu(u^\nu e^{i\gamma} + w^\nu) + \phi_\nu(u^\nu e^{-i\gamma} + w^\nu) \} du,$$

provided that $\sigma > 0$, $R(w^\nu) > 0$, and $\gamma < \frac{1}{2}\pi$.

* G. H. Hardy, *Quart. J. of Math.*, 47 (1916), 178-180.

We know that

$$\Gamma\left(\frac{s}{\nu}\right)n^{-s} = \int_0^\infty y^{s/\nu} e^{-yn^\nu} \frac{dy}{y} = \nu \int_0^\infty y^{s-1} e^{-(yn)^\nu} dy,$$

provided that $\sigma > 0$. Multiply both sides by $a_n e^{-(wn)^\nu}$ and sum. Then

$$(2.12) \quad \Gamma\left(\frac{s}{\nu}\right) \phi_\nu(w^\nu, s) = \nu \int_0^\infty y^{s-1} \phi_\nu(y^\nu + w^\nu) dy,$$

provided that $\sigma > 0$ and $R(w^\nu) > 0$. Term by term integration is justified by the fact that the series $\sum |a_n e^{-(wn)^\nu}|$ is convergent for $R(w^\nu) > 0$, and, *a fortiori*,

$$\sum |a_n e^{-(wn)^\nu}| \int_0^\infty y^{\sigma-1} e^{-(yn)^\nu} dy$$

is convergent for $\sigma > 0$.

Consider the integral*

$$\int y^{s-1} \phi_\nu(y^\nu + w^\nu) dy,$$

where $R(w^\nu) > 0$, taken round a closed contour consisting of the real axis from δ to ρ , the radius vector $\theta = \gamma/\nu$, where $\gamma = \frac{1}{2}\pi$, from $\rho e^{i\gamma/\nu}$ to $\delta e^{i\gamma/\nu}$, and arcs of circles with centres at the origins and radii δ and ρ . When $\rho \rightarrow \infty$ the first term of the series $\sum a_n e^{-(y^\nu + w^\nu)n^\nu}$ outweighs the rest, so that

$$(2.121) \quad \phi_\nu(y^\nu + w^\nu) = O\{e^{-\rho^\nu \cos \nu\theta}\} = O\{e^{-\rho^\nu \cos \gamma}\}$$

for $R(w^\nu) > 0$, on the arc of radius ρ , and hence the integral taken along that arc vanishes. The series $\sum a_n e^{-(wn)^\nu}$ is absolutely convergent for $R(w^\nu) > 0$, so that, when $\delta \rightarrow 0$,

$$\phi_\nu(y^\nu + w^\nu) = O(1)$$

for $R(w^\nu) > 0$ on the arc of radius δ ; hence the integral taken along that arc vanishes. Using Cauchy's Theorem, we have

$$\int_\delta^\rho y^{s-1} \phi_\nu(y^\nu + w^\nu) dy = \int_\delta^{\delta e^{i\gamma/\nu}} + \int_{\delta e^{i\gamma/\nu}}^{\rho e^{i\gamma/\nu}} + \int_{\rho e^{i\gamma/\nu}}^\rho y^{s-1} \phi_\nu(y^\nu + w^\nu) dy.$$

Now make $\delta \rightarrow 0$, $\rho \rightarrow \infty$, and put $y = u e^{i\gamma/\nu}$ on the radius vector $\theta = \gamma/\nu$; then, by (2.12),

$$(2.13) \quad \Gamma\left(\frac{s}{\nu}\right) \phi_\nu(w^\nu, s) = \nu e^{i\gamma s/\nu} \int_0^\infty u^{s-1} \phi_\nu(u^\nu e^{i\gamma/\nu} + w^\nu) du.$$

* Compare Hj. Mellin, *Math. Annalen*, 68 (1910), 321-322.

Similarly we can show that

$$(2.14) \quad \Gamma\left(\frac{s}{\nu}\right) \phi_{\nu}(w^{\nu}, s) = \nu e^{-i\gamma s/\nu} \int_0^{\infty} u^{s-1} \phi_{\nu}(u^{\nu} e^{-i\gamma/\nu} + w^{\nu}) du.$$

Multiplying (2.13) by $e^{-i\gamma s/\nu}$ and (2.14) by $e^{i\gamma s/\nu}$ and adding, we obtain (2.11).

The next lemma is not actually needed for the proof of Theorem I, but I shall refer to it later, and it follows most conveniently here.

2.2. Lemma 2.2. *If (1.31) is satisfied, then $\Sigma a_n n^{-s}$ is summable (A_{α}, ν) for $\sigma > c\nu$, and if $f(s)$ is the sum (A_{α}, ν) , then*

$$(2.21) \quad 2\Gamma\left(\frac{s}{\nu}\right) f(s) \cos \frac{\alpha s}{\nu} = \nu \int_0^{\infty} u^{s-1} \{ \phi_{\nu}(u^{\nu} e^{i\alpha}) + \phi_{\nu}(u^{\nu} e^{-i\alpha}) \} du.$$

The following proof is substantially the same as that given by Hardy* for the case in which $\alpha = 0$, $a_n = O(n^K)$.

Consider the integral in (2.11), putting $\gamma = \alpha$. From the Argand diagram it is easy to see that

$$(2.22) \quad |\arg(u^{\nu} e^{\pm i\alpha} + w^{\nu})| \leq \text{Max}(\alpha; |\arg w^{\nu}|),$$

so that, by (1.31),

$$(2.23) \quad u^{c\nu} \phi_{\nu}(u^{\nu} e^{\pm i\alpha} + w^{\nu}) = O(1),$$

when $u \rightarrow 0$, uniformly for $0 \leq |w| \leq w_0$, $|\arg w^{\nu}| \leq \alpha$. Also, as in (2.121),

$$(2.24) \quad \phi_{\nu}(u^{\nu} e^{\pm i\alpha} + w^{\nu}) = O(e^{-u^{\nu} \cos \alpha}),$$

when $u \rightarrow \infty$, uniformly for $0 \leq |w| \leq w_0$, $|\arg w^{\nu}| \leq \alpha$. Hence, when $\gamma = \alpha$, the integral in (2.11) is uniformly convergent for those values of w . Therefore

$$\lim_{w \rightarrow 0} 2\Gamma\left(\frac{s}{\nu}\right) \cos \frac{\alpha s}{\nu} \phi_{\nu}(w^{\nu}, s) = \nu \int_0^{\infty} u^{s-1} \{ \phi_{\nu}(u^{\nu} e^{i\alpha}) + \phi_{\nu}(u^{\nu} e^{-i\alpha}) \} du,$$

when $w \rightarrow 0$ so that $|\arg w^{\nu}| \leq \alpha$, provided that $\sigma > c\nu$, which is the result stated.

If $s = (m + \frac{1}{2})\pi\nu/\alpha$, where m is a positive integer, the sum is given by the corresponding formula involving $\sin \alpha s/\nu$.

2.3. Lemma 2.3. *Let*

$$\Phi(y, w) = \Sigma a_n e^{-yn^{\mu} - wn^{\nu}},$$

* G. H. Hardy, *Quart. J. of Math.*, 47 (1916), 177-178.

then

$$(2.31) \quad \mu \Phi(y^\mu, w^\nu) = \frac{\nu}{4\pi i} \int_0^\infty \{ \phi_\nu(u^\nu e^{i\gamma} + w^\nu) + \phi_\nu(u^\nu e^{-i\gamma} + w^\nu) \} \\ \times \frac{du}{u} \int_{\kappa-i\infty}^{\kappa+i\infty} \left(\frac{y}{u}\right)^{-s} \frac{\Gamma(s/\mu) ds}{\Gamma(s/\nu) \cos(\gamma s/\nu)},$$

provided that $r > 0$, $-\beta/\mu \leq \theta \leq \beta/\mu$, where $\beta < \frac{1}{2}\pi - (\frac{1}{2}\pi - \gamma)\mu/\nu$, $\gamma < \frac{1}{2}\pi$, and that $R(w^\nu) > 0$, $\kappa > cv$, the path of integration being chosen so that it does not pass through a zero of $\cos(\gamma s/\nu)$.

By Mellin's integral we have

$$e^{-y^\mu} = \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} y^{-s\mu} \Gamma(s) ds = \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} y^{-s} \Gamma\left(\frac{s}{\mu}\right) \frac{ds}{\mu},$$

provided that $\kappa > 0$, $-\frac{1}{2}\pi/\mu < \theta < \frac{1}{2}\pi/\mu$, and $r > 0$. Write ny for y , multiply both sides by a_n and sum. Then

$$(2.32) \quad \mu \Phi(y^\mu, w^\nu) = \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} y^{-s} \Gamma\left(\frac{s}{\mu}\right) \phi_\nu(w^\nu, s) ds,$$

provided that $\kappa > 0$, $R(w^\nu) > 0$, $r > 0$, and $-\frac{1}{2}\pi/\mu < \theta < \frac{1}{2}\pi/\mu$. Term by term integration is justified by the fact that $\sum |a_n n^{-s} e^{-(wn)^\nu}|$ is convergent for $\sigma > 0$ and $R(w^\nu) > 0$, and hence

$$\int_{\kappa-i\infty}^{\kappa+i\infty} |y^{-s}| \left| \Gamma\left(\frac{s}{\mu}\right) \right| \left| \sum |a_n n^{-s} e^{-(wn)^\nu}| \right| |ds|$$

is convergent.

Now substitute from (2.11) in (2.32), and we have

$$(2.33) \quad \mu \Phi(y^\mu, w^\nu) = \frac{\nu}{4\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} y^{-s} \frac{\Gamma(s/\mu)}{\Gamma(s/\nu) \cos(\gamma s/\nu)} \\ \times ds \int_0^\infty u^{s-1} \{ \phi_\nu(u^\nu e^{i\gamma} + w^\nu) + \phi_\nu(u^\nu e^{-i\gamma} + w^\nu) \} du,$$

provided that $R(w^\nu) > 0$, $r > 0$, $-\beta/\mu \leq \theta \leq \beta/\mu$, where

$$\beta < \frac{1}{2}\pi - (\frac{1}{2}\pi - \alpha)\mu/\nu, \quad \gamma < \frac{1}{2}\pi,$$

and that $\kappa > 0$, the path of integration being deflected if necessary so that it does not pass through a zero of $\cos(\gamma s/\nu)$. The double integral

$$\int_{\kappa-i\infty}^{\kappa+i\infty} \int_0^\infty \left| \frac{y^{-s} \Gamma(s/\mu) u^{s-1} \{ \phi_\nu(u^\nu e^{i\gamma} + w^\nu) + \phi_\nu(u^\nu e^{-i\gamma} + w^\nu) \}}{\Gamma(s/\nu) \cos(\gamma s/\nu)} \right| |ds| du$$

is convergent, provided that $R(w^\nu) > 0$, $r > 0$, $-\beta/\mu \leq \theta \leq \beta/\mu$, and $\kappa > 0$, as may be seen at once by comparison with the integral

$$\int_{-\infty}^\infty \int_0^\infty \exp \left\{ \left(\left| \theta \right| - \frac{\pi}{2\mu} + \frac{\pi}{2\nu} - \frac{\gamma}{\nu} + \epsilon \right) |t| \right\} \left(\frac{u}{r} \right)^\kappa \frac{e^{-u\nu} \cos \gamma}{u} |dt| du,$$

where ϵ is any positive number. Hence we may invert the order of integration in (2.33), and we obtain (2.31).

2.4. *Proof of Theorem I.* Put $\gamma = \alpha$ and take $\kappa > cv$ in (2.31). It is easy to see that

$$(2.41) \quad I\left(\frac{y}{u}, \kappa\right) = \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \left(\frac{y}{u}\right)^{-s} \frac{\Gamma(s/\mu) ds}{\Gamma(s/\nu) \cos(as/\nu)} = O\left\{\left(\frac{u}{r}\right)^\kappa\right\},$$

provided that $u > 0$, $r > 0$, and $-\beta/\mu \leq \theta \leq \beta/\mu$, where

$$\beta < \frac{1}{2}\pi - (\frac{1}{2}\pi - \alpha)\mu/\nu.$$

Hence the integral

$$(2.42) \quad \int_0^1 u^{-cv} \left| I\left(\frac{y}{u}, \kappa\right) \right| \frac{du}{u},$$

where $\kappa > cv$, is convergent. As in Lemma 2.2, (2.23) and (2.24) are satisfied uniformly for $0 \leq |w| \leq w_0$, $|\arg w^\nu| \leq \alpha$. From them, together with (2.41) and (2.42), it follows that the integral in (2.31), which can now be written

$$\int_0^\infty \{ \phi_\nu(u^\nu e^{i\alpha} + w^\nu) + \phi_\nu(u^\nu e^{-i\alpha} + w^\nu) \} I\left(\frac{y}{u}, \kappa\right) \frac{du}{u},$$

is uniformly convergent for $0 \leq |w| \leq w_0$, $|\arg w^\nu| \leq \alpha$, provided that $r > 0$, $-\beta/\mu \leq \theta \leq \beta/\mu$ and $\kappa > cv$. Hence

$$\lim_{w \rightarrow 0} \mu \Phi(y^\mu, w^\nu) = \frac{\nu}{2} \int_0^\infty \{ \phi_\nu(u^\nu e^{i\alpha}) + \phi_\nu(u^\nu e^{-i\alpha}) \} I\left(\frac{y}{u}, \kappa\right) \frac{du}{u},$$

when $w \rightarrow 0$ so that $|\arg w^\nu| \leq \alpha$, provided that $r > 0$, $-\beta/\mu \leq \theta \leq \beta/\mu$, and $\kappa > cv$, which completes the theorem.

2.5. It may seem surprising that the value of $\phi_\mu(y)$ does not change with the value of κ even if the path of integration is shifted from one side of the point $s = (m + \frac{1}{2})\pi\nu/\alpha$, where m is a positive integer, to the other, although the inner integrand has a pole at that point. In order to see what happens in this case consider the integral

$$(2.51) \quad \frac{1}{2\pi i} \int \left(\frac{y}{u}\right)^{-s} \frac{\Gamma(s/\mu) ds}{\Gamma(s/\nu) \cos(as/\nu)},$$

where $u > 0$, $r > 0$, and $-\beta/\mu \leq \theta \leq \beta/\mu$, taken round the rectangle $\kappa_1 - iT$, $\kappa_1 + iT$, $\kappa_2 + iT$, $\kappa_2 - iT$, where $\kappa_1 < (m + \frac{1}{2})\pi\nu/\alpha < \kappa_2$. The contributions to the integral of the horizontal sides tend to zero as $T \rightarrow \infty$, and, on evaluating the residue at the point $s = (m + \frac{1}{2})\pi\nu/\alpha$, we have

$$I\left(\frac{y}{u}, \kappa_1\right) = \frac{\Gamma\{(m + \frac{1}{2})\pi\nu/\mu\alpha\}}{\Gamma\{(m + \frac{1}{2})\pi\nu/\alpha\}} \left(\frac{u}{y}\right)^{(m + \frac{1}{2})\pi\nu/\alpha} + I\left(\frac{y}{u}, \kappa_2\right).$$

Hence
$$\phi_\mu(y^\mu, \kappa_1) = \frac{\nu \Gamma\left\{(m+\frac{1}{2})\pi\nu/\mu\alpha\right\}}{\mu \Gamma\left\{(m+\frac{1}{2})\pi/\alpha\right\}} \chi(y) + \phi_\mu(y^\mu, \kappa_2),$$

where

$$\begin{aligned} \chi(y) &= \frac{1}{2} \int_0^\infty \{ \phi_\nu(u^\nu e^{i\alpha}) + \phi_\nu(u^\nu e^{-i\alpha}) \} \left(\frac{u}{y} \right)^{(m+\frac{1}{2})\pi\nu/\alpha} \frac{du}{u} \\ &= \Gamma\left\{(m+\frac{1}{2})\pi/\alpha\right\} \cos(m+\frac{1}{2})\pi \cdot f\left\{(m+\frac{1}{2})\pi\nu/\alpha\right\} y^{-(m+\frac{1}{2})\pi\nu/\alpha} = 0, \end{aligned}$$

by Lemma 2.2. Hence

$$\phi_\mu(y^\mu, \kappa_1) = \phi_\mu(y^\mu, \kappa_2).$$

2.6. The formula which I have used implicitly is

$$(2.61) \quad \mu e^{-y^\mu} = \frac{\nu}{2\pi i} \int_0^\infty e^{-\nu u \cos \alpha} \cos(u^\nu \sin \alpha) \frac{du}{u} \int_{\kappa-i\infty}^{\kappa+i\infty} \left(\frac{y}{u} \right)^{-s} \frac{\Gamma(s/\mu) ds}{\Gamma(s/\nu) \cos(\alpha s/\nu)},$$

where $\kappa > 0$, $r > 0$, $-\beta/\mu \leq \theta \leq \beta/\mu$, $\beta < \frac{1}{2}\pi - (\frac{1}{2}\pi - \alpha)\mu/\nu$. When $\nu = 2\mu$ and $\alpha = 0$, this is equivalent to

$$(2.62) \quad e^{-2y} = \frac{2}{\pi^{\frac{1}{2}}} \int_0^\infty e^{-t^2 - y^2/t^2} dt.$$

For then, writing $2y$ instead of y^μ , and u instead of u^μ in (2.61), we have

$$e^{-2y} = \frac{1}{\mu \pi i} \int_0^\infty e^{-u^2} \frac{du}{u} \int_{\kappa-i\infty}^{\kappa+i\infty} \left(\frac{2y}{u} \right)^{-s/\mu} \frac{\Gamma(s/\mu) ds}{\Gamma(\frac{1}{2}s/\mu)}.$$

Put $u = y/t$, and apply the duplication formula for the gamma-function. Then

$$e^{-2y} = \frac{1}{2\pi^{\frac{3}{2}} i} \int_0^\infty e^{-y^2/t^2} dt \int_{\kappa-i\infty}^{\kappa+i\infty} t^{-s/\mu-1} \Gamma\left\{\frac{1}{2}\left(\frac{s}{\mu}+1\right)\right\} \frac{ds}{\mu}.$$

Now put $s' = \frac{1}{2}(s/\mu+1)$, and the inner integral becomes Mellin's integral. Hence we obtain (2.62).

If $\nu = 2\mu$, Theorems I-III may be proved more simply by means of (2.62).

3.1. *Proof of Theorem II.* We have to show that, if

$$\phi_\nu(y^\nu) \rightarrow l,$$

when $y \rightarrow 0$ so that $-a/\nu \leq \theta \leq a/\nu$, then

$$\phi_\mu(y^\mu) \rightarrow l,$$

when $y \rightarrow 0$ so that $-\beta/\mu \leq \theta \leq \beta/\mu$.

Putting $u = rz$ in (1.32) we have

$$(3.11) \quad \mu \phi_\mu(r^\mu e^{i\theta\mu}) = \frac{\nu}{2} \int_0^\infty \{ \phi_\nu(r^\nu z^\nu e^{i\alpha}) + \phi_\nu(r^\nu z^\nu e^{-i\alpha}) \} I\left(\frac{e^{i\theta}}{z}, \kappa\right) \frac{dz}{z},$$

provided that $r > 0$, $-\beta/\mu \leq \theta \leq \beta/\mu$ and $\kappa > 0$. Consider the integral (2.51) taken round a similar contour to that used before, where now

$$(3.12) \quad -\text{Min}(\tfrac{1}{2}\pi\nu/a; \mu) < \kappa_1 < \kappa_2 < \tfrac{1}{2}\pi\nu/a.$$

Since the integrand is regular inside the contour we have

$$(3.13) \quad I\left(\frac{e^{i\theta}}{z}, \kappa_1\right) = I\left(\frac{e^{i\theta}}{z}, \kappa_2\right)$$

for all values of κ_1, κ_2 which satisfy (3.12)*. This justifies the previous statement that (1.32) holds for $\kappa > -\text{Min}(\tfrac{1}{2}\pi\nu/a; \mu)$.

Hence

$$\int_0^\infty \left| I\left(\frac{e^{i\theta}}{z}, \kappa\right) \right| \frac{dz}{z} = \int_0^1 \left| I\left(\frac{e^{i\theta}}{z}, \kappa_2\right) \right| \frac{dz}{z} + \int_1^\infty \left| I\left(\frac{e^{i\theta}}{z}, \kappa_1\right) \right| \frac{dz}{z}.$$

Choose κ_1 and κ_2 , so that $\kappa_1 < 0 < \kappa_2$, then, by (2.41), we see that the integral

$$\int_0^\infty \left| I\left(\frac{e^{i\theta}}{z}, \kappa\right) \right| \frac{dz}{z}$$

is uniformly convergent for $-\beta/\mu \leq \theta \leq \beta/\mu$. By the definition of summability (A_a, ν)

$$\phi_\nu(r^\nu z^\nu e^{\pm i\alpha}) \rightarrow l$$

when $r \rightarrow 0$, uniformly for $0 \leq z \leq z_0$. Also

$$(3.14) \quad \phi_\nu(r^\nu z^\nu e^{\pm i\alpha}) = O(1),$$

when $r \rightarrow 0$, uniformly for $z \geq 0$. Hence we may proceed to the limit under the sign of integration in (3.11), and we have

$$(3.15) \quad \begin{aligned} \lim_{r \rightarrow 0} \mu \phi_\mu(r^\mu e^{i\theta\mu}) &= \nu l \int_0^\infty I\left(\frac{e^{i\theta}}{z}\right) \frac{dz}{z} \\ &= \frac{\nu l}{2\pi i} \int_0^1 \frac{dz}{z} \int_{\kappa_2-i\infty}^{\kappa_2+i\infty} \frac{z^s e^{i\theta s} \Gamma(s/\mu) ds}{\Gamma(s/\nu) \cos(as/\nu)} + \frac{\nu l}{2\pi i} \int_1^\infty \frac{dz}{z} \int_{\kappa_1-i\infty}^{\kappa_1+i\infty} \dots ds, \end{aligned}$$

where† $-\text{Min}(\tfrac{1}{2}\pi\nu/a; \mu) < \kappa_1 < 0 < \kappa_2 < \tfrac{1}{2}\pi\nu/a$.

The double integral

$$\int_0^1 \int_{\kappa_2-i\infty}^{\kappa_2+i\infty} \left| \frac{z^{s-1} e^{-i\theta s} \Gamma(s/\mu)}{\Gamma(s/\nu) \cos(as/\nu)} \right| dz |ds|$$

is convergent, as may be seen at once by comparison with the integral

$$\int_0^1 \int_{-\infty}^\infty z^{\kappa_2-1} \exp\left\{\left(|\theta| - \frac{\pi}{2\mu} + \frac{\pi}{2\nu} - \frac{a}{\nu} + \epsilon\right)|t|\right\} dz |dt|,$$

* Compare Hj. Mellin, *Math. Annalen*, 68 (1910), 318-319.

† Compare G. H. Hardy, *Acta Math.*, 42 (1920), 330-332.

provided that $-\beta/\mu \leq \theta \leq \beta/\mu$. Hence we may calculate the first integral in (3.15) by inverting the order of integration; and applying a similar process to the other integral we find that

$$(3.16) \quad \lim_{r \rightarrow 0} \mu \phi_\mu(r^\mu e^{i\theta\mu}) = \frac{\nu l}{2\pi i} \left\{ \int_{\kappa_2 - i\infty}^{\kappa_2 + i\infty} - \int_{\kappa_1 - i\infty}^{\kappa_1 + i\infty} \right\} \frac{e^{-i\theta s} \Gamma(s/\mu) ds}{s \Gamma(s/\nu) \cos(as/\nu)},$$

provided that $-\beta/\mu \leq \theta \leq \beta/\mu$.

Consider the integral

$$\frac{1}{2\pi i} \int \frac{e^{-i\theta s} \Gamma(s/\mu) ds}{s \Gamma(s/\nu) \cos(as/\nu)}$$

taken round the rectangle used for (3.13). The contributions to the integral of the horizontal sides tend to zero as $T \rightarrow \infty$, and the only singularity inside the rectangle is a simple pole at the origin with residue μ/ν . Hence

$$\lim_{r \rightarrow 0} \phi_\mu(r^\mu e^{i\theta\mu}) \rightarrow l,$$

provided that $-\beta/\mu \leq \theta \leq \beta/\mu$, which is the result stated.

4.1. *Proof of Theorem III.* We have to show that, if

$$(4.11) \quad y^{cv} \phi_\nu(y^\nu) \rightarrow l,$$

when $y \rightarrow 0$ so that $-a/\nu \leq \theta \leq a/\nu$, then

$$y^{cv} \phi_\mu(y^\mu) \rightarrow l \frac{\Gamma(1+cv/\mu)}{\Gamma(1+c)},$$

when $y \rightarrow 0$ so that $-\beta/\mu \leq \theta \leq \beta/\mu$.

Suppose that $(m-\frac{1}{2})\pi/\alpha < c < (m+\frac{1}{2})\pi/\alpha$, where m is a positive integer, then, putting $u=rz$ in (2.12), we have

$$\mu \phi_\mu(r^\mu e^{i\theta\mu}) = \frac{\nu}{2} \int_0^\infty \{ \phi_\nu(r^\nu z^\nu e^{i\alpha}) + \phi_\nu(r^\nu z^\nu e^{-i\alpha}) \} I\left(\frac{e^{i\theta}}{z}, \kappa\right) \frac{dz}{z},$$

so that

$$(4.12) \quad \begin{aligned} \mu(r e^{i\theta})^{cv} \phi_\mu(r^\mu e^{i\theta\mu}) \\ = \frac{\nu}{2} \int_0^\infty (rz)^{\nu!} \{ \phi_\nu(r^\nu z^\nu e^{i\alpha}) + \phi_\nu(r^\nu z^\nu e^{-i\alpha}) \} \left(\frac{e^{i\theta}}{z}\right)^{cv} I\left(\frac{e^{i\theta}}{z}, \kappa\right) \frac{dz}{z}, \end{aligned}$$

provided that $r > 0$, $-\beta/\mu \leq \theta \leq \beta/\mu$, where $\beta < \frac{1}{2}\pi - (\frac{1}{2}\pi - \alpha)\mu/\nu$ and $\kappa > cv$. As before, we consider the integral (2.61) taken round a similar rectangle, for which $(m-\frac{1}{2})\pi\nu/\alpha < \kappa_1 < cv < \kappa_2 < (m+\frac{1}{2})\pi\nu/\alpha$. It follows that

$$I\left(\frac{e^{i\theta}}{z}, \kappa_1\right) = I\left(\frac{e^{i\theta}}{z}, \kappa_2\right),$$

which justifies (1.32) for $\kappa > \text{Max}(cv - \frac{1}{2}\pi\nu/\alpha; -\mu)$.

Hence

$$\int_0^\infty \left| \left(\frac{e^{i\theta}}{z} \right)^{c\nu} I \left(\frac{e^{i\theta}}{z}, \kappa \right) \right| \frac{dz}{z} = \int_0^1 \left| \left(\frac{e^{i\theta}}{z} \right)^{c\nu} I \left(\frac{e^{i\theta}}{z}, \kappa_2 \right) \right| \frac{dz}{z} + \int_1^\infty \left| \left(\frac{e^{i\theta}}{z} \right)^{c\nu} I \left(\frac{e^{i\theta}}{z}, \kappa_1 \right) \right| \frac{dz}{z}.$$

From (2.41) we see that the integral on the left is convergent. By (4.11)

$$r^\nu z^\nu \phi_\nu(r^\nu z^\nu e^{\pm i\alpha}) \sim l e^{\pm i\alpha c},$$

so that $\frac{1}{2} r^\nu z^\nu \{ \phi_\nu(r^\nu z^\nu e^{i\alpha}) + \phi_\nu(r^\nu z^\nu e^{-i\alpha}) \} \rightarrow l \cos \alpha c$,

when $r \rightarrow 0$, uniformly for $0 \leq z \leq z_0$. Also

$$r^\nu z^\nu \phi_\nu(r^\nu z^\nu e^{\pm i\alpha}) = O(1),$$

when $r \rightarrow 0$, uniformly for $z \geq 0$. Hence we may proceed to the limit under the sign of integration in (4.12), and we have

$$\begin{aligned} \lim_{r \rightarrow 0} \mu (r e^{i\theta})^{c\nu} \phi_\mu(r^\mu e^{i\theta\mu}) &= \nu l \cos \alpha c \int_0^\infty \left(\frac{e^{i\theta}}{z} \right)^{c\nu} I \left(\frac{e^{i\theta}}{z} \right) \frac{dz}{z} \\ &= \frac{\nu l \cos \alpha c}{2\pi i} \int_0^1 \frac{dz}{z} \int_{\kappa_2 - i\infty}^{\kappa_2 + i\infty} \left(\frac{z}{e^{i\theta}} \right)^{s - c\nu} \frac{\Gamma(s/\mu) ds}{\Gamma(s/\nu) \cos(\alpha s/\nu)} + \frac{\nu l \cos \alpha c}{2\pi i} \int_1^\infty \frac{dz}{z} \int_{\kappa_1 - i\infty}^{\kappa_1 + i\infty} \dots ds, \end{aligned}$$

provided that $-\beta/\mu \leq \theta \leq \beta/\mu$.

We justify the inversion of the order of integration as for (3.15), so that

$$\lim_{r \rightarrow 0} \mu (r e^{i\theta})^{c\nu} \phi_\mu(r^\mu e^{i\theta\mu}) = \frac{\nu l \cos \alpha c}{2\pi i} \left\{ \int_{\kappa_2 - i\infty}^{\kappa_2 + i\infty} - \int_{\kappa_1 - i\infty}^{\kappa_1 + i\infty} \right\} \frac{e^{-i\theta(s - c\nu)} \Gamma(s/\mu) ds}{(s - c\nu) \Gamma(s/\nu) \cos(\alpha s/\nu)}.$$

The residue at $s = c\nu$ is

$$\frac{\Gamma(c\nu/\mu)}{\Gamma(c) \cos \alpha c},$$

hence, integrating round the rectangle used at the beginning of this section, we find that

$$\lim_{r \rightarrow 0} (r e^{i\theta})^{c\nu} \phi_\mu(r^\mu e^{i\theta\mu}) = \frac{\nu l \Gamma(c\nu/\mu)}{\mu \Gamma(c)} = \frac{l \Gamma(1 + c\nu/\mu)}{\Gamma(1 + c)},$$

provided that $-\beta/\mu \leq \theta \leq \beta/\mu$, which is the result stated. If $c = (m + \frac{1}{2})\pi/\alpha$, we use the corresponding formula involving $\sin(\alpha s/\nu)$, and proceed as before.

5.1. THEOREM IV. *If $\phi_\nu(y)$ is regular at the origin, then $\phi_\mu(y)$, where $\mu < \nu$ is an integral function, and*

$$\phi_\mu(y) = \sum_{p=0}^{\infty} \frac{f(p\mu)}{p!} (-y)^p,$$

where $f(s)$ is the function defined by the sum (A_a, ν) of the series $\sum a_n n^{-s}$.

It must be remembered that, by the Carlson-Landau Gap Theorem*, $\phi_\nu(y)$ cannot be continued beyond the imaginary axis for $\nu > 1$, so that this and the following theorem are only significant for $\nu \leq 1$.

5.2. *Proof.* Putting $\alpha = 0$ in Lemma 2.2 we have

$$(5.21) \quad \Gamma\left(\frac{s}{\nu}\right) f(s) = \nu \int_0^\infty u^{s-1} \phi_\nu(u^\nu) \frac{du}{u} = \int_0^\infty y^{s/\nu} \phi_\nu(y) \frac{dy}{y},$$

provided that $\sigma > 0$. We may suppose that $\phi_\nu(y)$ is regular for $r \leq r_0$, then

$$(5.22) \quad \int_0^\infty y^{s/\nu} \phi_\nu(y) \frac{dy}{y} = \frac{1}{2i \sin(\pi s/\nu)} \int_C (-y)^{s/\nu} \phi_\nu(y) \frac{dy}{y},$$

where C is a contour starting from the infinitely distant end of the positive real axis, encircling the origin once counter-clockwise, and returning to the starting point. The many-valued function $(-y)^{s/\nu}$ is made definite by the convention $(-y)^{s/\nu} = e^{s \log(-y)/\nu}$, where $\log(-y)$ is purely real on the negative real axis. The integral on the right-hand side of (5.22) converges uniformly for $\sigma < \sigma_0$, where σ_0 is any finite number, and therefore represents an integral function. Hence

$$(5.23) \quad \frac{1}{\pi} \Gamma\left(\frac{s}{\nu}\right) f(s) \sin \frac{\pi s}{\nu} = \frac{f(s)}{\Gamma(1-s/\nu)} = \frac{1}{2\pi i} \int_C (-y)^{s/\nu} \phi_\nu(y) \frac{dy}{y}$$

is an integral function. Comparing this result with (5.21) we see that $f(s)$ is an integral function†.

Since $\phi_\nu(y)$ is regular at the origin, (1.32) holds for every $\alpha < \frac{1}{2}\pi$. We invert the order of integration, justifying the process in the usual manner, and obtain

$$\mu \phi_\mu(y^\mu) = \frac{\nu}{4\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} y^{-s} \frac{\Gamma(s/\mu) ds}{\Gamma(s/\nu) \cos(\alpha s/\nu)} \int_0^\infty u^{s-1} \{ \phi_\nu(u^\nu e^{i\alpha}) + \phi_\nu(u^\nu e^{-i\alpha}) \} du,$$

provided that $r > 0$, $\kappa > 0$, $-\beta/\mu \leq \theta \leq \beta/\mu$, where $\beta < \frac{1}{2}\pi - (\frac{1}{2}\pi - \alpha)\mu/\nu$. Hence, by Lemma 2.2, we have

$$\mu \phi_\mu(y^\mu) = \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} y^{-s} \Gamma\left(\frac{s}{\mu}\right) f(s) ds,$$

* F. Carlson and E. Landau, *Göttinger Nachrichten*, 1921, 184-188.

† See M. Fekete, *Comptes rendus*, 150 (1910), 1034. G. H. Hardy, *Proc. London Math. Soc.* (2), 8 (1910), 290-295. E. Hille, *Proc. London Math. Soc.* (2), 25 (1925), 177-184.

so that

$$(5.24) \quad \phi_{\mu}(y) = \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} y^{-s} \Gamma(s) f(s\mu) ds,$$

provided that $\kappa > 0$, $r > 0$, and $-\beta \leq \theta \leq \beta$ for every $\beta < \frac{1}{2}\pi$. This formula may also be obtained more directly from (2.32) by proceeding to the limit under the sign of integration.

We now need two lemmas:—

5.3. LEMMA 5.3. *If $\phi_{\nu}(y)$ is regular inside and on the boundary of the circle $r = r_0$, and the angle $-\alpha \leq \theta \leq \alpha$, where $0 < \alpha < \frac{1}{2}\pi$, then the function defined by the integral*

$$\int_C (-y)^{s-1} \phi_{\nu}(y) dy$$

is regular and is

$$O\{r_0 e^{(\pi-\alpha)|t|}\}$$

when $|s| \rightarrow \infty$ and $\sigma < \sigma_0$.

This can be proved by integrating round a contour similar to that used in Theorem I. We have to deform the contour C into a contour D consisting of the circle $r = r_0$ and part of the real axis taken from ∞ to r_0 and from r_0 to ∞ . Then we turn the new contour D about the origin through an angle $\theta = \pm \alpha$ according as t is positive or negative. As this lemma is a particular form of a theorem due to Carlson* I shall not give the proof in full.

5.4. LEMMA 5.4. *If $f(s\mu)/\Gamma(1-s)$ is an integral function which is*

$$(5.41) \quad O\{r_0^{\sigma} e^{(\pi-\alpha)|t|}\},$$

where $0 < \alpha < \frac{1}{2}\pi$, for $\sigma < \sigma_0$, then the function defined by (5.24) can be expressed in the form

$$\sum_{p=0}^{\infty} \frac{f(-p\mu)}{p!} (-y)^p,$$

provided that $r < r_0$.

This can be proved by considering the integral

$$\frac{1}{2\pi i} \int y^{-s} \Gamma(s) f(s\mu) ds = \frac{1}{2\pi i} \int y^{-s} \frac{\pi f(s\mu)}{\sin \pi s \Gamma(1-s)} ds$$

* F. Carlson, "Sur une classe de séries de Taylor", *Thèse pour le doctorat* (Upsal, 1914), 45-54.

taken round the rectangle $\kappa - iT$, $\kappa + iT$, $-\lambda + \frac{1}{2} + iT$, $-\lambda + \frac{1}{2} - iT$, where λ is a positive integer. Make $T \rightarrow \infty$ first and then λ . The result is well known under slightly less general hypotheses*.

5.5. *Proof of Theorem IV (continued).* By Lemma 5.3 and formula (5.23)

$$\frac{f(s)}{\Gamma(1-s/\nu)} = O\{r_0^{\sigma/\nu} e^{(\pi-\alpha)|t|/\nu}\}$$

for every $\alpha < \frac{1}{2}\pi$ and for $\sigma < \sigma_0$. Therefore, as $|s| \rightarrow \infty$,

$$\begin{aligned} \frac{f(s\mu)}{\Gamma(1-s)} &= O\left\{r_0^{\sigma\mu/\nu} e^{(\pi-\alpha)\mu|t|/\nu} \left| \frac{\Gamma(1-s\mu/\nu)}{\Gamma(1-s)} \right| \right\} \\ &= O\{e^{k\sigma+(\pi-\beta)|t|} |s|^{\sigma(1-\mu/\nu)}\}, \end{aligned}$$

where k is a number depending on μ and ν , for every $\beta < \frac{1}{2}\pi$, provided that $\sigma < \sigma_0$. Since $\mu < \nu$, $1-\mu/\nu$ is positive, and (5.41) is satisfied for every finite r_0 . It follows from Lemma 5.4 that

$$\phi_\mu(y) = \sum_{p=0}^{\infty} \frac{f(-p\mu)}{p!} (-y)^p$$

for all finite values of y . Hence $\phi_\mu(y)$ is an integral function, and the theorem is complete.

6.1. THEOREM V. *If, near the origin, $\phi_\nu(y)$ is of the form*

$$(6.11) \quad y^{-c} \psi(y),$$

where $\psi(y)$ is regular at the origin, then $\phi_\mu(y)$, where $\mu < \nu$, is regular all over the plane except at the origin, and

$$(6.12) \quad \phi_\mu(y) = \sum_{q=0}^{\infty} \Gamma\left\{\frac{(c-q)\nu}{\mu}\right\} g_q y^{-(c-q)\nu/\mu} + \sum_{p=0}^{\infty} \frac{f(-p\mu)}{p!} (-y)^p,$$

where g_q is the residue of $f(s\mu)$ at the point $s = (c-q)\nu/\mu$, in any finite region of the plane, excluding the origin.

Of course the form of the series (6.12) will have to be modified slightly if $-p = (c-q)\nu/\mu$ for some positive integers p and q .

Proof. We may suppose that $\psi(y)$ is regular for $r \leq r_0$; then, by Lemma 2.2, we have

$$(6.13) \quad \Gamma\left(\frac{s}{\nu}\right) f(s) = \int_0^{\infty} y^{s/\nu} \phi_\nu(y) \frac{dy}{y},$$

* See G. H. Hardy, *Acta Math.*, 42 (1920), 330, and E. Lindelöf, *Le calcul des résidus* (1905), 109-113.

provided that $\sigma > c\nu$. As before, it follows that

$$\Gamma\left(\frac{s}{\nu}\right)f(s) = \frac{1}{2i \sin \left\{ \pi(s-c\nu)/\nu \right\}} \int_C (-y)^{(s-c\nu)/\nu} \psi(y) \frac{dy}{y},$$

so that

(6.14)
$$\Gamma\left(\frac{s}{\nu}\right)f(s) \sin \frac{\pi(s-c\nu)}{\nu}$$

is an integral function of the form

$$O\left\{r_0^{\sigma/\nu} e^{(\pi-\alpha)|t|/\nu}\right\}$$

for every $\alpha < \frac{1}{2}\pi$ and for $\sigma < \sigma_0$. Comparing (6.13) and (6.14) we see that the only singularities of $f(s)$ are simple poles at the points $s = c\nu, (c-1)\nu, (c-2)\nu, \dots$. Hence $\Gamma(s)f(s\mu)$ is regular all over the finite part of the plane, except for poles at $s = c\nu/\mu, (c-1)\nu/\mu, \dots$, and $0, -1, -2, \dots$. Also, except at these poles,

(6.15)
$$\begin{aligned} \Gamma(s)f(s\mu) &= O\left\{\frac{r_0^{\sigma\mu/\nu} e^{(\pi-\alpha)\mu|t|/\nu} |\Gamma(s)|}{|\Gamma(s\mu/\nu) \sin \left\{ \pi(s-c\nu)\mu/\nu \right\}|}\right\} \\ &= O\left\{e^{k\sigma+(\pi-\beta)|t|} |s|^{\sigma(1-\mu/\nu)}\right\}, \end{aligned}$$

where k is a number depending on μ and ν , provided that $\sigma < \sigma_0$.

We can show, as in Theorem IV, that

$$\phi_\mu(y) = \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} y^{-s} \Gamma(s) f(s\mu) ds,$$

provided that $\kappa > c\nu/\mu, r > 0, -\beta \leq \theta \leq \beta < \frac{1}{2}\pi$. As in Lemma 5.4, we integrate round a rectangle $\kappa-iT, \kappa+iT, -\lambda+iT, -\lambda-iT$. Make $T \rightarrow \infty$ and then make $\lambda \rightarrow \infty$ through any sequence of values which do not coincide with $(c-q)\nu/\mu$, or $-p$ for any positive integers p and q . Evaluating the residues we obtain the two series required, which, by (6.15), converge in any finite region of the plane, excluding the origin. If $-p = (c-q)\nu/\mu$ for some positive integers p and q , the function $\Gamma(s)f(s\mu)$ will have a pole of the second order at the point $s = -p$. The residue will have to be calculated, and the terms containing y^p must be modified accordingly.

6.2. A simple example illustrating the above theorem is given by

$$\phi_1(y) = \sum_1^\infty n^\gamma e^{-y^n},$$

then

$$\begin{aligned} f(s) &= \sum_1^\infty n^{\gamma-s} \\ &= \zeta(s-\gamma). \end{aligned}$$

By Mellin's integral we have

$$\begin{aligned}\phi_1(y) &= \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} y^{-s} \Gamma(s) \xi(s-\gamma) ds \\ &= \Gamma(\gamma+1) y^{-\gamma-1} + \sum_{p=0}^{\infty} \frac{\xi(-p-\gamma)}{p!} (-y)^p.\end{aligned}$$

Similarly, if $\mu < 1$,

$$\begin{aligned}\phi_\mu(y) &= \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} y^{-s} \Gamma(s) \xi(s\mu-\gamma) ds \\ &= \Gamma\left(\frac{\gamma+1}{\mu}\right) y^{-(\gamma+1)/\mu} + \sum_{p=0}^{\infty} \frac{\xi(-p\mu-\gamma)}{p!} (-y)^p.\end{aligned}$$

The series for $\phi_1(y)$ is convergent for $0 < r < 2\pi$, and the series for $\phi_\mu(y)$ converges in any finite region, excluding the origin.

In conclusion, I should like to express my thanks to Prof. Hardy and Prof. Titchmarsh for many valuable suggestions and criticisms.

THE SOLIDIFICATION OF MOLTEN STEEL*

By N. M. H. LIGHTFOOT.

[Received 16 April, 1929.—Read 25 April, 1929.]

The temperature distribution, during solidification, in steel ingots of various shapes and subject to different initial and boundary conditions has been investigated by Saitô†. A number of simplifying assumptions have been made, and one of them is that the latent heat of solidification can be neglected. The present investigation is an attempt to take account of the evolution of latent heat, and to determine its effect on the rate of solidification. The metal solidifies gradually from the outer walls, and the rate of advance of crystallization is thought to play an important part in the segregation of impurities from the liquid steel. For this reason particular reference has been made to steel in numerical examples, but the investigation applies equally well to the solidification of any other substance.

In the same paper, Saitô investigates the temperature in a thick plate of steel, taking account of the liberation of latent heat at the temperature $700^{\circ}\text{C}.$, when a change of state takes place. He obtains an equa-

* The results obtained in this paper have already been published without any mathematical analysis. See "Third Report of the Committee on Heterogeneity of Steel Ingots, Section V.—The effect of latent heat on the solidification of steel ingots", *Journal of the Iron and Steel Institute*, 1 (1929), 364.

Since the paper was written, the attention of the author has been drawn to the fact that an analogous problem on the formation of ice has been discussed by Ingersoll and Zobel, *Mathematical theory of heat conduction* (1913), 150. The problem is practically identical with Case I of this paper, and, apart from differences of notation, the results of § 154 are the same. Their solution is in fact more complete, inasmuch as the thermal constants for ice and water are not assumed to be the same.

The method of this paper, if longer, is more direct, and it leads simply to the approximate solution obtained in Case III, which is the most important in connection with steel ingots.

† "On the distribution of temperature in steel ingots during cooling", *Tôhoku Imperial University, Science Reports* 10 (1921).

tion giving the position at any time of the plane at which the transformation takes place, namely the plane at which the temperature is 700°C. , and gives an approximate graphical solution. The present investigation applies equally well to this case and to the case involving the latent heat of solidification, and Saitô's results will form a useful basis for comparison.

It will be assumed throughout that the diffusivity κ , the specific heat c , and the density ρ of the steel are constant for all temperatures and are the same for liquid as for solid steel; and no account will be taken of convection currents in the liquid steel. Even with these assumptions it has been found possible to deal with only a few of the simplest types of initial and boundary conditions.

An exact solution of the problem has been obtained for a semi-infinite mass of molten steel, originally at constant temperature, bounded by the plane $x = 0$, this plane being kept at constant (zero) temperature. This solution holds approximately near the surface of a cylindrical ingot, but, of course, gives no indication of the temperature distribution near the axis.

An exact solution has also been obtained for a similar mass of molten steel in contact, at the plane $x = 0$, with a semi-infinite mass of steel originally at zero temperature. This solution gives a first approximation to the effect of the ingot mould; but since, in practice, the mould is of finite extent and since heat escapes by radiation from its outer surface, after the lapse of some time the approximation no longer has any validity. A formula for the velocity of solidification in this case has been obtained by Feild*, latent heat being taken into account. The derivation of the formulae used by him is not clear, and his formula for the velocity is different from that obtained in this paper.

An approximate solution has been obtained for the case of a slab of finite thickness, which gives an indication of the velocity of the wall of solidification when it reaches the central plane.

I. Semi-infinite mass of molten steel, the boundary $x = 0$ being constantly at temperature zero.

We consider a mass of molten steel bounded by the plane $x = 0$ and extending to infinity in the direction of x positive. The initial temperature is taken to be constant and equal to Φ , and it is assumed that the

* "Solidification of steel in the ingot mould", *Trans. American Soc. Steel Treating*, 11 (1927), 264.

boundary is kept constantly at zero temperature. The temperature at which solidification takes place is V , and the latent heat of fusion is λ .

The equation of heat-conduction reduces to

$$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2}, \quad (1)$$

of which the solution appropriate to the above initial and boundary conditions (but neglecting the effect of latent heat) is*

$$v = \Phi \Theta \left(\frac{x}{2\sqrt{\kappa t}} \right) = U, \quad \text{say}, \quad (2)$$

where

$$\Theta(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi.$$

Thus in this case the position of the plane at which the temperature is V , at time t , is given by

$$x = 2k\sqrt{\kappa t},$$

where k is given by the equation

$$V = \Phi \Theta(k). \quad (3)$$

When latent heat is taken into account, these equations are modified. We assume that the position of the plane of solidification, at which $v = V$, is given by

$$x = f(t).$$

As this plane passes any particular point, heat is evolved in consequence of the solidification of the metal, the amount of heat liberated per unit area in time dt being $\lambda \rho f'(t) dt$. Thus the evolution of heat can be accounted for by the assumption of a "moving plane source" at $x = f(t)$.

The temperature due to an instantaneous plane source of strength Q , situated at $x = x'$ when $t = t'$, the boundary $x = 0$ being kept constantly at zero temperature, is given whenever $t > t'$ by

$$v = \frac{Q}{2\sqrt{\{\pi\kappa(t-t')\}}} \left[\exp\left(-\frac{(x-x')^2}{4\kappa(t-t')}\right) - \exp\left(-\frac{(x+x')^2}{4\kappa(t-t')}\right) \right],$$

where the amount of heat concentrated at the plane $x = x'$ when $t = t'$ is $Q\rho c$ per unit area†.

* Carslaw, *The conduction of heat* (1921), 33-35.

† Carslaw, *loc. cit.*

Thus the "strength" of the moving source at time t' is $(\lambda/c) f'(t') dt'$ and its position at that time is $x' = f(t')$. The temperature at any time t due to this source is therefore

$$v = \frac{\lambda}{2c} \int_0^t \frac{f'(t')}{\sqrt{\{\pi\kappa(t-t')\}}} \left[\exp\left(-\frac{\{x-f(t')\}^2}{4\kappa(t-t')}\right) - \exp\left(-\frac{\{x+f(t')\}^2}{4\kappa(t-t')}\right) \right] dt' \\ = W, \quad \text{say.} \quad (4)$$

To this must be added the temperature given by (2) due to the original supply of heat, giving for the total temperature

$$v = U + W.$$

The unknown function $f(t)$ is to be determined from the condition that $v = V$ when $x = f(t)$. It is found that the assumption

$$f(t) = 2k\sqrt{\kappa t}$$

enables this condition to be satisfied, though k is not now determined by (3).

Substituting this value for $f(t')$ in (4), and writing

$$t' = \tau t, \quad \tau = \left(\frac{1-z^2}{1+z^2} \right)^2, \quad y = \frac{x}{2k\sqrt{\kappa t}},$$

$$a = \frac{1}{2}k(y-1), \quad \beta = \frac{1}{2}k(y+1),$$

we obtain

$$W = \frac{2\lambda k}{c\sqrt{\pi}} \{I(a, \beta) - I(\beta, a)\}, \quad (5)$$

where

$$I(m, n) = \int_0^1 \exp\left\{-\left(\frac{m}{z} + nz\right)^2\right\} \frac{dz}{1+z^2}.$$

For positive values of x and t , β is always positive, but a may assume negative as well as positive values.

By taking $(a/z) + \beta z$ as a new variable and making some obvious transformations, it is easily proved that, when $a < 0$,

$$I(a, \beta) = \frac{1}{2} \left\{ \int_0^\infty \frac{e^{-k^2 u^2}}{1+u^2} du + \int_0^y \frac{e^{-k^2 u^2}}{1+u^2} du - e^{-4a\beta} \int_{1/y}^\infty \frac{e^{-k^2 y^2 u^2}}{1+u^2} du \right\}, \quad (6)$$

and, when $a \geq 0$,

$$I(a, \beta) = \frac{1}{2} \left\{ e^{-4a\beta} \int_0^\infty \frac{e^{-k^2 y^2 u^2}}{1+u^2} du + e^{-4a\beta} \int_0^{1/y} \frac{e^{-k^2 y^2 u^2}}{1+u^2} du - \int_y^\infty \frac{e^{-k^2 u^2}}{1+u^2} du \right\}, \quad (7)$$

while, whether α is positive or negative,

$$I(\beta, \alpha) = \frac{1}{2} \left\{ \int_y^\infty \frac{e^{-k^2 u^2}}{1+u^2} du + e^{-4\alpha\beta} \int_{1/y}^\infty \frac{e^{-k^2 y^2 u^2}}{1+u^2} du \right\}. \quad (8)$$

From (5)-(8) we deduce that :

When $\alpha < 0$,

$$W = \frac{2\lambda k}{c\sqrt{\pi}} \left\{ \int_0^y \frac{e^{-k^2 u^2}}{1+u^2} du - e^{-4\alpha\beta} \int_{1/y}^\infty \frac{e^{-k^2 y^2 u^2}}{1+u^2} du \right\},$$

and, when $\alpha > 0$,

$$W = \frac{2\lambda k}{c\sqrt{\pi}} \left\{ e^{-4\alpha\beta} \int_0^{1/y} \frac{e^{-k^2 y^2 u^2}}{1+u^2} du - \int_y^\infty \frac{e^{-k^2 u^2}}{1+u^2} du \right\}.$$

Now*

$$\int_0^\infty \frac{e^{-k^2 u^2}}{1+u^2} du = \frac{1}{2} \pi e^{k^2} \{1 - \Theta(k)\},$$

and, when $\xi \leq 1$,

$$\int_0^\xi \frac{e^{-k^2 u^2}}{1+u^2} du = \sum_{n=0}^\infty \frac{(-1)^n \xi^{2n+1}}{2n+1} \left\{ 1 + \frac{k^2}{1!} + \frac{k^4}{2!} + \dots + \frac{k^{2n}}{n!} \right\}.$$

When, however, $\xi \geq 1$, we have

$$\begin{aligned} \int_0^\xi \frac{e^{-k^2 u^2}}{1+u^2} du &= \frac{1}{2} \pi e^{k^2} \{1 - \Theta(k) \Theta(k\xi)\} \\ &\quad - e^{k^2(1-\xi^2)} \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1) \xi^{2n+1}} \left\{ 1 + \frac{k^2 \xi^2}{1!} + \frac{k^4 \xi^4}{2!} + \dots + \frac{k^{2n} \xi^{2n}}{n!} \right\}. \end{aligned}$$

From these results we find that :

If $y < 1$,

$$W = \frac{\lambda k \sqrt{\pi}}{c} e^{k^2} \Theta(ky) \{1 - \Theta(k)\},$$

and, if $y > 1$,

$$W = \frac{\lambda k \sqrt{\pi}}{c} e^{k^2} \Theta(k) \{1 - \Theta(ky)\}.$$

That is, if $x < 2k\sqrt{(\kappa t)}$,

$$\begin{aligned} v &= \Phi \Theta\left(\frac{x}{2\sqrt{(\kappa t)}}\right) + \frac{\lambda k \sqrt{\pi}}{c} e^{k^2} \Theta\left(\frac{x}{2\sqrt{(\kappa t)}}\right) \{1 - \Theta(k)\} \\ &= v_1, \quad \text{say,} \end{aligned} \quad (9)$$

* This formula, which is given by De Morgan, *Differential and integral calculus* (1842), 676, is most easily verified by multiplying by e^{-k^2} and differentiating with respect to k . The two following formulae can also be verified by differentiation, the first with respect to ξ , the second with respect to k and $k\xi$.

and, if $x > 2k\sqrt{(\kappa t)}$,

$$v = \Phi \Theta\left(\frac{x}{2\sqrt{(\kappa t)}}\right) + \frac{\lambda k \sqrt{\pi}}{c} e^{k^2} \Theta(k) \left\{1 - \Theta\left(\frac{x}{2\sqrt{(\kappa t)}}\right)\right\} \quad (10)$$

$$= v_2, \text{ say.}$$

Both v_1 and v_2 satisfy the equation of heat-conduction. The initial and boundary conditions are satisfied, as we see from (9) that $v = 0$ when $x = 0$, and from (10) that $v = \Phi$ when $t = 0$. Also the temperature is constant when $x = 2k\sqrt{(\kappa t)}$, that is at the advancing wall of solidification. By equating the constant temperature to V , we obtain the equation which determines k , namely

$$V = \Phi \Theta(k) + \frac{\lambda k \sqrt{\pi}}{c} e^{k^2} \Theta(k) \{1 - \Theta(k)\}. \quad (11)$$

On substituting the value of k obtained from (11) in the expressions (9) and (10), we obtain equations giving the temperature at any point at any time.

The temperature v is continuous at the moving plane $x = 2k\sqrt{(\kappa t)}$, but the temperature-gradient is discontinuous in consequence of the evolution of heat. The rate of conduction of heat from this moving plane, per unit area, is*

$$\begin{aligned} \kappa c \rho \left[\frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial x} \right]_{x=2k\sqrt{(\kappa t)}} &= \lambda \rho k \sqrt{\frac{\kappa}{t}} \\ &= \lambda \rho \left[\frac{dx}{dt} \right]_{v=V}, \end{aligned}$$

and this is exactly the rate of evolution of latent heat per unit area due to the solidification of the metal.

By using (11), the equations for the temperature may be rewritten in the form

$$v_1 = \frac{V}{\Theta(k)} \Theta\left(\frac{x}{2\sqrt{(\kappa t)}}\right), \quad (12)$$

$$v_2 = \frac{V - \Phi \Theta(k)}{1 - \Theta(k)} + \frac{\Phi - V}{1 - \Theta(k)} \Theta\left(\frac{x}{2\sqrt{(\kappa t)}}\right). \quad (13)$$

It is seen from (13) that if $\Phi = V$, that is if the molten steel is originally just on the point of solidification, then $v_2 = V$ for all values of x and t , as would be expected. It is also evident that the tempera-

* Carslaw, *loc. cit.*, 5-7.

ture in the solid steel is the same as would be obtained if the latent heat were neglected and the original constant temperature taken to be $V/\Theta(k)$, which is greater than Φ , instead of Φ ; while the temperature in the liquid steel is the same as in the problem in which the original temperature is Φ , but the constant temperature at the boundary is

$$\frac{V - \Phi \Theta(k)}{1 - \Theta(k)}$$

instead of zero, latent heat again being neglected.

As a numerical illustration we take

$$\Phi = 1600^{\circ} \text{C.}, \quad V = 1500^{\circ} \text{C.},$$

$$\lambda = 48 \text{ calories per gram,}$$

$$c = 0.12, \quad \kappa = 0.0974,$$

From (11) it is found* that $k = 0.893$. If latent heat is neglected, the value of k found from (3) is 1.317. The speed at which the plane of solidification passes a point at distance x from the boundary is

$$\left[\frac{dx}{dt} \right]_{x=2k\sqrt{(\kappa t)}} = \frac{2k^2\kappa}{x}.$$

Thus the effect of latent heat is to reduce this speed in the ratio

$$(0.893)^2 : (1.317)^2 = 0.4598 : 1.$$

The rate of advance of solidification at any point is thus less than half the value obtained when latent heat is neglected.

II. *Semi-infinite mass of molten steel in contact with semi-infinite mass of steel originally at temperature zero.*

This is the case considered by Feild†. We assume κ , ρ , and c to be the same for the solid as for the molten steel.

The initial conditions are

$$v = 0 \quad \text{when} \quad x < 0,$$

$$v = \Phi \quad \text{when} \quad x > 0;$$

* Tables of $\Theta(x)$ are given by Silberstein, *Synopsis of applicable mathematics* (1923), 116-120.

† *Loc. cit.*

and the temperature when latent heat is neglected is easily seen to be

$$v = \frac{\Phi}{2} \left\{ 1 + \Theta \left(\frac{x}{2\sqrt{(\kappa t)}} \right) \right\} \\ = U, \quad \text{say.}$$

The evolution of latent heat can be accounted for as before, except that in this case the condition that the boundary $x = 0$ is at temperature zero does not apply.

We have therefore

$$W = \frac{\lambda}{2c} \int_0^t \frac{f'(t')}{\sqrt{\{\pi\kappa(t-t')\}}} \exp \left(-\frac{\{x-f(t')\}^2}{4\kappa(t-t')} \right) dt'.$$

Assuming again that

$$f(t) = 2k\sqrt{(\kappa t)},$$

we obtain the following results by methods similar to those already used.

When $x < 2k\sqrt{(\kappa t)}$,

$$v = \frac{\Phi}{2} \left\{ 1 + \Theta \left(\frac{x}{2\sqrt{(\kappa t)}} \right) \right\} + \frac{\lambda k \sqrt{\pi}}{2c} e^{k^2} \{1 - \Theta(k)\} \left\{ 1 + \Theta \left(\frac{x}{2\sqrt{(\kappa t)}} \right) \right\} \quad (14) \\ = v_1, \quad \text{say,}$$

and, when $x > 2k\sqrt{(\kappa t)}$,

$$v = \frac{\Phi}{2} \left\{ 1 + \Theta \left(\frac{x}{2\sqrt{(\kappa t)}} \right) \right\} + \frac{\lambda k \sqrt{\pi}}{2c} e^{k^2} \{1 + \Theta(k)\} \left\{ 1 - \Theta \left(\frac{x}{2\sqrt{(\kappa t)}} \right) \right\} \quad (15) \\ = v_2, \quad \text{say.}$$

As before, it can be verified from (14) and (15) that the rate of conduction of heat from the moving plane $x = 2k\sqrt{(\kappa t)}$ is exactly equal to the rate of evolution of latent heat.

The equation for k is

$$V = \{1 + \Theta(k)\} \left[\frac{\Phi}{2} + \frac{\lambda k \sqrt{\pi}}{2c} e^{k^2} \{1 - \Theta(k)\} \right]. \quad (16)$$

This equation enables us to put the expressions for the temperature into the simpler forms

$$v_1 = \frac{V}{1 + \Theta(k)} \left\{ 1 + \Theta \left(\frac{x}{2\sqrt{(\kappa t)}} \right) \right\}, \quad (17)$$

$$v_2 = \frac{V - \Phi \Theta(k)}{1 - \Theta(k)} + \frac{\Phi - V}{1 - \Theta(k)} \Theta \left(\frac{x}{2\sqrt{(\kappa t)}} \right). \quad (18)$$

The temperature distribution for $x > 0$ is the same as it would be in Case I if the constant temperature at the boundary were $V/\{1+\Theta(k)\}$ instead of zero. It is apparent from (18) that, if $\Phi = V$, then $v_2 = V$ for all values of x and t .

Using the same numerical values as before, we find that $k = 0.6265$. When latent heat is neglected, it is found that $k = 1.085$. Thus in this case the speed at which the plane of solidification passes any particular point is only about one-third of the value which would be obtained if latent heat were neglected. The temperature at the plane $x = 0$ is constant and equal to

$$\frac{V}{1+\Theta(k)} = 923^\circ \text{C.}, \text{ approximately.}$$

With the numerical values used by Feild, namely

$$V = \Phi = 1500^\circ \text{C.},$$

$$\lambda = 70, \quad c = 0.15,$$

$$\text{thermal conductivity, } K = \kappa c \rho = 0.07,$$

$$\rho = 7.5,$$

the value of k is found to be 0.682. Thus the position of the wall of solidification at time t is

$$x = 2k\sqrt{(\kappa t)} = 0.3402\sqrt{t}.$$

The value obtained by Feild is

$$x = 0.88\sqrt{t}.$$

III. *Slab of finite thickness.*

The initial and boundary conditions are taken to be

$$v = \Phi \quad \text{when } t = 0, \quad 0 < x < a,$$

$$v = 0 \quad \text{at } x = 0 \text{ and } x = a \text{ for all } t.$$

In the special case in which $\Phi = V$, it is possible to obtain an exact solution of this problem. For we have seen in Cases I and II that, when $\Phi = V$, there is no temperature gradient in the molten metal. Thus solidification proceeds from each of the boundaries $x = 0$ and $x = a$

independently, and the equation for k is still (11). The temperature is given by

$$v_1 = \frac{V}{\Theta(k)} \Theta\left(\frac{x}{2\sqrt{(\kappa t)}}\right), \quad 0 < x < 2k\sqrt{(\kappa t)},$$

$$v_2 = V, \quad 2k\sqrt{(\kappa t)} < x < \frac{1}{2}a,$$

and it is symmetrical about the plane $x = \frac{1}{2}a$. These equations hold until the metal has completely solidified.

It has not been found possible to solve the more general problem when the temperature is initially above that of solidification, but it is obvious that near the beginning of the process the rate of solidification is almost the same as in Case I, though in this case solidification takes place from both boundaries. It is interesting to see for how long, in an actual numerical case, this approximation can be used.

The temperature, when latent heat is neglected, can be put in the form

$$v = \Phi \left[\Theta\left(\frac{x}{2\sqrt{(\kappa t)}}\right) + \sum_{m=1}^{\infty} (-1)^m \left\{ \Theta\left(\frac{ma+x}{2\sqrt{(\kappa t)}}\right) - \Theta\left(\frac{ma-x}{2\sqrt{(\kappa t)}}\right) \right\} \right] \quad (19)$$

$$= U, \quad \text{say.}$$

The series converges for all values of x and t , and it satisfies the differential equation (1). As $t \rightarrow 0$, $v \rightarrow \Phi$ for $0 < x < a$, but $v = 0$ when $x = 0$ or $x = a$. Thus all the required conditions are satisfied.

To account for the evolution of latent heat and to keep the boundaries permanently at zero temperature, we must assume an infinite series of moving plane sources and sinks of strength $\pm(\lambda/c)f'(t)dt$ at time t , the positions of the sources being given by

$$x = 2na + f(t) \quad \text{and} \quad x = (2n+1)a - f(t),$$

and the positions of the sinks by

$$x = (2n+1)a + f(t) \quad \text{and} \quad x = 2na - f(t),$$

where n assumes all integral values, positive and negative.

In the region $0 < x < a$, this distribution gives sources at $x = f(t)$ and at $x = a - f(t)$, the assumed positions of the two walls of solidification. As before, the temperature at each of these sources must be constant and equal to V .

The temperature at any time t due to this distribution is

$$W = \frac{\lambda}{2c} \int_0^t \frac{f'(t') S dt'}{\sqrt{\{\pi\kappa(t-t')\}}},$$

where

$$\begin{aligned} S &= \sum_{n=-\infty}^{\infty} \left[\exp\left(-\frac{\{x-2na-f(t')\}^2}{4\kappa(t-t')}\right) - \exp\left(-\frac{\{x-2na+f(t')\}^2}{4\kappa(t-t')}\right) \right. \\ &\quad \left. + \exp\left(-\frac{\{x-(2n+1)a+f(t')\}^2}{4\kappa(t-t')}\right) - \exp\left(-\frac{\{x-(2n+1)a-f(t')\}^2}{4\kappa(t-t')}\right) \right] \\ &= \sum_{m=-\infty}^{\infty} (-1)^m \left[\exp\left(-\frac{\{x-ma-f(t')\}^2}{4\kappa(t-t')}\right) - \exp\left(-\frac{\{x-ma+f(t')\}^2}{4\kappa(t-t')}\right) \right]. \end{aligned}$$

If now we assume as an approximation

$$f(t) = 2k\sqrt{(\kappa t)},$$

we find that, when $0 < x < 2k\sqrt{(\kappa t)}$,

$$\begin{aligned} W_1 &= \frac{\lambda k \sqrt{\pi}}{c} e^{k^2} \left[\Theta\left(\frac{x}{2\sqrt{(\kappa t)}}\right) \{1 - \Theta(k)\} \right. \\ &\quad \left. - \Theta(k) \sum_{m=1}^{\infty} (-1)^m \left\{ \Theta\left(\frac{ma+x}{2\sqrt{(\kappa t)}}\right) - \Theta\left(\frac{ma-x}{2\sqrt{(\kappa t)}}\right) \right\} \right], \quad (20) \end{aligned}$$

and that, when $2k\sqrt{(\kappa t)} < x < \frac{1}{2}a$,

$$\begin{aligned} W_2 &= \frac{\lambda k \sqrt{\pi}}{c} e^{k^2} \left[\Theta(k) \left\{1 - \Theta\left(\frac{x}{2\sqrt{(\kappa t)}}\right)\right\} \right. \\ &\quad \left. - \Theta(k) \sum_{m=1}^{\infty} (-1)^m \left\{ \Theta\left(\frac{ma+x}{2\sqrt{(\kappa t)}}\right) - \Theta\left(\frac{ma-x}{2\sqrt{(\kappa t)}}\right) \right\} \right]. \quad (21) \end{aligned}$$

The temperature in the region $\frac{1}{2}a < x < a$ is obtained by replacing x by $a-x$ in (20) and (21), since the temperature distribution is symmetrical about the plane $x = \frac{1}{2}a$.

The discontinuity in the temperature gradient at $x = 2k\sqrt{(\kappa t)}$ is easily seen to account exactly for the latent heat evolved, but it is obvious that, when $x = 2k\sqrt{(\kappa t)}$, the temperature is not constant. If, however, we choose k to satisfy equation (11), we deduce from (19), (20), and (21) that, when $0 < x < 2k\sqrt{(\kappa t)}$,

$$v_1 = \frac{V}{\Theta(k)} \Theta\left(\frac{x}{2\sqrt{(\kappa t)}}\right) + \frac{\Phi - V}{1 - \Theta(k)} \sum_{m=1}^{\infty} (-1)^m \left\{ \Theta\left(\frac{ma+x}{2\sqrt{(\kappa t)}}\right) - \Theta\left(\frac{ma-x}{2\sqrt{(\kappa t)}}\right) \right\}, \quad (22)$$

and that, when $2k\sqrt{(\kappa t)} < x < \frac{1}{2}a$,

$$v_2 = \frac{V - \Phi \Theta(k)}{1 - \Theta(k)} + \frac{\Phi - V}{1 - \Theta(k)} \left[\Theta\left(\frac{x}{2\sqrt{(\kappa t)}}\right) + \sum_{m=1}^{\infty} (-1)^m \left\{ \Theta\left(\frac{ma+x}{2\sqrt{(\kappa t)}}\right) - \Theta\left(\frac{ma-x}{2\sqrt{(\kappa t)}}\right) \right\} \right]. \quad (23)$$

If $\Phi = V$, (22) and (23) reduce to

$$\begin{aligned} v_1 &= \frac{V}{\Theta(k)} \Theta\left(\frac{x}{2\sqrt{(\kappa t)}}\right) \\ &= V, \quad \text{when } x = 2k\sqrt{(\kappa t)}, \\ v_2 &= V. \end{aligned}$$

This verifies that, in this particular case, the form assumed for $f(t)$ satisfies the conditions of the problem. Solidification, in this case, proceeds from each boundary exactly as if the metal were of infinite extent.

If $V < \Phi$, we find that, when $x = 2k\sqrt{(\kappa t)}$,

$$v_1 = v_2 = V + \frac{\Phi - V}{1 - \Theta(k)} \sum_{m=1}^{\infty} (-1)^m \left\{ \Theta\left(\frac{ma}{2\sqrt{(\kappa t)}} + k\right) - \Theta\left(\frac{ma}{2\sqrt{(\kappa t)}} - k\right) \right\}. \quad (24)$$

When t is very small, the variable part of the expression on the right in (24) is also very small, so that the conditions of the problem are approximately satisfied. Taking the same numerical values as in Case I, we find without difficulty that the temperature at the plane of solidification does not differ from 1500°C. by more than about 17°C. until this plane reaches the position $x = \frac{3}{8}a$. Thus, until this position is reached, equations (11), (22), and (23) give a very good approximation to the temperature. The value of the temperature at the plane $x = 2k\sqrt{(\kappa t)}$, derived from equation (24), is less than V , so the actual velocity of the plane of solidification at any point is greater than the assumed value.

After this stage has been reached, it is necessary to assume a different functional form for $f(t)$. In accordance with what has been stated about the velocity of the plane of solidification, it seems reasonable to assume that

$$\left. \begin{aligned} f(t) &= 2k\sqrt{(\kappa t)}, & (0 < t < t_1) \\ f(t) &= b - 2K\sqrt{\{\kappa(t_2 - t)\}} & (t > t_1) \end{aligned} \right\}, \quad (25)$$

where t_1 may be chosen more or less arbitrarily; *e.g.*, in the numerical case just considered, we may take t_1 to be the time required for the plane

of solidification to reach the position $x = \frac{3}{2}a$, the error until this stage with the assumption (25) being comparatively small. The constants b and t_2 are to be chosen in terms of k , t_1 , and K so as to make $f(t)$ and $f'(t)$ continuous at $t = t_1$. Hence

$$b = 2k\sqrt{\kappa t_1} \cdot \left\{1 + \frac{K^2}{k^2}\right\}, \quad t_2 = t_1 \left\{1 + \frac{K^2}{k^2}\right\}.$$

This leaves the constant K undetermined. Its value can be adjusted in a numerical case so as to make the temperature at the plane $x = f(t)$ equal to V for some particular value of t , for example the value for which $x = f(t) = \frac{1}{2}a$.

The form assumed for $f(t)$ makes it possible to evaluate the rather complicated integrals which occur; it gives a curve similar in general form to that of Saitô's graphical approximation*.

With the assumption (25) we find that, after the instant $t = t_1$ until the plane of solidification reaches the centre of the slab,

$$v = U + \frac{\lambda k}{2c} \int_0^{t_1} \frac{S_1 dt'}{\sqrt{\{\pi t'(t-t')\}}} + \frac{\lambda K}{2c} \int_{t_1}^t \frac{S_2 dt'}{\sqrt{\{\pi(t_2-t')(t-t')\}}},$$

where

$$S_1 = \sum_{m=-\infty}^{\infty} (-1)^m \left\{ \exp \left(-\frac{[x - ma - 2k\sqrt{\kappa t'}]^2}{4\kappa(t-t')} \right) - \exp \left(-\frac{[x - ma + 2k\sqrt{\kappa t'}]^2}{4\kappa(t-t')} \right) \right\},$$

$$S_2 = \sum_{m=-\infty}^{\infty} (-1)^m \left\{ \exp \left(-\frac{[x - ma - b + 2K\sqrt{\{\kappa(t_2-t')\}}]^2}{4\kappa(t-t')} \right) - \exp \left(-\frac{[x - ma + b - 2K\sqrt{\{\kappa(t_2-t')\}}]^2}{4\kappa(t-t')} \right) \right\}.$$

The first integral may be transformed by the substitution

$$t' = \tau t,$$

and the second by the substitution

$$t_2 - t' = \tau(t_2 - t).$$

If, for brevity, we write

$$\frac{t_1}{t} = \mu, \quad \frac{t_2 - t_1}{t_2 - t} = \nu,$$

* *Loc. cit.*

the result of making the substitution is

$$v = U + \frac{\lambda k}{2c\sqrt{\pi}} \int_0^\mu \frac{(S_3 - S_4) d\tau}{\sqrt{\{\tau(1-\tau)\}}} + \frac{\lambda K}{2c\sqrt{\pi}} \int_1^\nu \frac{(S_5 + S_6) d\tau}{\sqrt{\{\tau(\tau-1)\}}}, \quad (26)$$

where

$$S_3 = \sum_{m=0}^{\infty} (-1)^m \left\{ \exp\left(-\frac{k^2(y'_m - \sqrt{\tau})^2}{1-\tau}\right) - \exp\left(-\frac{k^2(y'_m + \sqrt{\tau})^2}{1-\tau}\right) \right\},$$

$$S_4 = \sum_{m=1}^{\infty} (-1)^m \left\{ \exp\left(-\frac{k^2(y_m - \sqrt{\tau})^2}{1-\tau}\right) - \exp\left(-\frac{k^2(y_m + \sqrt{\tau})^2}{1-\tau}\right) \right\},$$

$$S_5 = \sum_{m=0}^{\infty} (-1)^m \left\{ \exp\left(-\frac{K^2(\eta_m - \sqrt{\tau})^2}{\tau-1}\right) - \exp\left(-\frac{K^2(\xi'_m - \sqrt{\tau})^2}{\tau-1}\right) \right\},$$

$$S_6 = \sum_{m=1}^{\infty} (-1)^m \left\{ \exp\left(-\frac{K^2(\eta'_m + \sqrt{\tau})^2}{\tau-1}\right) - \exp\left(-\frac{K^2(\xi_m + \sqrt{\tau})^2}{\tau-1}\right) \right\}$$

and

$$\begin{aligned} y_m &= \frac{ma-x}{2k\sqrt{\{\kappa t\}}}, & y'_m &= \frac{ma+x}{2k\sqrt{\{\kappa t\}}}, \\ \eta_m &= \frac{ma+b-x}{2K\sqrt{\{\kappa(t_2-t)\}}}, & \eta'_m &= \frac{ma-b+x}{2K\sqrt{\{\kappa(t_2-t)\}}}, \\ \xi_m &= \frac{ma-b-x}{2K\sqrt{\{\kappa(t_2-t)\}}}, & \xi'_m &= \frac{ma+b+x}{2K\sqrt{\{\kappa(t_2-t)\}}}. \end{aligned}$$

By appropriate changes of variable*, it is easy to show that

$$\begin{aligned} & \int_0^\mu \left\{ \exp\left(-\frac{k^2(y-\sqrt{\tau})^2}{1-\tau}\right) - \exp\left(-\frac{k^2(y+\sqrt{\tau})^2}{1-\tau}\right) \right\} \frac{d\tau}{\sqrt{\{\tau(1-\tau)\}}} \\ &= 2 \left\{ e^{-4\alpha\beta} \left(\int_{-1/y}^{\gamma/(ky)} \frac{e^{-k^2 y^2 u^2}}{1+u^2} du - \int_{1/y}^{\delta/(ky)} \frac{e^{-k^2 y^2 u^2}}{1+u^2} du \right) \right. \\ & \quad \left. - \left(\int_y^{\epsilon/k} \frac{e^{-k^2 u^2}}{1+u^2} du + \int_y^{\xi/k} \frac{e^{-k^2 u^2}}{1+u^2} du \right) \right\}, \quad (27) \end{aligned}$$

where α and β have the same values as in Case I, and

$$\gamma = \frac{\alpha}{p} - \beta p, \quad \delta = \frac{\beta}{p} - \alpha p, \quad \epsilon = \frac{\alpha}{p} + \beta p, \quad \xi = \frac{\beta}{p} + \alpha p,$$

* For example, by writing $\tau = \left(\frac{1-z^2}{1+z^2}\right)^2$, the integrand on the left transforms into two terms of the same form as the integrand of $I(m, n)$ in Case I.

p being defined by the equation

$$p^2 = \frac{\sqrt{t} - \sqrt{t_1}}{\sqrt{t} + \sqrt{t_1}}.$$

This integral has the same form whether α be positive or negative. Similarly it can be shown that, if $0 < \eta < 1$,

$$\begin{aligned} \int_1^\nu \exp\left(-\frac{K^2(\eta - \sqrt{\tau})^2}{\tau - 1}\right) \frac{d\tau}{\sqrt{\{\tau(\tau - 1)\}}} \\ = 2 \left\{ \int_{\delta'/K}^\infty \frac{e^{-K^2 u^2}}{u^2 - 1} du - e^{4\alpha'\beta'} \int_{-\xi'/(K\eta)}^\infty \frac{e^{-K^2 \eta^2 u^2}}{u^2 - 1} du \right\}. \end{aligned} \quad (28)$$

If, however, $\eta > 1$, the value of the integral on the left is

$$2 \left\{ e^{4\alpha'\beta'} \int_{\xi'/(K\eta)}^\infty \frac{e^{-K^2 \eta^2 u^2}}{u^2 - 1} du - \int_{-\delta'/K}^\infty \frac{e^{-K^2 u^2}}{u^2 - 1} du \right\}, \quad (29)$$

while, whenever $\eta > -1$,

$$\begin{aligned} \int_1^\nu \exp\left(-\frac{K^2(\eta + \sqrt{\tau})^2}{\tau - 1}\right) \frac{d\tau}{\sqrt{\{\tau(\tau - 1)\}}} \\ = 2 \left\{ \int_{-\gamma'/K}^\infty \frac{e^{-K^2 u^2}}{u^2 - 1} du + e^{4\alpha'\beta'} \int_{\epsilon'/(K\eta)}^\infty \frac{e^{-K^2 \eta^2 u^2}}{u^2 - 1} du \right\}, \end{aligned} \quad (30)$$

where $\alpha' = \frac{1}{2}K(\eta - 1)$, $\beta' = \frac{1}{2}K(\eta + 1)$,

$$\gamma' = \frac{\alpha'}{q} - \beta'q, \quad \delta' = \frac{\beta'}{q} - \alpha'q, \quad \epsilon' = \frac{\alpha'}{q} + \beta'q, \quad \xi' = \frac{\beta'}{q} + \alpha'q, \quad (31)$$

q being defined by the equation

$$\frac{q^2 + 1}{q^2 - 1} = \sqrt{\frac{t_2 - t_1}{t_2 - t}}.$$

When the ranges of integration of the integrals on the right of (28), (29), and (30) contain infinities of the integrands, the integrals are to be interpreted as Cauchy's principal values. It is to be observed that

$$P \int_0^\infty \frac{e^{-K^2 u^2}}{1 - u^2} du = \sqrt{\pi} \cdot e^{-K^2} \int_0^K e^{\xi^2} d\xi, \quad (32)$$

where P denotes the principal value; and, if $\xi < 1$,

$$\int_0^\xi \frac{e^{-K^2 u^2}}{1 - u^2} du = \sum_{n=0}^\infty \frac{\xi^{2n+1}}{2n+1} \left\{ 1 - \frac{K^2}{1!} + \frac{K^4}{2!} - \dots + (-1)^n \frac{K^{2n}}{n!} \right\}, \quad (33)$$

while, if $\xi > 1$,

$$\int_{\xi}^{\infty} \frac{e^{-K^2 u^2}}{u^2 - 1} du = \sum_{n=0}^{\infty} \frac{e^{-K^2(1+\xi^2)}}{(2n+1)\xi^{2n+1}} \left\{ 1 + \frac{(K\xi)^2}{1!} + \frac{(K\xi)^4}{2!} + \dots + \frac{(K\xi)^{2n}}{n!} \right\} \\ - \sqrt{\pi} \cdot e^{-K^2} \{ 1 - \Theta(K\xi) \} \int_0^K e^{\xi^2} d\xi. \quad (34)$$

These results, with those obtained in Case I, enable us to calculate the temperature at any time and place when the constant K is known. The labour involved in determining K by trial, so as to make the temperature at the plane of solidification assume the correct value at some prescribed time, is very great; the result of such a calculation is given below.

It is easy to see that $\eta_0 > 1$ when $x < f(t)$, and $\eta_0 < 1$ when $x > f(t)$; η_m ($m > 0$) and ξ'_m always exceed 1, and ξ_m and η'_m always exceed -1 . Hence the only term in the expression (26) which has a discontinuous gradient is the term which involves η_0 .

It follows that

$$\left[\frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial x} \right]_{x=f(t)} = \frac{\lambda K}{c\sqrt{\pi}} \left[\frac{\partial}{\partial x} \left\{ \exp(4\alpha'_0 \beta'_0) \int_{\xi'_0/(K\eta_0)}^{\infty} \frac{\exp(-K^2 \eta_0^2 u^2)}{u^2 - 1} du \right. \right. \\ \left. \left. - \int_{-\delta'_0/K}^{\infty} \frac{\exp(-K^2 u^2)}{u^2 - 1} du - \int_{\delta'_0/K}^{\infty} \frac{\exp(-K^2 u^2)}{u^2 - 1} du \right. \right. \\ \left. \left. + \exp(4\alpha'_0 \beta'_0) \int_{-\xi'_0/(K\eta_0)}^{\infty} \frac{\exp(-K^2 \eta_0^2 u^2)}{u^2 - 1} du \right\} \right]_{\eta_0=1},$$

where α'_0 , β'_0 , δ'_0 , ξ'_0 are derived from the corresponding expressions in (31) by substituting η_0 for η .

Hence

$$\left[\frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial x} \right]_{x=f(t)} = \frac{2\lambda K}{c\sqrt{\pi}} \left[\frac{\partial}{\partial x} \left\{ \exp(4\alpha'_0 \beta'_0) \int_0^{\infty} \frac{\exp(-K^2 \eta_0^2 u^2)}{u^2 - 1} du \right. \right. \\ \left. \left. - \int_0^{\infty} \frac{\exp(-K^2 u^2)}{u^2 - 1} du \right\} \right]_{\eta_0=1} \\ = -\frac{2\lambda K}{c} \left[\frac{\partial}{\partial x} \left(e^{-K^2} \int_0^{K\eta_0} e^{\xi^2} d\xi \right) \right]_{\eta_0=1} \quad \text{by (32),} \\ = -\frac{2\lambda K^2}{c} \left[\frac{\partial \eta_0}{\partial x} \right]_{\eta_0=1} \\ = \frac{\lambda K}{c} \frac{1}{\sqrt{\{\kappa(t_2 - t)\}}}.$$

Also

$$f'(t) = K \sqrt{\frac{\kappa}{t_2 - t}},$$

and so finally

$$\kappa c \rho \left[\frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial x} \right]_{x=f(t)} = \lambda \rho f'(t),$$

that is to say, the rate of conduction of heat from the moving plane is exactly equal to the rate of evolution of latent heat.

As a numerical illustration, take the same values for Φ , V , ... as in Case I. We assume that

$$f(t) = 2k \sqrt{\kappa t} \quad \text{when } f(t) \leq \frac{3}{8}a,$$

$$f(t) = b - 2K \sqrt{\kappa(t_2 - t)} \quad \text{when } f(t) \geq \frac{3}{8}a,$$

$$k = 0.893.$$

As already shown, the error in this assumption when $f(t) \leq \frac{3}{8}a$ is comparatively small. In order that $f(t)$ and $f'(t)$ may be continuous at $f(t) = \frac{3}{8}a$, we must have

$$b = \frac{3}{8}a \left(1 + \frac{K^2}{k^2} \right), \quad \frac{\kappa t_2}{a^2} = \frac{9}{256k^2} \left(1 + \frac{K^2}{k^2} \right).$$

The value of K , obtained by trial, which makes $v = V$ when $x = f(t) = \frac{1}{2}a$ is 2.635. Then

$$\frac{b}{a} = 3.639, \quad \frac{\kappa t_2}{a^2} = 0.4275.$$

It has been verified that, with these values for the constants, the temperature at the moving plane of solidification differs from 1500°C . by not more than about 1.5 per cent. for any value of t . Thus the position of the solid boundary at any time t is given approximately by the formulae

$$\left. \begin{aligned} \frac{x}{a} &= 1.786 \sqrt{\frac{\kappa t}{a^2}}, & 0 \leq x \leq \frac{3}{8}a \\ \frac{x}{a} &= 3.639 - 5.270 \sqrt{\left(0.4275 - \frac{\kappa t}{a^2} \right)}, & \frac{3}{8}a \leq x \leq \frac{1}{2}a \end{aligned} \right\}. \quad (34)$$

The relation between x/a and $\kappa t/a^2$, given by (34), is shown graphic-

ally in Fig. 1 (i). Fig. 1 (ii) shows the relation between the same quantities at the solid boundary when latent heat is neglected. The

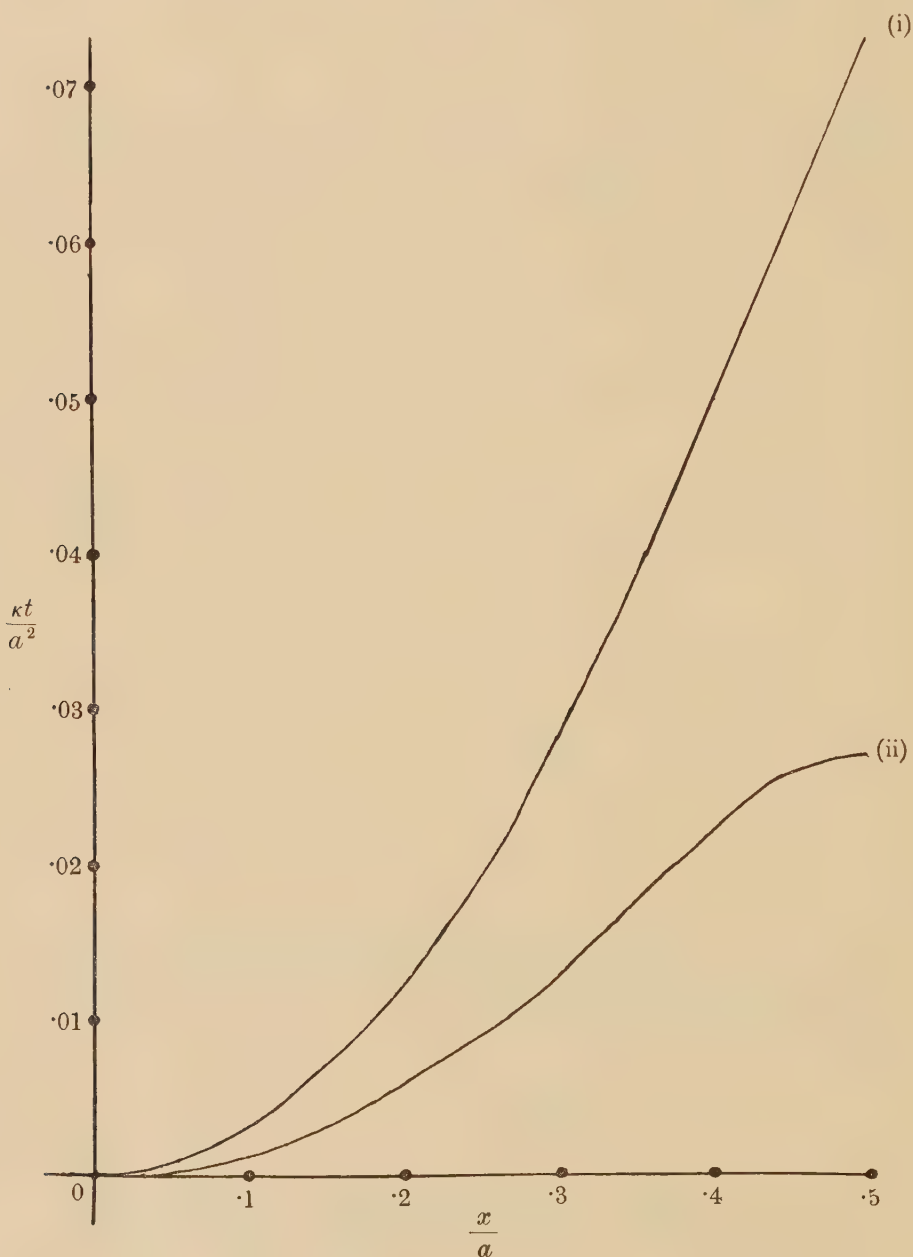


Fig. 1.

entirely different character of the two curves near the central plane of the slab indicates the importance of the effect of the evolution of latent

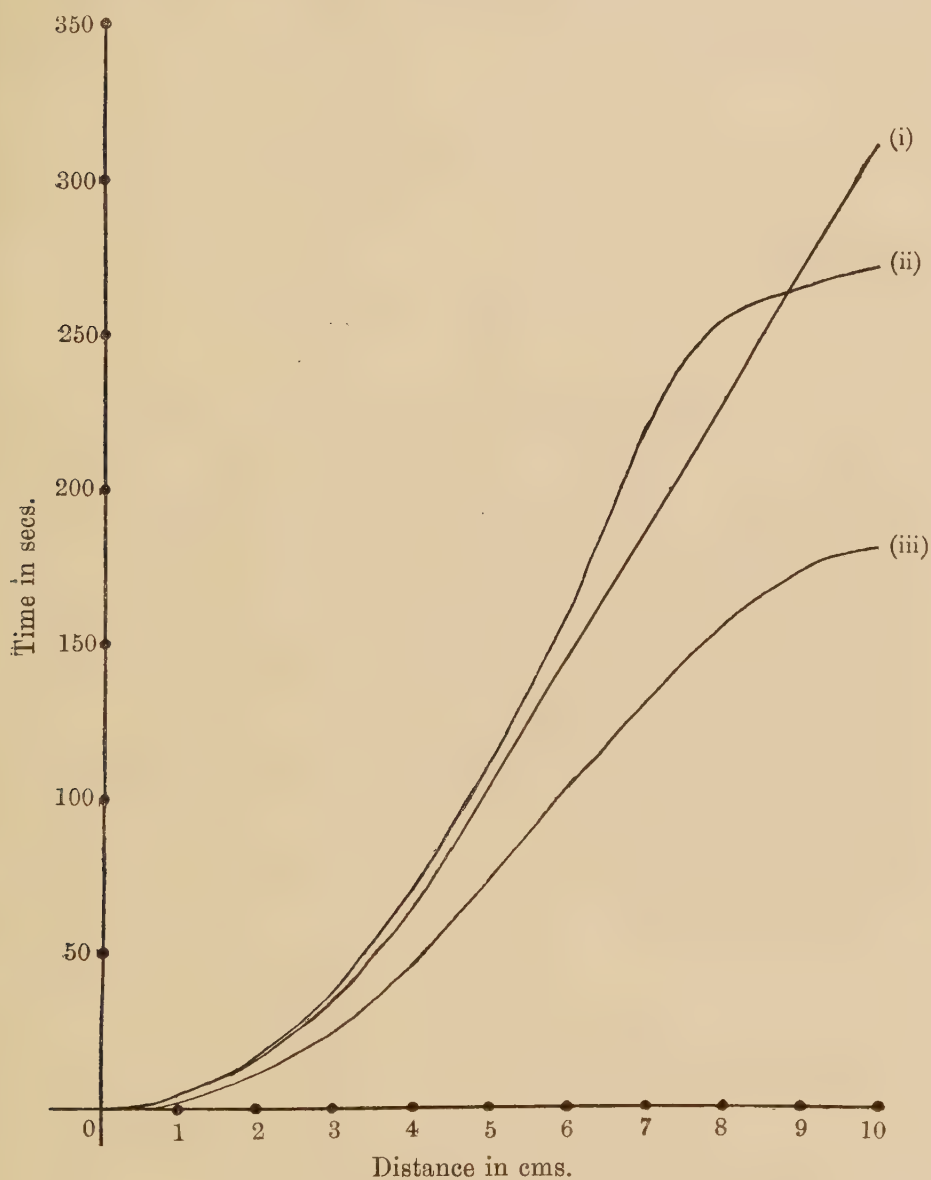


Fig. 2.

heat upon the rate of advance of solidification. This rate is practically constant during the latter part of the process, instead of tending to infinity, as it appears to do when latent heat is neglected.

It was thought to be of some interest to apply the method given above to the case for which Saitô* obtained a graphical approximation in connection with the latent heat evolved at 700° C.

The numerical values used are

$$\Phi = 850^{\circ} \text{C.}, \quad V = 700^{\circ} \text{C.},$$

$$a = 20 \text{ cm.}, \quad \kappa = 0.0974,$$

$$\lambda = 18.2, \quad c = 0.12.$$

The formulae giving the approximate position of the plane which is at temperature 700° C. are

$$\frac{x}{a} = 1.571 \sqrt{\frac{\kappa t}{a^2}}, \quad 0 \leq x \leq \frac{1}{4}a,$$

$$\frac{x}{a} = 6.926 - 8.12 \sqrt{\left(0.7015 - \frac{\kappa t}{a^2}\right)}, \quad \frac{1}{4}a \leq x \leq \frac{1}{2}a.$$

It has been verified that, with these formulae, the temperature at the critical plane is always within about 1.5 per cent. of 700° C. Fig. 2 (i) shows graphically the position of this plane at any time. Figs. 2 (ii) and 2 (iii) are reproduced from Saitô's paper, and show respectively his approximation when latent heat is taken into account and the position of the plane when latent heat is neglected. It seems likely that a closer approximation obtained by Saitô's method would agree more closely with the result obtained in this paper.

It appears, in this case also, that the rate of advance of the critical plane is practically constant as the plane approaches the centre of the slab.

* *Loc. cit.*

CONGRUENCES IN REAL QUATERNIONS

By A. R. RICHARDSON.

[Received 13 March, 1929.—Read 14 March, 1929.]

The theory of quaternionic congruences with respect to a modulus $m \cdot 1$, where m is a real prime and 1 is the unit quaternion, has been considered by A. Hurwitz*, and leads to the important theorem that every orthogonal substitution of determinant $+1$ on four variables is obtained from the quaternionic equation $y = axb$. Since such a modulus is not a prime quaternion†, and since it is commutative with all real quaternions, the general theory still remains to be discussed. We shall take m to be a real prime quaternion; hence its norm N is a real prime which, for the sake of shortness in stating results, will be assumed to be also a complex prime, *i.e.* a real prime of the form $4n-1$.

The quaternions will be written as $A+Bj$, where A and B are of the form $a+bi$, a and b being real numbers, and i the usual quaternionic element. As quaternions $a+bi$ are simply isomorphic with the corresponding complex numbers $a+\sqrt{(-1)}b$, we shall use the same notation for both sets. Conjugates will be denoted by corresponding letters with bar.

We write $x \equiv a \pmod{m}$ if $x-a = \mu m$, where μ is a real quaternion integer. Such congruences will be termed left-handed to distinguish them from the more general case in which

$$x-a = \mu_1 m + \mu_2 mi + \mu_3 mj + \mu_4 mk.$$

THEOREM 1. *If u is a unity quaternion, and if $x \equiv a \pmod{m}$, then $x \equiv a \pmod{um}$.*

For $x-a = \mu m = \lambda(um)$ where $\lambda u = \mu$, *i.e.* $\lambda = \mu u^{-1}$, and is also a quaternion integer. Hence $x \equiv a \pmod{um}$.

* *Zahlentheorie der Quaternionen* (1919), Ch. 7 and 8.

† L. E. Dickson, *Algebras and their arithmetics* (1923), 152-153.

Since a unity u can always be found such that um has real integral coordinates*, we shall suppose now that m has real integral coordinates. Its norm N is unaltered.

Further, any integral quaternion q may be written†

$$q_0\rho + q_1i + q_2j + q_3k,$$

where the q 's are real integers and

$$\rho = \frac{1}{2}(1+i+j+k),$$

and since $q \equiv q_0(1+N)\rho + q_1i + q_2j + q_3k \pmod{m}$,

and $1+N$ is an even integer, a complete set of residues $\pmod{m}$ will be taken to consist of quaternions with real integral coefficients.

THEOREM 2. *If $x \equiv a$ and $a \equiv y \pmod{m}$ then $x \equiv y \pmod{m}$, and if $x \equiv a$ and $y \equiv \beta \pmod{m}$ then $x \pm y \equiv a \pm \beta \pmod{m}$, and $yx \equiv ya \pmod{m}$, but xy is NOT necessarily congruent to ay .*

The proofs are obvious.

THEOREM 3. *The necessary and sufficient condition that*

$$A + Bj \equiv 0 \pmod{m = p + qj}$$

is that $Aq \equiv Bp \pmod{N}$.

The condition is necessary, for if

$$A + Bj = (x + yj)(p + qj) = (xp - y\bar{q}) + (xq + y\bar{p})j,$$

then

$$A = xp - y\bar{q}, \quad B = xq + y\bar{p}.$$

Hence $Aq = xpq - yq\bar{q} \equiv xpq + yp\bar{p} \pmod{N} \equiv Bp \pmod{N}$.

An equivalent condition is that $A\bar{p} + B\bar{q} \equiv 0 \pmod{N}$. The condition is also sufficient. Let $Aq \equiv Bp \pmod{N}$, then $A \equiv Bp q^{-1} \pmod{N}$, for, since N is a complex prime and m a prime quaternion, $q \not\equiv 0 \pmod{N}$.

Hence $A + Bj \equiv Bq^{-1}(p + qj) \pmod{N} \equiv 0 \pmod{m}$.

Since N is a complex prime, the congruence $Aq \equiv Bp \pmod{N}$ has only one solution. Similar results follow if N is not a complex prime, but then there will be more than one solution. The complete set of complex residues mod N are known to form a Galois field of order N^2 , and we

* *Ibid.*, 153.

† *Ibid.*, 148.

shall denote a primitive root of this field by α , and the complete set of residues by $0, \alpha, \alpha^2, \dots, \alpha^{N^2-1} \equiv 1$.

The condition of Theorem 3 may then be written $A \equiv \beta B \pmod{N}$, where β is a definite power of α .

THEOREM 4. *The quaternions $0, \alpha, \dots, \alpha^{N^2-1}$ form a complete set of residues mod m .*

No two are congruent, for if $\alpha^r \equiv \alpha^s \pmod{m}$, then, by Theorem 3, $\alpha^r - \alpha^s \equiv 0 \pmod{N}$, which is impossible.

Next, every integral quaternion is congruent to one of them.

The condition for $A + Bj \equiv \alpha^s$ or $A - \alpha^s + Bj \equiv 0 \pmod{m}$ is

$$A - \alpha^s \equiv \beta B \pmod{N}, \text{ by Theorem 3.}$$

But any number $A - \beta B$ of the field is congruent to a power α^s of its primitive root $\alpha \pmod{N}$, and the result follows*.

We shall show later that there are many other complete sets of residues mod m which are not equivalent to the above.

THEOREM 5. *Complete sets of residues mod m which are also congruent mod N are equivalent with respect to addition and multiplication.*

This result is true, since $N = m\overline{m}$.

Hence the system of non-equivalent complete sets of residues mod m is to be found among the set of numbers $\alpha^r + \alpha^s j$. These will now be written in the form of a table in which congruent elements appear in the same column, and elements in which the multiplier of j is the same appear in the same row. In other words, the congruent elements in different rows differ by the same complex multiple of m .

TABLE I, corresponding to the multiplier β .

0,	1,	α ,	...	α^r ,	...	α^{N^2-2} ,	$+0.j$,
β ,	$\beta+1$,	$\beta+\alpha$,	...	$\beta+\alpha^r$,	...	$\beta+\alpha^{N^2-2}$,	$+1.j$,
...	...	...	...	...	...	...	...
$\beta\alpha^s$,	$\beta\alpha^s+1$,	$\beta\alpha^s+\alpha$,	...	$\beta\alpha^s+\alpha^r$,	...	$\beta\alpha^s+\alpha^{N^2-2}$,	$+\alpha^s.j$,
...	...	...	...	...	...	...	...
$\beta\alpha^{N^2-2}$,	$\beta\alpha^{N^2-2}+1$,	$\beta\alpha^{N^2-2}+\alpha$,	...	$\beta\alpha^{N^2-2}+\alpha^r$,	...	$\beta\alpha^{N^2-2}+\alpha^{N^2-2}$,	$+\alpha^{N^2-2}.j$.

Thus $\beta\alpha^s + \alpha^r + \alpha^s j \equiv \alpha^r \pmod{m}$.

* I owe the simplification of this proof to a referee.

Since $\beta\alpha^s + \alpha^r$ is an element of the field it is a power of α , and it is evident that the first elements of any row in the table may be obtained by permuting the elements of the first row.

THEOREM 6. *The permutations by which the rows of Table I can be obtained from its first row form an Abelian group of order and degree N^2 .*

Let P_s denote the permutation by which the row beginning with $\beta\alpha^s$ is obtained from the first. Hence P_s changes α^r into $\beta\alpha^s + \alpha^r$, and P_t changes this into $\beta\alpha^s + \alpha^r + \beta\alpha^t$. Therefore $P_s P_t$ changes α^r into $\beta(\alpha^s + \alpha^t) + \alpha^r$, or, since $\alpha^s + \alpha^t = \alpha^p$, $P_s P_t = P_p$.

The set of permutations is therefore a group which we shall denote by G .

The group is Abelian, since $P_t P_s$ is also equal to P_p .

The multiplication table for the group is easily written out from an inspection of the addition table for the elements of the Galois field mod N , and is identical with it if β is chosen suitably. Similarly the same group G applied to the first column of Table I will give the other columns. A formula giving p in terms of s and t would be interesting.

THEOREM 7. *The N^2 complete sets of residues of Table I are not simply isomorphic as regards multiplication.*

$$\text{Let} \quad A_s + B_s j \equiv \alpha^s \equiv A'_s + B'_s j$$

$$\text{and} \quad A_r + B_r j \equiv \alpha^r \equiv A'_r + B'_r j,$$

and let $\bar{\alpha} = \alpha^p$. Then

$$(A_s + B_s j)(A_r + B_r j) \equiv (A_s + B_s j) \alpha^r \equiv A_s \alpha^r + B_s \alpha^{rp} j$$

$$\text{and} \quad (A'_s + B'_s j)(A'_r + B'_r j) \equiv (A'_s + B'_s j) \alpha^r \equiv A'_s \alpha^r + B'_s \alpha^{rp} j.$$

These are congruent only if

$$A_s \alpha^r - A'_s \alpha^r \equiv \beta \alpha^{rp} (B_s - B'_s) \pmod{N},$$

$$\text{i.e. if} \quad A_s - A'_s \equiv \alpha^{rp-r} \beta (B_s - B'_s) \pmod{N}.$$

$$\text{But} \quad A_s - A'_s \equiv \beta (B_s - B'_s) \pmod{N};$$

therefore we must have

$$\beta (B_s - B'_s) \equiv \beta \alpha^{rp-r} (B_s - B'_s) \pmod{N}.$$

Such a relation cannot hold for all values of r unless $B_s \equiv B'_s$. Hence the N^2 rows are non-equivalent in contrast with the cases where m is a real or complex integer and with those discussed by Hurwitz.

It is interesting to note that, with the exception of the first, each row is a non-commutative and, in general, *non-associative* algebra. The question arises whether all non-associative finite algebras can be generated in a similar way.

Further, numerical examples show that a sub-classification is possible, since in some rows primitive roots exist, that is numbers ρ such that each residue in the row is a non-associative power of ρ . In other rows this is not possible, although each element can be expressed in terms of a small number of generating elements. An examination of these algebras would be of interest.

Relations between different tables.

There are N^2-1 different tables corresponding to the N^2-1 possible values of β . Not all of these correspond to congruences with norm N , but only those for which $1+\beta\bar{\beta} \equiv 0 \pmod{N}$. All those for which $1+\beta\bar{\beta} \equiv 0 \pmod{N}$ will be prime quaternions*.

Corresponding rows in these tables consist of the same elements arranged in a different order.

THEOREM 8. *The N^2-1 different tables may be obtained by applying the substitutions of the group G to one of them.*

Consider the first parts of the elements of corresponding rows of two tables belonging to β_1 and $\beta_2 \equiv \beta_1 \alpha^t \pmod{N}$ respectively.

$$\text{Then} \quad \beta_2 \alpha^s + \alpha^r + \alpha^s j \equiv (\beta_1 \alpha^{s+t} + \alpha^r) + \alpha^s j.$$

The part in brackets is the general term in the $(s+t+1)$ -th row of the first table and can be obtained from the $(s+1)$ -th row by a permutation of G .

If we associate with each row of a given table the permutation P_r by which we can pass from the corresponding row of a fixed table, then the relations which exist between the different tables may be expressed thus.

THEOREM 9. *The permutations on the suffixes of the elements P_r of G form the cyclic group of order N^2-1 .*

For a change from a table belonging to β to one belonging to $\beta\alpha$ alters the first part of the row $\dots, \beta\alpha^s + \alpha^r, \dots$ into the first part of the next row $\dots, \beta\alpha^{s-1} + \alpha^r, \dots$

* L. E. Dickson, *loc. cit.*, 152, Lemma 6.

THEOREM 10. *The permutations of G are of period N and are expressible as products of N cycles each of period N .*

For $P_s = (0, \beta a^s, 2\beta a^s, \dots, r\beta a^s)(\delta, \delta + \beta a^s, \dots, \delta + t\beta a^s) \dots$

where

$$(r+1)\beta a^s \equiv 0 \pmod{N},$$

$$(t+1)\beta a^s + \delta \equiv \delta \pmod{N}, \text{ and so on.}$$

Hence, since N is a complex prime,

$$(r+1) \equiv 0 \pmod{N},$$

$$(t+1) \equiv 0 \pmod{N}, \text{ and so on.}$$

Thus the theorem is proved.

In view of the fact that ideals are powerless to restore unique factorization in quaternions*, it would be of interest to establish relations between the tables belonging to m_1 and m_2 and that belonging to $m_1 m_2$.

Further, it is desirable to interpret, from the point of view of congruences, tables generated by groups of permutations other than G .

* A. Speiser, *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich*, 71 (1926), 8-49.

ON THE REPRESENTATIONS OF A NUMBER AS THE SUM OF TWO PRODUCTS

By T. ESTERMANN.

[Received 7 June, 1929.—Read 13 June, 1929.]

1. Introduction.

This paper deals with the number $\nu_2(n)$ of solutions of the equation $xy+zu=n$ in positive integers x, y, z , and u , for any given positive integer n . It is an application of Kloosterman's refinements of Hardy's and Littlewood's well known method, which have been simplified in my paper "Vereinfachter Beweis eines Satzes von Kloosterman", *Hamburger Abh.*, 7 (1929), 83–98, hereafter quoted as "I". The result is

$$\nu_2(n) = n \sum_{r=0}^2 \log^r n \sum_{j=0}^{2-r} a_{r,j} \sigma_{-1}^{(j)}(n) + O\{n^{\frac{1}{2}} \log^{\frac{2}{3}} n \sigma_{-\frac{3}{4}}(n)\},$$

where $\sigma_{-1}^{(j)}(n) = \sum_{d|n} d^{-1} \log^j d$ (0^0 meaning 1), $\sigma_{-\frac{3}{4}}(n) = \sum_{d|n} d^{-\frac{3}{4}}$, and

$a_{0,0}, a_{0,1}, \dots, a_{2,0}$ are certain constants. Their values are given in formula (70), where γ is Euler's constant, a_q the q -th derivative of $1/\Gamma(s)$ for $s=2$, and b_h the h -th derivative of $1/\zeta(s)$ for $s=2$; the 0-th derivative meaning the function itself.

The error term in the above formula is obviously $O(n^{\frac{1}{2}+\epsilon})$ for any positive ϵ , and the reader will easily find that it implies Ingham's* elementary formulae for $\nu_2(n)$, which he denotes by $T(n)$.

I am indebted to Prof. Hecke for his suggestion to use the functions

$$(1) \quad \xi(s; a, k) = \sum_{m \equiv a \pmod{k}} m^{-s}$$

instead of the L functions, which I used in my paper "On the representations of a number as the sum of three products".† Prof. Hecke's suggestion has led to a very considerable simplification.

* *Journal London Math. Soc.*, 2 (1927), 207–208.

† *Proc. London Math. Soc.*, 29 (1929), 453–478.

2. Application of I (Section 4) and of Mellin's formula.

Putting

$$(2) \quad F(\tau) = \sum_{n=1}^{\infty} d(n) e^{2\pi i n \tau},$$

where $d(n)$ is the number of positive divisors of n , we have

$$(3) \quad F^2(\tau) = \sum_{n=2}^{\infty} \nu_2(n) e^{2\pi i n \tau}.$$

Applying I (Section 4) with $N = 1$, $\eta = n^{-1}$, and $M = [\sqrt{n}]$, we obtain

$$(4) \quad \nu_2(n) = e^{2\pi} \sum_{k=1}^{\sqrt{n}} c_{n,k},$$

where

$$(5) \quad c_{n,k} = \int_{-k^{-1}n^{-\frac{1}{2}}}^{k^{-1}n^{-\frac{1}{2}}} e^{-2\pi i n \mathfrak{S}} S d\mathfrak{S},$$

$$(6) \quad S = \sum_{\substack{0 < h \leq k \\ (h,k)=1}} g \xi_k^{-nh} F^2\left(\frac{h}{k} + y\right),$$

and

$$(7) \quad y = \frac{i}{n} + \mathfrak{S};$$

the factor g being an auxiliary function of n , k , h , and $\mathfrak{S}$, which assumes no values other than 0 and 1, and ξ_k being an abbreviation for $e^{2\pi i/k}$.

Now, by (2) and Mellin's formula,

$$(8) \quad F\left(\frac{h}{k} + y\right) = \sum_{m=1}^{\infty} d(m) \xi_k^{mh} e^{2\pi i m y} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(s) z^{-s} D(s; \xi_k^h) ds,$$

where

$$(9) \quad z = -2\pi i y$$

and

$$(10) \quad D(s; \xi_k^h) = \sum_{m=1}^{\infty} d(m) \xi_k^{hm} m^{-s}.$$

3. The functional equation for the D functions.

Henceforth (except in Section 10) we always take

$$(11) \quad (h, k) = 1.$$

Beside the functions $\zeta(s; a, k)$, defined by (1), I now introduce the functions

$$(12) \quad \zeta(s; \xi_k^a) = \sum_{m=1}^{\infty} \xi_k^{am} m^{-s}.$$

Then, putting

$$(13) \quad G(s) = -i(2\pi)^{s-1} \Gamma(1-s),$$

we have

$$(14) \quad \zeta(s; a, k) = G(s) k^{-s} [e^{\frac{1}{2}i\pi s} \zeta(1-s; \xi_k^a) - e^{-\frac{1}{2}i\pi s} \zeta(1-s; \xi_k^{-a})]$$

and

$$(15) \quad \zeta(s; \xi_k^a) = G(s) k^{1-s} [e^{\frac{1}{2}i\pi s} \zeta(1-s; -a, k) - e^{-\frac{1}{2}i\pi s} \zeta(1-s; a, k)].$$

(14) can be proved in the same way as the functional equation of Riemann's ζ -function, and (15) is an immediate consequence of (14).

Now, by (10), (1), and (12),

$$(16) \quad \begin{aligned} D(s; \xi_k^b) &= \sum_{l, r} \xi_k^{blr} (lr)^{-s} = \sum_{b=1}^k \sum_{\substack{l, r \\ l \equiv b \pmod{k}}} \xi_k^{blr} (lr)^{-s} \\ &= \sum_{b=1}^k \sum_{r=1}^{\infty} \xi_k^{bbr} r^{-s} \sum_{l \equiv b \pmod{k}} l^{-s} = \sum_{b=1}^k \zeta(s; \xi_k^{bb}) \zeta(s; b, k). \end{aligned}$$

Hence, applying (14) and (15), we obtain by an easy calculation

$$(17) \quad \begin{aligned} D(s; \xi_k^b) &= 2G^2(s) k^{1-2s} \left[\cos(\pi s) \sum_{b=1}^k \zeta(1-s; -hb, k) \zeta(1-s; \xi_k^b) \right. \\ &\quad \left. - \sum_{b=1}^k \zeta(1-s; hb, k) \zeta(1-s; \xi_k^b) \right]. \end{aligned}$$

Now, if the least positive residue mod k of $-hb$ be denoted by a , and if $[h, k]$ be defined as in I by

$$(18) \quad h[h, k] \equiv 1 \pmod{k}, \quad 0 < [h, k] \leq k,$$

we have, by (11) and (16),

$$\begin{aligned} \sum_{b=1}^k \zeta(1-s; -hb, k) \zeta(1-s; \xi_k^b) &= \sum_{a=1}^k \zeta(1-s; a, k) \zeta(1-s; \xi_k^{-[h, k]a}) \\ &= D(1-s; \xi_k^{-[h, k]}), \end{aligned}$$

and similarly

$$\sum_{b=1}^k \zeta(1-s; hb, k) \zeta(1-s; \xi_k^b) = D(1-s; \xi_k^{[h, k]});$$

hence, by (17),

$$(19) \quad D(s; \xi_k^h) = 2G^2(s) k^{1-2s} [\cos(\pi s) D(1-s; \xi_k^{-[h, k]}) - D(1-s; \xi_k^{[h, k]})].$$

4. The pole of the D functions at the point $s = 1$.

It easily follows from (16) that $D(s; \xi_k^h)$ is regular everywhere except at $s = 1$. To investigate the function in the neighbourhood of this point, I observe that, by (1),

$$(20) \quad \zeta(s; k, k) = k^{-s} \zeta(s);$$

hence, by (16),

$$D(s; \xi_k^h) - k^{-s} \zeta(s) \sum_{b=1}^k \zeta(s; \xi_k^{hb}) = \sum_{b=1}^{k-1} \zeta(s; \xi_k^{hb}) [\zeta(s; b, k) - k^{-s} \zeta(s)].$$

Here each term of the sum on the right is regular at $s = 1$; so, therefore, is the left-hand side, which means that $D(s; \xi_k^h)$ has the same meromorphic part as $k^{-s} \zeta(s) \sum_{b=1}^k \zeta(s; \xi_k^{hb})$. Now, by (12),

$$\sum_{b=1}^k \zeta(s; \xi_k^{hb}) = k \sum_{m \equiv 0 \pmod{k}} m^{-s} = k^{1-s} \zeta(s).$$

Hence $D(s; \xi_k^h)$ has the same meromorphic part as $k^{1-2s} \zeta^2(s)$, which means that

$$(21) \quad D(s; \xi_k^h) = k^{-1} [(s-1)^{-2} + 2(\gamma - \log k)(s-1)^{-1} + \dots],$$

where γ is Euler's constant.

5. Decomposition of $F(h/k+y)$.

It follows from (8) that

$$(22) \quad F\left(\frac{h}{k} + y\right) = Z + U,$$

where Z is the residue of $\Gamma(s) z^{-s} D(s; \xi_k^h)$ at $s = 1$, and

$$(23) \quad U = D(0; \xi_k^h) + \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(s) z^{-s} D(s; \xi_k^h) ds.$$

Since $\Gamma(s) = 1 - \gamma(s-1) + \dots$ and $z^{-s} = z^{-1} [1 - \log z \cdot (s-1) + \dots]$, we have, by (21),

$$(24) \quad Z = (kz)^{-1} (\gamma - 2 \log k - \log z).$$

6. Calculation of $D(0; \xi_k^h)$.

By (13), $G(0) = (2\pi i)^{-1}$, and by (21),

$$\lim_{s \rightarrow 0} \{ [\cos (\pi s)-1] D(1-s ; \xi_k^{-[h, k]})\}=-\frac{\pi^2}{2 k} ;$$

hence, by (19),

(25)
$$D(0 ; \xi_k^h)=\frac{1}{4}-\frac{k}{2 \pi^2} E,$$

where

(26)
$$E=\lim _{s \rightarrow 1}[D(s ; \xi_k^{-[h, k]})-D(s ; \xi_k^{[h, k]})] .$$

Now, by (16),

$$\begin{aligned} D(s ; \xi_k^{-[h, k]})-D(s ; \xi_k^{[h, k]}) &= \sum_{b=1}^k\left[\zeta(s ; \xi_k^{-[h, k] b})-\zeta(s ; \xi_k^{[h, k] b})\right] \zeta(s ; b, k) \\ &= \sum_{0< b < \frac{1}{2} k}\left[\zeta(s ; \xi_k^{-[h, k] b})-\zeta(s ; \xi_k^{[h, k] b})\right]\left[\zeta(s ; b, k)-\zeta(s ;-b, k)\right] ; \end{aligned}$$

hence, by (26),

(27)
$$E=\sum_{0< b < \frac{1}{2} k}\left[\zeta(1 ; \xi_k^{-[h, k] b})-\zeta(1 ; \xi_k^{[h, k] b})\right] E_b,$$

where

(28)
$$E_b=\lim _{s \rightarrow 1}\left[\zeta(s ; b, k)-\zeta(s ;-b, k)\right]=\frac{1}{b}-\frac{1}{k-b}+\frac{1}{k+b}-\frac{1}{2 k-b}+\ldots,$$

which implies that

(29)
$$0< E_b<\frac{1}{b} .$$

Denoting by $R_k(m)$ the least non-negative residue mod k of m , we have, by (12),

$$\begin{aligned} \zeta(1 ; \xi_k^{-[h, k] b})-\zeta(1 ; \xi_k^{[h, k] b}) &= \log \frac{1-\xi_k^{[h, k] b}}{1-\xi_k^{-[h, k] b}} \\ &= \log \left(-\xi_k^{[h, k] b}\right)=\frac{2 \pi i}{k}\left\{R_k([h, k] b)-\frac{1}{2} k\right\} ; \end{aligned}$$

hence, by (27),

(30)
$$E=\frac{2 \pi i}{k} \sum_{0< b < \frac{1}{2} k}\left\{R_k([h, k] b)-\frac{1}{2} k\right\} E_b .$$

An easy calculation shows that, for any integer m ,

$$(31) \quad R_k(m) - \frac{1}{2}k = \sum_{r=1}^k c_r \xi_k^{rm},$$

where

$$(32) \quad c_k = -\frac{1}{2}, \quad c_r = (\xi_k^{-r} - 1)^{-1} \quad (0 < r < k),$$

and this implies that

$$(33) \quad \sum_{r=1}^k |c_r| \leq \frac{1}{2} + \sum_{r=1}^{\frac{1}{2}k} \sin^{-1} \left(\frac{\pi r}{k} \right) \leq \frac{1}{2} + \frac{1}{2}k \sum_{r=1}^{\frac{1}{2}k} r^{-1} < k \log(2k).$$

By (25), (30), and (31),

$$(34) \quad D(0; \xi_k^h) = \frac{1}{4} + \frac{1}{\pi i} \sum_{\substack{0 < b < \frac{1}{2}k \\ 0 < r \leq k}} c_r \xi_k^{rb[h, k]} E_b = \sum_{l=1}^k d_l \xi_k^{l[h, k]},$$

where

$$(35) \quad d_k = \frac{1}{4} + \frac{1}{\pi i} \sum_{\substack{0 < b < \frac{1}{2}k \\ 0 < r \leq k \\ rb \equiv 0 \pmod{k}}} c_r E_b, \quad d_l = \frac{1}{\pi i} \sum_{\substack{0 < b < \frac{1}{2}k \\ 0 < r \leq k \\ rb \equiv l \pmod{k}}} c_r E_b \quad (0 < l < k).$$

By (35), (33), and (29),

$$(36) \quad \sum_{l=1}^k |d_l| \leq \frac{1}{4} + \frac{1}{\pi} \sum_{r=1}^k |c_r| \sum_{0 < b < \frac{1}{2}k} E_b < k \log^2(2k).$$

7. Calculation of the integral in (23).

I put

$$(37) \quad \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(s) z^{-s} D(s; \xi_k^h) ds = I.$$

Then, by (19) and (10),

$$(38) \quad I = \sum_{m=1}^{\infty} d(m) \xi_k^{-[h, k]m} e_m - \sum_{m=1}^{\infty} d(m) \xi_k^{[h, k]m} f_m,$$

$$\text{where} \quad e_m = \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} 2\Gamma(s) z^{-s} G^2(s) k^{1-2s} \cos(\pi s) m^{s-1} ds$$

$$\text{and} \quad f_m = \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} 2\Gamma(s) z^{-s} G^2(s) k^{1-2s} m^{s-1} ds.$$

By (13), $2\Gamma(s) G^2(s) = -(2\pi)^{2s-1} \sin^{-1}(\pi s) \Gamma(1-s)$; hence

$$(39) \quad e_m = -\frac{k}{2\pi m} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(1-s)}{\sin(\pi s)} \cos(\pi s) \left(\frac{4\pi^2 m}{k^2 z} \right)^s ds$$

and

$$(40) \quad f_m = -\frac{k}{2\pi m} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(1-s)}{\sin(\pi s)} \left(\frac{4\pi^2 m}{k^2 z}\right)^s ds.$$

Henceforth let $A_1, A_2, \dots$ denote sufficiently large absolute constants.

LEMMA. *Let $\Re u > 0$. Then*

$$(41) \quad \left| \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(1-s)}{\sin(\pi s)} u^s ds \right| < A_1 |u|^{-\frac{1}{2}}$$

and

$$(42) \quad \left| \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(1-s)}{\sin(\pi s)} \cos(\pi s) u^s ds \right| < A_2 |u|^{-\frac{1}{2}} + 2\pi |u| e^{-\Re u}.$$

Of these results, (41) is trivial. To prove (42), I first assume that $\Re u \leq 0$. Then the inequality

$$(43) \quad \left| \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(1-s)}{\sin(\pi s)} e^{i\pi s} u^s ds \right| < A_2 |u|^{-\frac{1}{2}}$$

is also trivial. Now $\cos(\pi s) = e^{i\pi s} - i \sin(\pi s)$; hence, by (43),

$$(44) \quad \left| \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(1-s)}{\sin(\pi s)} \cos(\pi s) u^s ds \right| < A_2 |u|^{-\frac{1}{2}} + \left| \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(1-s) u^s ds \right|.$$

$$\text{Now} \quad \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(1-s) u^s ds = \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \Gamma(w) u^{1-w} dw = 2\pi i u e^{-u},$$

which, together with (44), proves (42). The proof is similar in the case $\Re u > 0$.

Now let $k \leq \sqrt{n}$ and $|z| \leq k^{-1} n^{-\frac{1}{2}}$. Then, by (9) and (7),

$$\Re \frac{4\pi^2}{k^2 z} = \frac{4\pi^2}{k^2 |z|^2} \frac{2\pi}{n} \geq \pi;$$

$$\text{hence} \quad \left(\Re \frac{4\pi^2}{k^2 z} \right)^{-2} \leq \pi^{-1} \left(\Re \frac{4\pi^2}{k^2 z} \right)^{-1} = \frac{k^2 |z|^2 n}{8\pi^4};$$

hence (since $e^{-x} < 2x^{-2}$ for any positive x)

$$\exp \left(-\Re \frac{4\pi^2 m}{k^2 z} \right) < \frac{k^2}{4\pi^4} |z|^2 n m^{-2};$$

hence, by (39), (40), (41), and (42),

$$|e_m| + |f_m| < (A_1 + A_2) \frac{k^2}{4\pi^2} |z|^{\frac{1}{2}} m^{-\frac{3}{2}} + \frac{k}{\pi^2} |z| n m^{-2} < A_3 k |z| n m^{-2};$$

hence

$$(45) \quad \sum_{m=1}^{\infty} (|e_m| + |f_m|) d(m) < A_4 k |z| n.$$

Putting

$$(46) \quad g_l = \sum_{m \equiv -l \pmod{k}} d(m) e_m - \sum_{m \equiv l \pmod{k}} d(m) f_m,$$

we have, by (38),

$$(47) \quad I = \sum_{l=1}^k g_l \xi_k^{l[h, k]},$$

and, by (45),

$$(48) \quad \sum_{l=1}^k |g_l| < A_4 k |z| n.$$

8. Calculation of $F^2(h/k+y) - Z^2$.

By (23) and (37), $U = D(0; \xi_k^h) + (2\pi i)^{-1} I$; hence, by (34) and (47),

$$(49) \quad U = \sum_{l=1}^k i_l \xi_k^{l[h, k]},$$

where $i_l = d_l + (2\pi i)^{-1} g_l$ and therefore, by (36) and (48),

$$(50) \quad \sum_{l=1}^k |i_l| < A_5 k |z| n \log^2(2k).$$

Henceforth let $n \geq 2$. Still taking $k \leq \sqrt{n}$ and $|\mathfrak{S}| \leq k^{-1} n^{-\frac{1}{2}}$, we have, by (9) and (7), $k^2 |z|^2 n \leq 8\pi^2$. Hence, by (22), (24), (49), and (50),

$$(51) \quad F^2\left(\frac{h}{k} + y\right) - Z^2 = \sum_{l=1}^k j_l \xi_k^{l[h, k]},$$

where $j_1, j_2, \dots, j_k$ are certain numbers independent of h , which satisfy the inequality

$$(52) \quad \sum_{l=1}^k |j_l| < A_6 n \log^4 n.$$

9. The inequality for the error term.

I put

$$(53) \quad T = \sum_{\substack{0 < h \leq k \\ (h, k) = 1}} g \xi_k^{-nh} Z^2,$$

$$(54) \quad V = \sum_{\substack{0 < h \leq k \\ (h, k) = 1}} \xi_k^{-nh} Z^2 = c_k(n) Z^2,$$

where $c_k(n)$ is Ramanujan's* sum,

$$(55) \quad d_{n,k} = \int_{-k^{-1}n^{-\frac{1}{2}}}^{k^{-1}n^{-\frac{1}{2}}} e^{-2\pi i n \mathfrak{S}} T d\mathfrak{S},$$

and

$$(56) \quad e_{n,k} = \int_{-\infty}^{\infty} e^{-2\pi i n \mathfrak{S}} V d\mathfrak{S}.$$

It follows from the formulae (8), (9), (60), and (61) of I† that

$$(57) \quad \left| \sum_{\substack{0 < h \leq k \\ (h,k)=1}} g \xi_k^{\varepsilon - nh + l[h,k]} \right| < d^{\frac{3}{4}} \left(\frac{k}{(n,k)} \right) k^{\frac{3}{4}}(n,k)^{\frac{1}{4}} \log(4k)$$

and

$$(58) \quad \left| \sum_{\substack{0 < h \leq k \\ (h,k)=1}} (g-1) \xi_k^{\varepsilon - nh} \right| < 2d^{\frac{3}{4}} \left(\frac{k}{(n,k)} \right) k^{\frac{3}{4}}(n,k)^{\frac{1}{4}} \log(4k).$$

Since $k \leq \sqrt{n}$, we have, by Section 4.21 of I, (53), and (54),

$$(59) \quad T = V \quad (|\mathfrak{S}| \leq \tfrac{1}{2} k^{-1} n^{-\frac{1}{2}}).$$

By (6), (53), (51), (52), and (57),

$$(60) \quad |S - T| < A_7 n \log^5 n d^{\frac{3}{4}} \left(\frac{k}{(n,k)} \right) k^{\frac{3}{4}}(n,k)^{\frac{1}{4}},$$

and, by (53), (54), (58), (24), (9), and (7),

$$(61) \quad |T - V| < A_8 d^{\frac{3}{4}} \left(\frac{k}{(n,k)} \right) k^{-\frac{1}{4}}(n,k)^{\frac{1}{4}} \log n (\log^2 n + \log^2 |\mathfrak{S}|) |\mathfrak{S}|^{-2} \\ (|\mathfrak{S}| > \tfrac{1}{2} k^{-1} n^{-\frac{1}{2}}).$$

By (5), (55), and (60),

$$(62) \quad |c_{n,k} - d_{n,k}| < A_7 n^{\frac{1}{2}} \log^5 n d^{\frac{3}{4}} \left(\frac{k}{(n,k)} \right) \left(\frac{(n,k)}{k} \right)^{\frac{1}{4}}.$$

By Section 4.22 of I, (53), and (55),

$$d_{n,k} = \int_{-\infty}^{\infty} e^{-2\pi i n \mathfrak{S}} T d\mathfrak{S};$$

hence, by (56), (59), and (61),

$$(63) \quad |d_{n,k} - e_{n,k}| < A_9 n^{\frac{1}{2}} \log^3 n d^{\frac{3}{4}} \left(\frac{k}{(n,k)} \right) \left(\frac{(n,k)}{k} \right)^{\frac{1}{4}}.$$

* *Trans. Camb. Phil. Soc.*, 22 (1918), 259.

† See introduction.

It can easily be shown (see the end of I) that

$$\sum_{k=1}^{\sqrt{n}} d^3\left(\frac{k}{(n, k)}\right) \left(\frac{(n, k)}{k}\right)^{\frac{1}{4}} < A_{10} n^{\frac{3}{4}} \sigma_{-\frac{3}{4}}(n) \log^{\frac{3}{4}} n;$$

hence, by (4), (62), and (63),

$$(64) \quad \left| \nu_2(n) - \sum_{k=1}^{\sqrt{n}} e^{2\pi} e_{n, k} \right| < A_{11} n^{\frac{7}{8}} \log^{\frac{21}{8}} n \sigma_{-\frac{3}{4}}(n).$$

It easily follows from (56), (54), (24), (9), and (7) that

$$\left| \sum_{k > \sqrt{n}}^n e_{n, k} \right| < A_{12} n^{\frac{7}{8}};$$

hence, by (64),

$$(65) \quad \left| \nu_2(n) - \sum_{k=1}^n e^{2\pi} e_{n, k} \right| < A_{13} n^{\frac{7}{8}} \log^{\frac{21}{8}} n \sigma_{-\frac{3}{4}}(n).$$

10. The dominant term.

Putting $w = nz$, we have, by (56), (54), (24), (9), and (7),

$$\begin{aligned} (66) \quad e^{2\pi} e_{n, k} &= c_k(n) \int_{-\infty}^{\infty} e^{w} n^2 (kw)^{-2} (\gamma - 2 \log k - \log w + \log n)^2 d\mathfrak{Z} \\ &= nc_k(n) \frac{1}{2\pi i} \int_{2\pi - i\infty}^{2\pi + i\infty} e^{w} (kw)^{-2} \\ &\quad \times \sum_{l+m+q+r=2} \frac{2}{l! m! q! r!} \gamma^l (-2 \log k)^m (-\log w)^q \log^r n dw. \end{aligned}$$

Now it is known that

$$\frac{1}{2\pi i} \int_{2\pi - i\infty}^{2\pi + i\infty} e^w w^{-s} dw = \frac{1}{\Gamma(s)} \quad (s > 1);$$

$$\text{hence} \quad \frac{1}{2\pi i} \int_{2\pi - i\infty}^{2\pi + i\infty} e^w w^{-2} (-\log w)^q dw = a_q \quad (q = 0, 1, 2),$$

where a_q is the q -th derivative of $\Gamma^{-1}(s)$ for $s = 2$ (the 0-th derivative meaning the function itself). Hence, by (66),

$$(67) \quad e^{2\pi} e_{n, k} = nc_k(n) k^{-2} \sum_{l+m+q+r=2} \frac{2}{l! m! q! r!} \gamma^l (-2 \log k)^m a_q \log^r n,$$

It has been proved by Ramanujan* and Hardy† that

$$\sum_{k=1}^{\infty} c_k(n) k^{-s} = \frac{1}{\zeta(s)} \sigma_{1-s}(n) \quad (s > 1);$$

hence

$$(68) \quad \sum_{k=1}^{\infty} c_k(n) k^{-2} (-\log k)^m = \sum_{h+j=m} \frac{m!}{h! j!} b_h (-1)^j \sigma_{-1}^{(j)}(n),$$

where b_h and $\sigma_{-1}^{(j)}(n)$ have the meaning explained in the introduction. By (67) and (68),

$$(69) \quad \sum_{k=1}^{\infty} e^{2\pi} e_{n,k} = n \sum_{l+h+j+q+r=2} \frac{2}{l! h! j! q! r!} \gamma^l 2^h b_h (-2)^j \sigma_{-1}^{(j)}(n) a_q \log^r n \\ = n \sum_{r=0}^2 \log^r n \sum_{j=0}^{2-r} a_{r,j} \sigma_{-1}^{(j)}(n),$$

where

$$(70) \quad a_{r,j} = \frac{(-2)^j}{r! j!} \sum_{l+h+q-2-r-j} \frac{2}{l! h! q!} \gamma^l 2^h b_h a_q.$$

(65) and (69) imply the result stated.

* *Loc. cit.*, 264 (7.2).

† *Proc. Camb. Phil. Soc.*, 20 (1920), 266.

ON ABEL'S INTEGRAL EQUATION AND FRACTIONAL INTEGRALS

By L. S. BOSANQUET.

[Received and read 13 June, 1929.]

The object of this note is to solve Abel's integral equation and prove the consistency theorems for fractional integrals and mean continuity, when the functions employed are non-absolutely integrable. The integrals introduced exist in the general Denjoy* sense, and theorems are proved in which no further hypothesis is required.

The first theorem, which includes the classical theorem of Abel, is as follows.

THEOREM 1. *If $f(u) = \int_a^u (u-v)^{a-1} \phi(v) dv$ exists as a general Denjoy integral, for $a < u \leq t$, then $\int_a^t \phi(v) dv = \frac{\sin \pi a}{\pi} \int_a^t (t-u)^{-a} f(u) du$, where $0 < a < 1$, and a is any finite number.*

Let $\phi_a(t) = \frac{1}{\Gamma(a)} \int_a^t (t-u)^{a-1} \phi(u) du$, where $a > 0$, $t > a$, and a is

* I.e. are DKY-integrals. See Hobson, *Theory of functions of a real variable*, 1 (1926), 653-661, and *Encyc. Math. Wiss.*, II 3, 7 (1924), 35 c, 1065-1071.

If $f(t)$ has a general Denjoy integral in (a, b) , then $F(x) = \int_a^x f(t) dt$ is continuous in (a, b) , and, if $\phi(t)$ is a function of bounded variation in (a, b) ,

$$\int_a^b f(t) \phi(t) dt = \left[F(t) \phi(t) \right]_a^b - \int_a^b F(t) d\phi(t).$$

The mean value theorems are also valid for these integrals.

any finite number. Then $\phi_a(t)$ is called the Riemann* fractional integral of $\phi(t)$ of order a . If, in the definition of $\phi_a(t)$, we take $a = -\infty$, $\phi_a(t)$ is called the Weyl† fractional integral.

The second theorem is the consistency theorem for Riemann fractional integrals, and includes Theorem 1 as a particular case, when $0 < a < 1$ and $\beta = 1 - a$.

THEOREM 2. If $\phi_a(u) = \frac{1}{\Gamma(a)} \int_a^u (u-v)^{a-1} \phi(v) dv$ exists as a general Denjoy integral, for $a < u \leq t$, then $\phi_{a+\beta}(t) = \frac{1}{\Gamma(\beta)} \int_a^t (t-u)^{\beta-1} \phi_a(u) du$, where $a > 0$ and $\beta > 0$.

Theorem 1 is included in the following more general theorem.

THEOREM 3. If $\phi(v)$ is integrable in the general Denjoy sense in the interval (a, t) , then $\int_a^t \phi(v) dv = \frac{\sin \pi a}{\pi} \int_a^t (t-u)^{-a} du \int_a^u (u-v)^{a-1} \phi(v) dv$, where $0 < a < 1$, and a is any finite number.

In the same way Theorem 2 is included in

THEOREM 4. If $\int_a^t (t-v)^{a+\beta-1} \phi(v) dv$ exists as a general Denjoy integral, then

$$\frac{1}{\Gamma(a+\beta)} \int_a^t (t-v)^{a+\beta-1} \phi(v) dv = \frac{1}{\Gamma(a)\Gamma(\beta)} \int_a^t (t-u)^{\beta-1} du \int_a^u (u-v)^{a-1} \phi(v) dv,$$

where $a > 0$ and $\beta > 0$.

Theorems 1 and 2 differ from Theorems 3 and 4 only in the fact that the hypotheses in the former theorems are more stringent. Thus, for example, in Theorem 2, since $(t-v)^\beta$ is of bounded variation in the interval (a, t) , the existence of $\int_a^t (t-v)^{a-1} \phi(v) dv$ implies that of $\int_a^t (t-v)^\beta (t-v)^{a-1} \phi(v) dv$. In Theorems 3 and 4, nothing is said about

* Riemann, *Gesammelte Werke* (1876), 331-344.

† Weyl, "Bemerkung zum Begriff des Differentialquotienten gebrochener Ordnung", *Vierteljahrsschrift d. naturf. Gesellschaft*, 62 (1917), 296-302.

the existence of $\int_a^u (u-v)^{a-1} \phi(v) dv$, but it appears incidentally that this exists in the general Denjoy sense for almost all values of u in (a, t) .

Proof of Theorem 3. Starting from the identity

$$\int_v^t (t-u)^{-a} (u-v)^{a-1} du = \frac{\pi}{\sin \pi a},$$

as in the ordinary solution of Abel's integral equation, we have

$$\begin{aligned} \frac{\pi}{\sin \pi a} \int_a^t \phi(v) dv &= \int_a^t \phi(v) dv \int_v^t (t-u)^{-a} (u-v)^{a-1} du \\ &= \int_a^t (t-u)^{-a} du \int_a^u (u-v)^{a-1} \phi(v) dv, \end{aligned}$$

provided that it is legitimate to invert the order of integration. In order to justify this inversion, we require a series of lemmas. It is assumed throughout the following seven lemmas that $\phi(v)$ satisfies the hypotheses of Theorem 3, and that the integrand of every repeated integral is $\phi(v)(t-u)^{-a}(u-v)^{a-1}$.

LEMMA 1. *If $\phi(v)$ is absolutely integrable in an interval (p, q) , where $a \leq p < q \leq t$, then $\int_p^q dv \int_v^t du = \int_p^t du \int_p^u dv$.*

Proof. The first repeated integral exists as an absolutely convergent double integral, and therefore, by Fubini's theorem*, the inversion is justified.

LEMMA 2. *If $\phi(v)$ is absolutely integrable over a given perfect set of points, and $[p, q]$ denotes the intersection of this set with the interval (p, q) , where $a \leq p < q \leq t$, then*

$$\int_{[p, q]} dv \int_{[v, q]} du = \int_{[p, q]} du \int_{[p, u]} dv.$$

Proof. The proof is identical with that of Lemma 1.

LEMMA 3. *If $a \leq p < q < r < s \leq t$, then $\int_p^q dv \int_r^s du = \int_r^s du \int_p^q dv$.*

* Hobson, *loc. cit.*, 1, 577.

Proof. Integrating by parts with respect to v , we have

$$\begin{aligned} & \int_p^q \phi(v) dv \int_r^s (t-u)^{-\alpha} (u-v)^{\alpha-1} du \\ &= \left[\phi_1(v) \int_r^s (t-u)^{-\alpha} (u-v)^{\alpha-1} du \right]_{v=p}^q - \int_p^q \phi_1(v) dv \frac{d}{dv} \int_r^s (t-u)^{-\alpha} (u-v)^{\alpha-1} du \\ &= \int_r^s (t-u)^{-\alpha} du \left[\phi_1(v) (u-v)^{\alpha-1} \right]_{v=p}^q - \int_r^s (t-u)^{-\alpha} du \int_p^q \phi_1(v) \frac{d}{dv} (u-v)^{\alpha-1} dv \\ &= \int_r^s (t-u)^{-\alpha} du \int_p^q (u-v)^{\alpha-1} \phi(v) dv, \end{aligned}$$

by another integration by parts.

The integrations by parts are legitimate, for, since $u-v \geq r-q > 0$, the functions $\int_r^s (t-u)^{-\alpha} (u-v)^{\alpha-1} du$ and $(u-v)^{\alpha-1}$ are both bounded, monotone functions of v . The inversion of the order of integration is legitimate, because $\phi_1(v)$ is continuous.

It should be remarked here that the inner integral on the right-hand side is thus proved to exist for almost all values of u in (p, q) , and no more is necessarily true*.

LEMMA 4. *Lemma 3 remains true if $q = r$.*

Proof. Applying Lemma 3, we have

$$\int_p^q dv \int_q^s du = \lim_{\epsilon \rightarrow 0} \int_p^{q-\epsilon} dv \int_q^s du = \lim_{\epsilon \rightarrow 0} \int_q^s du \int_p^{q-\epsilon} dv = \int_q^s du \int_p^q dv,$$

provided that
$$\lim_{\epsilon \rightarrow 0} \left| \int_q^s du \int_{q-\epsilon}^q dv \right| = 0.$$

Let ϵ be chosen so that $\left| \int_{\xi}^q \phi(v) dv \right| < \delta$, if $q-\epsilon \leq \xi \leq q$. Then, for $u > q$, we have, by the second mean value theorem,

$$\begin{aligned} \left| \int_{q-\epsilon}^q (u-v)^{\alpha-1} \phi(v) dv \right| &= (u-q)^{\alpha-1} \left| \int_{\xi(u)}^q \phi(v) dv \right| \quad (q-\epsilon \leq \xi \leq q) \\ &< (u-q)^{\alpha-1} \delta. \end{aligned}$$

* See also Tonelli, *Math. Annalen*, 99 (1928), 183-199.

Therefore

$$\left| \int_q^s (t-u)^{-a} du \int_{q-\epsilon}^u (u-v)^{a-1} \phi(v) dv \right| < \delta \int_q^s (t-u)^{-a} (u-q)^{a-1} du \\ \leq \delta \int_q^t (t-u)^{-a} (u-q)^{a-1} du = \delta \frac{\pi}{\sin \pi a},$$

and, since δ is arbitrary, the result follows.

LEMMA 5. *If it has been proved that $\int_{p_n}^{p_{n+1}} dv \int_r^{p_{n+1}} du = \int_{r_n}^{p_{n+1}} du \int_{p_n}^u dv$, for $n = 1, 2, 3, \dots, N-1$, where $a \leq p = p_1 < p_2 < \dots < p_N = q \leq t$, then $\int_p^q dv \int_v^q du = \int_p^q du \int_p^u dv$.*

Proof. Applying Lemmas 3 and 4, we have

$$\int_p^q dv \int_v^q du = \sum_{n=1}^{N-1} \int_{p_n}^{p_{n+1}} dv \int_v^{p_{n+1}} du + \sum_{n=1}^{N-1} \sum_{m=n+1}^{N-1} \int_{p_n}^{p_{n+1}} dv \int_{p_m}^{p_{m+1}} du \\ = \sum_{m=1}^{N-1} \int_{p_m}^{p_{m+1}} du \int_{p_m}^u dv + \sum_{m=2}^{N-1} \sum_{n=1}^{m-1} \int_{p_m}^{p_{m+1}} du \int_{p_n}^{p_{n+1}} dv = \int_p^q du \int_p^u dv,$$

and this proves the lemma.

LEMMA 6. *If it has been proved that $\int_{p'}^{q'} dv \int_v^{q'} du = \int_{p'}^{q'} du \int_{p'}^u dv$, for every pair of numbers p', q' , such that $a \leq p < p' < q' < q \leq t$, then $\int_p^q dv \int_v^q du = \int_p^q du \int_p^u dv$.*

Proof. We have

$$\int_p^q dv \int_v^q du = \lim_{\substack{p' \rightarrow p \\ q' \rightarrow q}} \int_{p'}^{q'} dv \int_v^q du = \lim_{\substack{p' \rightarrow p \\ q' \rightarrow q}} \left[\int_{p'}^{q'} dv \int_v^{q'} du + \int_{p'}^{q'} dv \int_{q'}^q du \right] \\ = \lim_{\substack{p' \rightarrow p \\ q' \rightarrow q}} \left[\int_{p'}^{q'} du \int_{p'}^u dv + \int_{p'}^{q'} dv \int_{q'}^q du \right] \\ = \lim_{\substack{p' \rightarrow p \\ q' \rightarrow q}} \left[\int_{p'}^{q'} du \int_p^u dv - \int_{p'}^{q'} du \int_p^{p'} dv + \int_{p'}^{q'} dv \int_{q'}^q du \right] = \int_p^q du \int_p^u dv,$$

$$\text{if } \overline{\lim}_{\substack{p' \rightarrow p \\ q' \rightarrow q}} \left| \int_{p'}^{q'} du \int_p^{p'} dv \right| = 0 \quad \text{and} \quad \overline{\lim}_{\substack{p' \rightarrow p \\ q' \rightarrow q}} \left| \int_{p'}^{q'} dv \int_{q'}^q du \right| = 0.$$

To prove the first result, let p' be chosen so that $\left| \int_{\xi}^{\xi'} \phi(v) dv \right| < \delta$, if $p \leq \xi < \xi' \leq p'$. Then, arguing as in Lemma 4, we have

$$\begin{aligned} \left| \int_{p'}^{q'} du \int_p^{p'} dv \right| &= \left| \int_{p'}^{q'} (t-u)^{-\alpha} du (u-p')^{\alpha-1} \int_{\xi(u)}^{p'} \phi(v) dv \right| \quad (p \leq \xi \leq p'), \\ &< \delta \int_{p'}^{q'} (t-u)^{-\alpha} (u-p')^{\alpha-1} du < \delta \frac{\pi}{\sin \pi \alpha}, \end{aligned}$$

and, since δ is arbitrary, the result follows.

To prove the second result, choose q'' so that $\left| \int_{\xi}^{\xi'} \phi(v) dv \right| < \delta$, if $q'' \leq \xi < \xi' \leq q$, and suppose that $p' < q'' < q'$. Let

$$M = \max_{p \leq \xi < \xi' \leq q''} \left| \int_{\xi}^{\xi'} \phi(v) dv \right|,$$

and write

$$I = \int_{p'}^{q'} dv \int_{q'}^q du = \int_{p'}^{q''} dv \int_{q'}^q du + \int_{q''}^{q'} dv \int_{q'}^q du = I_1 + I_2.$$

Then, since $\int_{q'}^q (t-u)^{-\alpha} (u-v)^{\alpha-1} du$ is a monotone, increasing function of v for $v < q'$, we have, by the second mean value theorem,

$$\begin{aligned} |I_2| &= \left| \int_{q'}^q (t-u)^{-\alpha} (u-q')^{\alpha-1} du \int_{\xi}^{q'} \phi(v) dv \right| \quad (q'' \leq \xi \leq q') \\ &< \delta \int_{q'}^q (t-u)^{-\alpha} (u-q')^{\alpha-1} du \leq \delta \frac{\pi}{\sin \pi \alpha}, \end{aligned}$$

$$\begin{aligned} \text{while} \quad |I_1| &= \left| \int_{q'}^q (t-u)^{-\alpha} (u-q'')^{\alpha-1} du \int_{\xi}^{q''} \phi(v) dv \right| \quad (p \leq \xi \leq q'') \\ &\leq M \int_{q'}^q (t-u)^{-\alpha} (u-q'')^{\alpha-1} du < \delta, \end{aligned}$$

provided that $q-q'$ is sufficiently small. Since δ is arbitrary, the result follows.

LEMMA 7. Suppose $\phi(v)$ is absolutely integrable over a given perfect set of points, and let (p_n, q_n) ($n = 1, 2, 3, \dots$) be the set of contiguous or semi-contiguous intervals, which are contained in the interval (p, q) ,

where $a \leq p < q \leq t$. Suppose also that $\sum_{n=1}^{\infty} \left| \int_{p_n}^{q_n} \phi(v) dv \right|$ is convergent. Then, if it has been proved that $\int_{p_n}^{q_n} dv \int_v^{q_n} du = \int_{p_n}^{q_n} du \int_{p_n}^u dv$ ($n = 1, 2, 3, \dots$), it follows that $\int_p^q dv \int_v^q du = \int_p^q du \int_p^u dv$.

Proof. Applying Lemmas 2, 3, and 4, we have

$$\begin{aligned} \int_p^q dv \int_v^q du &= \int_{[p, q]} dv \int_{[v, q]} du + \sum_{n=1}^{\infty} \int_{p_n}^{q_n} dv \int_v^{q_n} du + \sum_{n=1}^{\infty} \sum_{p_m \geq q_n} \int_{p_n}^{q_n} dv \int_{p_m}^{q_m} du \\ &= \int_{[p, q]} du \int_{[p, u]} dv + \sum_{n=1}^{\infty} \int_{p_n}^{q_n} du \int_{p_n}^u dv + \sum_{m=1}^{\infty} \sum_{q_n \leq p_m} \int_{p_m}^{q_m} du \int_{p_n}^{q_n} dv \\ &= \int_p^q du \int_p^u dv, \end{aligned}$$

and this proves the lemma.

The proof of the theorem may now be completed by applying Lemmas 1, 5, 6, and 7. We say that the inversion has been justified for the triangle (p, q) if it has been proved that $\int_p^q dv \int_v^q du = \int_p^q du \int_p^u dv$. If (p, q) is contiguous to a certain closed set of points, we say that the triangle (p, q) is contiguous to that set. It remains then to justify the inversion for the triangle (a, t) .

Since $\phi(v)$ has a general Denjoy integral in (a, t) , there is a non-dense closed set of points H such that, if (p, q) is any interval contiguous to H , and (p', q') is any interior interval, $\phi(v)$ is absolutely integrable in the interval (p', q') . It follows, by Lemmas 1 and 6, that the inversion is valid for any triangle contiguous to H .

Let $H = P_1 + M_1$, where P_1 is perfect and M_1 is enumerable. Then the inversion may be justified for any triangle contiguous to P_1 as follows. Let (p, q) be a sub-interval of (a, t) which contains no points of P_1 . Then, if (p, q) is contiguous to M_1' , the first derivative of M_1 , and (p', q') is any interior interval, (p', q') contains only a finite number of points of M_1 . Therefore, by Lemma 5, the inversion is valid for the triangle (p', q') , and hence, by Lemma 6, for the triangle (p, q) . In the same way it may be justified if (p, q) is contiguous to M_1'' , and so on. Now M_1 is reducible, and therefore there is a number $\bar{\gamma}$ of the first or second class such that $M^{(\bar{\gamma})} \equiv 0$, and the inversion may be justified if (p, q) is contiguous to $M^{(\gamma)}$, where γ is any number of the first or second class, by means of a

finite or transfinite induction*. It follows that the inversion is valid for any triangle contiguous to P_1 .

Since $\phi(v)$ has a general Denjoy integral in the interval (a, t) , the set P_1 contains a sub-set H_1 , which is non-dense in H , and is such that, if (p, q) is any interval contiguous to H_1 , the conditions of Lemma 7 are satisfied in (p, q) . The perfect set over which $\phi(v)$ is absolutely integrable consists, in this case, of those points of P_1 which do not belong to H_1 . It follows that the inversion is valid for any triangle contiguous to H_1 .

Let $H_1 = P_2 + M_2$, where P_2 is perfect and M_2 is enumerable. Then the inversion may be justified for any triangle contiguous to P_2 , by an argument exactly similar to the one used above.

Now the set P_2 contains a sub-set H_2 , non-dense in H_1 , such that the conditions of Lemma 7 are satisfied in any contiguous interval (p, q) , and thus the inversion is valid for any triangle contiguous to H_2 .

In the same way, if $H_2 = P_3 + M_3$, where P_3 is perfect and M_3 is enumerable, the inversion may be justified for any triangle contiguous to P_3 , and P_3 contains a sub-set H_3 , non-dense in H_2 , such that the inversion is valid for any triangle contiguous to H_3 , and so on.

Now H_γ is non-dense in $H_{\gamma'}$, if $\gamma' < \gamma$, and therefore[†] there is a number $\bar{\gamma}$, of the first or second class, such that $P_{\bar{\gamma}} \equiv 0$. The inversion may be justified for any triangle contiguous to $H_{\bar{\gamma}}$, where $\bar{\gamma}$ is a number of the first or second class, by means of a finite or transfinite induction, and hence the inversion is valid for the whole triangle (a, t) .

This completes the proof of the theorem.

Proof of Theorem 4. Starting from the identity

$$\int_v^t (t-u)^{\beta-1} (u-v)^{\alpha-1} du = (t-v)^{\alpha+\beta-1} \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

the proof is similar to that of Theorem 3. A similar sequence of lemmas is proved, in which the integrand is always $\phi(v)(t-u)^{\beta-1}(u-v)^{\alpha-1}$ and the hypotheses about $\phi(v)$ are replaced by the same hypotheses about $(t-v)^{\alpha+\beta-1}\phi(v)$. The proofs are all similar, the main difference being that in the sixth lemma we use the result that

$$\int_{q'}^q (t-u)^{\beta-1} (u-v)^{\alpha-1} du = (t-v)^{\alpha+\beta-1} \chi(v),$$

where the total variation of $\chi(v)$ in (p', q') is less than $2\Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$.

* The argument is similar to that used by Hobson, *loc. cit.*, 1, § 465 (631-634).

† Hobson, *loc. cit.*, 1, § 82, 115.

THEOREM 5. *Theorems 3 and 4 remain true if $a = -\infty$.*

Proof. We have

$$\begin{aligned} \int_{-\infty}^t dv \int_v^t du &= \lim_{T \rightarrow \infty} \int_{-T}^t dv \int_v^t du = \lim_{T \rightarrow \infty} \int_{-T}^t du \int_{-T}^u dv \\ &= \lim_{T \rightarrow \infty} \left[\int_{-T}^t du \int_{-\infty}^u dv - \int_{-T}^t du \int_{-\infty}^{-T} dv \right] = \int_{-\infty}^t du \int_{-\infty}^u dv, \end{aligned}$$

provided $\overline{\lim}_{T \rightarrow \infty} \left| \int_{-T}^t du \int_{-\infty}^{-T} dv \right| = 0$.

$$\begin{aligned} \text{Now } \int_{-T}^t du \int_{-\infty}^{-T} dv &= \lim_{T' \rightarrow \infty} \int_{-T}^t du \int_{-T'}^{-T} dv = \lim_{T' \rightarrow \infty} \int_{-T'}^{-T} dv \int_{-T}^t du \\ &= \int_{-\infty}^{-T} dv \int_{-T}^t du, \quad \text{by Lemma 4.} \end{aligned}$$

To complete the proof in the general case, we observe that

$$\frac{1}{(t-v)^{\alpha+\beta-1}} \int_{-T}^t (t-u)^{\beta-1} (u-v)^{\alpha-1} du$$

is an increasing function of v , for $v < -T$. Therefore, by the second mean value theorem, we have

$$\begin{aligned} \left| \int_{-\infty}^{-T} \phi(v) dv \int_{-T}^t (t-u)^{\beta-1} (u-v)^{\alpha-1} du \right| \\ = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \left| \int_{-Z}^{-T} \phi(v)(t-v)^{\alpha+\beta-1} dv \right| \quad (Z \geq T), \end{aligned}$$

and this is less than δ , provided that T be sufficiently large. Since δ is arbitrary the result follows.

Theorems 1 and 2 cannot be extended in this way, for their hypotheses would not be included in those of Theorem 5. But the following theorem is the consistency theorem for Weyl fractional integrals.

THEOREM 6. *If $\phi_\mu(t) = \frac{1}{\Gamma(\mu)} \int_{-\infty}^t (t-v)^{\mu-1} \phi(v) dv$ exists as a general Denjoy integral for $\mu = \alpha$ and $\mu = \alpha + \gamma$, then $\phi_\mu(t)$ exists for $\alpha \leq \mu \leq \alpha + \gamma$, and $\phi_{\alpha+\beta}(t) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^t (t-v)^{\beta-1} \phi_\alpha(v) dv$, where $\alpha > 0$ and $0 < \beta \leq \gamma$.*

This is a corollary of Theorem 5.

Mean Continuity. Suppose that $\alpha \int_0^1 (1-u)^{\alpha-1} \phi(tu) du \rightarrow l$, when $t \rightarrow 0$ where $\alpha > 0$, the integral being a general Denjoy integral. Then $\phi(t)$ is said to tend (C, α) to l , when $t \rightarrow 0$, and to be continuous (C, α) for $t = 0$. $\phi(t)$ is said to tend $(C, 0)$ to l if it tends to l in the ordinary sense.

The consistency theorem for mean continuity is as follows.

THEOREM 7. *If $\phi(t) \rightarrow l$ (C, α) , when $t \rightarrow 0$, then $\phi(t) \rightarrow l$ $(C, \alpha + \beta)$, when $t \rightarrow 0$, where $\alpha \geq 0$ and $\beta > 0$.*

Proof. Since the theorem is true if $\phi(t)$ is a constant, we need only consider the case in which $l = 0$. We may also suppose that $t > 0$. Then, if $\alpha > 0$, it follows from the hypothesis that

$$\phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-v)^{\alpha-1} \phi(v) dv = o(t^\alpha).$$

Therefore, by Theorem 2,

$$\begin{aligned} \phi_{\alpha+\beta}(t) &= \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-v)^{\alpha+\beta-1} \phi(v) dv = \frac{1}{\Gamma(\beta)} \int_0^t (t-v)^{\beta-1} \phi_\alpha(v) dv \\ &= \int_0^t (t-v)^{\beta-1} o(v^\alpha) dv = o(t^{\alpha+\beta}). \end{aligned}$$

If $\alpha = 0$, we write $\phi_0(t) = \phi(t) = o(1)$ and argue as before. This completes the proof.

ON THE SUMMABILITY OF FOURIER SERIES

By L. S. BOSANQUET.

[Part I: Received 12 June, 1928. Read 14 June, 1928.—Part II: Received 15 September, 1928.
Read 8 November, 1928. Revised 7 December, 1929†.]

1. *Introduction.*

The object of this paper is to extend some of the results of Hardy and Littlewood concerning the summability of Fourier series. They have proved‡ that the necessary and sufficient condition that the Fourier series of a periodic function $f(t)$, integrable in the Cauchy-Lebesgue sense, should be summable for $t = x$, to sum s , by some Cesàro mean, is that there is an integer κ such that

$$\frac{1}{t} \int_0^t \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \dots \int_0^{t_{\kappa-1}} \phi(t_\kappa) dt_\kappa \rightarrow 0$$

when $t \rightarrow 0$, where

$$\phi(t) = \frac{1}{2} \{ f(x+t) + f(x-t) - 2s \}.$$

This condition may be conveniently expressed by saying§ that $\phi(t) \rightarrow 0$ (C, κ) when $t \rightarrow 0$. The condition is also equivalent to

$$\frac{\kappa}{t^\kappa} \int_0^t (t-u)^{\kappa-1} \phi(u) du \rightarrow 0$$

when $t \rightarrow +0$, a form which is easily extensible to non-integral values of κ . More precisely, they proved, on the one hand, that if $\phi(t) \rightarrow 0$ (C, r) when $t \rightarrow 0$, where r is an integer, then the Fourier series is summable ($C, r+1$) for $t = x$, to sum s , and, on the other hand, that if the Fourier series is summable (C, r) for $t = x$, to sum s , then $\phi(t) \rightarrow 0$ ($C, r+2$).

† Two papers, with the titles "On the summability of Fourier series and Denjoy-Fourier series", I and II, have been incorporated in this paper.

‡ Hardy and Littlewood, 8.

§ $\phi(t)$ is also said to be continuous (C, κ) for $t = 0$.

Hardy and Littlewood proved later† that if $f(t)$ is integrable in the Lebesgue sense and $\phi(t) \rightarrow 0$ (C, a), then the Fourier series is summable ($C, a + \delta$) for $t = x$, to sum s , where $0 < a < 1$ and δ is any positive number. This theorem was previously known‡ for the cases in which $a = 0$ § and $a = 1$, and in § 2 of this paper it is shown to be true for all non-negative values of a . They also proved | that if the Fourier series is summable ($C, -\gamma$) for $t = x$, to sum s , where $0 < \gamma < 1$, then $\phi(t) \rightarrow 0$ ($C, 1$), and in § 3 it is proved further that if the Fourier series is summable (C, a) for $t = x$, to sum s , where $a \geq -1$, then $\phi(t) \rightarrow 0$ ($C, a + 1 + \delta$), where δ is any positive number. The first of these theorems remains true, provided that $a + \delta \geq 1$, when general Denjoy integrals are used, and this result cannot be strengthened even if the integrals are restricted to be Cauchy-Lebesgue integrals. The second theorem also remains true if the integrals are general Denjoy integrals, and this theorem is included in a more general theorem on series.

There is an analogous theory of summability for a certain class of Fourier integrals, and the corresponding theorems are given.

The problem of showing that these are "best possible" theorems, in the sense that δ cannot be replaced by zero, is one for further consideration. It has been shown that there are continuous functions whose Fourier series diverge at certain points¶, that there are functions which are continuous ($C, 1$) at points at which their Fourier series are not summable ($C, 1$)††, and that there are Fourier series which are convergent at points at which $\phi(t)$ is not continuous ($C, 1$)‡‡. There is, therefore, every reason to expect further results of a similar kind.

The general Denjoy integral §§. The general Denjoy integral is that described by Hobson as the *DKY*-integral. If $f(t)$ is integrable in the general Denjoy sense in (a, b) , then $F(x) = \int_a^x f(t) dt$ is continuous in (a, b) , and, if $\phi(t)$ is a function of bounded variation in (a, b) ,

$$\int_a^b f(t) \phi(t) dt = [F(t) \phi(t)]_a^b - \int_a^b F(t) d\phi(t).$$

The mean value theorems are also valid.

† Hardy and Littlewood, 10.

‡ See Hobson, 13, 2, 567, where other references are given, and Young, 18, 268.

§ I.e., when $\phi(t) \rightarrow 0$ in the ordinary sense.

|| Hardy and Littlewood, 11, 305.

¶ See Hobson, 13, 2, 539, where other references are given.

†† Hahn, 4, 359.

‡‡ Hardy and Littlewood, 6 and 7.

§§ See *Encyk. Math. Wiss.*, II 3, 7 (1924), 35c, 1965–1071, or Hobson, 13, 1, 653–661.

Fractional integrals and mean continuity. Let

$$\phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \phi(u) du,$$

the integral being a general Denjoy integral. Then $\phi_\alpha(t)$ is the fractional integral of $\phi(t)$ of order α . This notation is used throughout.

Suppose that

$$\alpha \int_0^1 (1-v)^{\alpha-1} \phi(tv) dv \rightarrow l$$

when $t \rightarrow 0$, where $\alpha > 0$ and the integral exists as a general Denjoy integral for all sufficiently small values of t . Then $\phi(t)$ is said to tend (C, α) to l when $t \rightarrow 0$ and to be continuous (C, α) for $t = 0$. If $\phi(t) \rightarrow l$ when $t \rightarrow 0$, $\phi(t)$ is said to tend $(C, 0)$ to l when $t \rightarrow 0$.

The consistency theorems for fractional integrals and mean continuity have been investigated in another paper†.

Fourier integrals of finite type. Suppose that $f(t)$ is defined and integrable in the general Denjoy sense in the interval $(-\pi, \pi)$, and let

$$a_r = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos rt dt, \quad b_r = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin rt dt.$$

Then
$$\int_0^\infty (a_r \cos rt + b_r \sin rt) dr$$

is called the Fourier integral of $f(t)$ of finite type.‡

Certain well known functions are used in the sequel. Let

$$\gamma_q(t) = \int_0^1 (1-v)^{q-1} \cos tv dv \quad (q > 0).$$

Then, for $t > 0$, $\gamma_q(t) = t^{-q} \Gamma(q) C_q(t)$, where $C_q(t)$ is Young's function§. It is known|| that, for large positive values of t ,

$$\gamma_q(t) = \frac{A}{t^2} + O\left(\frac{1}{t^3}\right) + \frac{B \cos(t - \frac{1}{2}q\pi)}{t^q} + O\left(\frac{1}{t^{q+1}}\right),$$

where A and B are constants, and asymptotic formulae for the derivatives

† Bosanquet, 2.

‡ Pollard, 14.

§ Young, 17. See also Hobson, 2, 564. $C_q(t)$ is the fractional integral of $\cos t$ of order q .

|| See Hardy and Littlewood, 9.

of $\gamma_q(t)$ are obtained by formal differentiation of this formula. It follows from the definition that $\gamma_q(t)$ and its derivatives are bounded functions[†].

2.

In this section the following theorems are proved.

THEOREM 1. *If $f(t)$ is integrable in the Lebesgue sense in the interval $(-\pi, \pi)$ and is periodic with period 2π , and if*

$$\phi(t) = \frac{1}{2} \{f(x+t) + f(x-t) - 2s\} \rightarrow 0 \quad (C, \alpha)$$

when $t \rightarrow 0$, then the Fourier series of $f(t)$ is summable (C, β) for $t = x$, to sum s , where $\beta > \alpha$ and $\alpha \geq 0$.

THEOREM 2. *If $f(t)$ is integrable in the general Denjoy sense in the interval $(-\pi, \pi)$ and is periodic with period 2π , and if*

$$\phi(t) = \frac{1}{2} \{f(x+t) + f(x-t) - 2s\} \rightarrow 0 \quad (C, \alpha)$$

when $t \rightarrow 0$, then the Fourier series of $f(t)$ is summable (C, β) for $t = x$, to sum s , where $\beta > \alpha$ if $\alpha \geq 1$ and $\beta \geq 1$ if $0 \leq \alpha < 1$.

THEOREM 3. *If $f(t)$ satisfies the conditions of Theorems 1 or 2 in the interval $(-\pi, \pi)$, then its Fourier integral of finite type is summable (C, β) for $t = x$, to sum s , where $\beta > \alpha$ and $\alpha \geq 0$.*

It may be shown by examples[‡] that the restriction $\beta \geq 1$ is essential to the truth of Theorem 2, in the sense that the number 1 cannot be replaced by any smaller number. It will be seen that this is due to the possible behaviour of the function at points other than those in the neighbourhood of $t = x$.

No restriction, such as $\beta \geq 1$, is necessary in Theorem 3, for the behaviour of a Fourier integral of finite type depends only on the behaviour of the function in the neighbourhood of the point under consideration.

Two proofs of Theorem 1 are given. The second is a direct extension of the proof given by Hardy and Littlewood[§] for the case $0 < \alpha < 1$, and is not given in full detail. The first proof is based on an alternative (unpublished) proof for that case by Hardy, and is of a more direct nature.

In the proofs of Theorems 1-3 the following lemmas are required.

[†] See also Pollard, 17 and 18.

[‡] Titchmarsh, 16.

[§] Hardy and Littlewood, 10.

LEMMA 1. If $\phi(t)$ is integrable in the general Denjoy sense in an interval $(0, \delta)$ and if $\phi(t) \rightarrow 0$ (C, α) when $t \rightarrow +0$, then $\phi_{h+1}(t) = o(t^{h+1})$ when $t \rightarrow +0$, where h is the greatest integer not greater than α .

This is simply a consequence of the consistency theorem for mean continuity†.

LEMMA 2. If $\phi(t)$ satisfies the conditions of Lemma 1, and $0 < \xi < \eta \leq 2\xi$, then

$$\phi_{h+1}(\eta) - \phi_{h+1}(\xi) = o\{\xi^\alpha(\eta - \xi)^{1+h-\alpha}\},$$

when $\xi \rightarrow 0$.

Proof. We have, by the consistency theorem for fractional integrals,

$$\begin{aligned} \phi_{h+1}(\eta) - \phi_{h+1}(\xi) &= \frac{1}{\Gamma(1+h-\alpha)} \int_0^\eta (\eta-u)^{h-\alpha} \phi_\alpha(u) du \\ &\quad - \frac{1}{\Gamma(1+h-\alpha)} \int_0^\xi (\xi-u)^{h-\alpha} \phi_\alpha(u) du \\ &= \frac{1}{\Gamma(1+h-\alpha)} \int_\xi^\eta (\eta-u)^{h-\alpha} \phi_\alpha(u) du \\ &\quad - \frac{1}{\Gamma(1+h-\alpha)} \int_0^\xi \{(\xi-u)^{h-\alpha} - (\eta-u)^{h-\alpha}\} \phi_\alpha(u) du, \end{aligned}$$

and the first term is

$$o\left\{\xi^\alpha \int_\xi^\eta (\eta-u)^{h-\alpha} du\right\} = o\{\xi^\alpha(\eta - \xi)^{1+h-\alpha}\},$$

while the second is

$$\begin{aligned} o\left[\xi^\alpha \int_0^\xi \{(\xi-u)^{h-\alpha} - (\eta-u)^{h-\alpha}\} du\right] &= o\left[\xi^\alpha \{\xi^{1+h-\alpha} - \eta^{1+h-\alpha} + (\eta-\xi)^{1+h-\alpha}\}\right] \\ &= o\{\xi^\alpha(\eta - \xi)^{1+h-\alpha}\}, \end{aligned}$$

for $(\eta^\kappa - \xi^\kappa)/(\eta - \xi)^\kappa$ is bounded when $0 < \xi < \eta \leq 2\xi$, uniformly in ξ , where $0 < \kappa \leq 1$.

First proof of Theorem 1. Making the standard simplifications, we may suppose, without loss of generality, that $x = 0$, $s = 0$, and that $f(t)$ is an even function whose Fourier series has no constant term, so that $\phi(t) = f(t)$ and $\phi_1(t)$ is itself periodic with period 2π . We suppose then that

$$f(t) = \phi(t) \sim \sum_{n=1}^{\infty} a_n \cos nt.$$

† See Bosanquet, 2, 143.

We may also suppose without loss of generality that $\beta < h+1$, where h is the greatest integer not greater than α . The proof for larger values of β is similar.

Now the β -th Rieszian mean of the Fourier series, for $t = 0$, is

$$\frac{2\omega}{\pi} \int_0^\infty \gamma_{1+\beta}(\omega u) \phi(u) du^\dagger.$$

Writing

$$J(\omega) = \omega \int_0^\infty \gamma_{1+\beta}(\omega u) \phi(u) du = \omega \int_0^\delta + \omega \int_\delta^\infty = J^* + J^{**},$$

where δ is a positive number, we have to show that $J(\omega) \rightarrow 0$ when $\omega \rightarrow \infty$.

Let ϵ be an arbitrary positive number. Then, by Lemma 2, δ may be chosen such that

$$\left| \int_\xi^\eta \phi_h(u) du \right| < \epsilon \{ \xi^\alpha (\eta - \xi)^{1+h-\alpha} \},$$

provided that $0 < \xi < \eta \leq 2\xi < \delta$.

Let $\kappa = \text{Min}(\alpha, 1)$. Then, since $\phi(u)$ is periodic, we have

$$\begin{aligned} |J^{**}| &= \left| \omega \int_\delta^\infty \gamma_{1+\beta}(\omega u) \phi(u) du \right| = \omega^{-\kappa} \int_\delta^\infty O(u^{-1-\kappa}) |\phi(u)| du \\ &= O(\omega^{-\kappa}) \{ \eta^{-1-\kappa} + 1^{-1-\kappa} + 2^{-1-\kappa} + \dots \} \int_{-\pi}^\pi |\phi(u)| du = O(\omega^{-\kappa}) \end{aligned}$$

when $\omega \rightarrow \infty$, δ remaining fixed.

Integrating J^* by parts h times, we have

$$\begin{aligned} J^* &= \omega \int_0^\delta \gamma_{1+\beta}(\omega u) \phi(u) du \\ &= \omega [\gamma_{1+\beta}(\omega u) \phi_1(u)]_0^\delta - \omega^2 \int_0^\delta \gamma'_{1+\beta}(\omega u) \phi_1(u) du \\ &= O(\omega^{-\kappa}) - \omega^2 \int_0^\delta \gamma'_{1+\beta}(\omega u) \phi_1(u) du \\ &\quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ &= O(\omega^{-\kappa}) + (-1)^h \omega^{h+1} \int_0^\delta \gamma_{1+\beta}^{(h)}(\omega u) \phi_h(u) du. \end{aligned}$$

† Hobson, **13**, 2, 567.

Now write

$$\omega^{h+1} \int_0^\delta \gamma_{1+\beta}^{(h)}(\omega u) \phi_h(u) du = \omega^{h+1} \int_0^{1/\omega} + \sum_{\nu=1}^{\tau-1} \omega^{h+1} \int_{\nu/\omega}^{(\nu+1)/\omega} + \omega^{h+1} \int_{\tau/\omega}^\delta,$$

where τ is the greatest integer not greater than $\omega\eta$.

Since $\beta < h+1$, it follows that $\gamma_{1+\beta}^{(h+1)}(t) = O(t^{-1-\beta})$ for large t , and hence, for $\nu \geq 1$, the total variation of $\gamma_{1+\beta}^{(h)}(t)$ in the interval $(\nu, \nu+1)$ is $\int_\nu^{\nu+1} O(t^{-1-\beta}) dt = \nu^{-1-\beta} O(1)$, uniformly in ν . Therefore, if $1 \leq \nu \leq \tau-1$, we have, by a double application of the second mean value theorem,

$$\begin{aligned} |J_\nu| &= \omega^{h+1} \left| \int_{\nu/\omega}^{(\nu+1)/\omega} \gamma_{1+\beta}^{(h)}(\omega u) \phi_h(u) du \right| \\ &< A \omega^{h+1} \nu^{-1-\beta} \int_{\lambda/\omega}^{\mu/\omega} \phi_h(u) du \quad (\text{where } \nu \leq \lambda < \mu \leq \nu+1) \\ &< \epsilon A \omega^{h+1} \nu^{-1-\beta} \{(\nu/\omega)^\alpha (1/\omega)^{1+h-\alpha}\} = \epsilon A \nu^{\alpha-\beta-1}, \end{aligned}$$

where A denotes some constant independent of ν . In the same way we can show that $|J_\tau| < \epsilon A \tau^{\alpha-\beta-1}$, for some A , and so

$$\sum_{\nu=1}^{\tau} |J_\nu| < \epsilon A \sum_{\nu=1}^{\tau} \nu^{\alpha-\beta-1} < \epsilon A \sum_{\nu=1}^{\infty} \nu^{\alpha-\beta-1} = \epsilon B,$$

where B is a constant.

Finally, since $\gamma_{1+\beta}^{(h)}(t)$ is of bounded variation in the interval $(0, 1)$, we have, by a double application of the second mean value theorem,

$$\begin{aligned} J_0 &= \omega^{h+1} \int_0^{1/\omega} \gamma_{1+\beta}^{(h)}(\omega u) \phi_h(u) du \\ &= \omega^{h+1} O(1) \int_\lambda^\mu \phi_h(u) du \quad (\text{where } 0 \leq \lambda < \mu \leq 1/\omega) \\ &= \omega^{h+1} O(1) o\{(1/\omega)^{h+1}\} \quad (\text{by Lemma 1}) \\ &= o(1) \end{aligned}$$

when $\omega \rightarrow \infty$.

Thus $|J(\omega)|$ may be made arbitrarily small by choice of ω sufficiently large, and therefore $J(\omega) \rightarrow 0$ when $\omega \rightarrow \infty$. This completes the proof.

Proof of Theorem 2. With the notation and simplifications of Theorem 1, the β -th Rieszian mean of the Fourier series is given by the same expression.†

† Pollard, 15

Now it has been shown[†] that $J^{**} \rightarrow 0$, when $\omega \rightarrow \infty$, whenever $\beta \geq 1$, and that the number 1 cannot be replaced by any smaller number. The rest of the proof is the same as that of Theorem 1.

Proof of Theorem 3. It has been shown[‡] that the necessary and sufficient condition that the Fourier integral should be summable (C, β) to sum s , for $t = x$, is

$$\lim_{\delta \rightarrow 0} \overline{\lim}_{\omega \rightarrow \infty} \left| \omega \int_0^\delta \gamma_{1+\beta}(\omega u) \phi(u) du \right| = 0.$$

The proof is thus reduced to showing that δ can be chosen such that $|J^*| < \epsilon$ provided that ω be sufficiently large, where ϵ is an arbitrary positive number and $\beta > \alpha$. The rest of the proof is therefore a repetition of the argument of Theorem 1.

Second proof of Theorem 1. Let the same preliminary simplifications be made and suppose that $0 < \alpha < \beta < \beta + \delta < h + 1$. Then, following Hardy and Littlewood[§], we have

$$\begin{aligned} J_{\beta+\delta}(\phi, \omega) &= \omega^{-\beta-\delta} \int_0^\infty u^{-1-\beta-\delta} C_{1+\beta+\delta}(\omega u) \phi(u) du \\ &= \frac{\omega^{-\delta}}{\Gamma(\beta)} \int_0^\infty u^{-1-\beta-\delta} \phi(u) du \int_0^u C_{1+\delta}(\omega v) (u-v)^{\beta-1} dv \\ &= \frac{\omega^{-\delta}}{\Gamma(\beta)} \int_0^\infty C_{1+\delta}(\omega v) dv \int_v^\infty u^{-1-\beta-\delta} (u-v)^{\beta-1} \phi(u) du \\ &= \frac{\omega^{-\delta}}{\Gamma(\beta)} \int_0^\infty v^{-1-\delta} C_{1+\delta}(\omega v) \psi(v) dv = \frac{1}{\Gamma(\beta)} J_\delta(\psi, \omega), \end{aligned}$$

where
$$\psi(v) = v^{1+\delta} \int_v^\infty u^{-1-\beta-\delta} (u-v)^{\beta-1} \phi(u) du.$$

If now it be proved that $\psi(v) \rightarrow 0$ when $v \rightarrow 0$, whenever $\beta > \alpha$ and $\delta > 0$, the theorem is reduced, apart from trivialities^{||}, to the case in which $\alpha = 0$ [¶].

[†] Pollard, **15**. See also Titchmarsh, **16**.

[‡] Pollard, **14**, and Grimshaw, **3**.

[§] Hardy and Littlewood, **10**.

^{||} *Ibid.*, 682.

[¶] See foot-note §, p. 145.

By h integrations by parts, we have

$$\begin{aligned}\psi(v) &= v^{1+\delta} \int_v^\infty u^{-1-\beta-\delta} (u-v)^{\beta-1} \phi(u) du \\ &= Av^{1+\delta} \int_v^\infty \phi_h(u) \frac{d^h}{du^h} \{u^{-1-\beta-\delta} (u-v)^{\beta-1}\} du \\ &= Av^{1+\delta} \int_v^{2v} + Av^{1+\delta} \int_{2v}^\infty = \psi_1(v) + \psi_2(v),\end{aligned}$$

where A is a constant.

To show that $\psi_2(v) \rightarrow 0$, when $v \rightarrow 0$, we have, integrating by parts and applying Lemma 1,

$$\begin{aligned}\int_{2v}^\infty \phi_h(u) \frac{d^h}{du^h} \{u^{-1-\beta-\delta} (u-v)^{\beta-1}\} du \\ &= \left[\phi_{h+1}(u) \frac{d^h}{du^h} \{u^{-1-\beta-\delta} (u-v)^{\beta-1}\} \right]_{2v}^\infty \\ &\quad - \int_{2v}^\infty \phi_{h+1}(u) \frac{d^{h+1}}{du^{h+1}} \{u^{-1-\beta-\delta} (u-v)^{\beta-1}\} du \\ &= \left[\phi_{h+1}(u) \sum_{s=0}^h O \{u^{-1-\beta-\delta-s} (u-v)^{\beta-1-h+s}\} \right]_{2v}^\infty \\ &\quad + \int_{2v}^1 o(u^{h+1}) \sum_{s=0}^{h+1} O \{u^{-1-\beta-\delta-s} (u-v)^{\beta-2-h+s}\} du + \int_1^\infty O(u^h) O(u^{-\beta-\delta-h}) du \\ &= o(v^{h+1}) O(v^{-h-\delta-2}) + o(v^{-1-\delta}) + O(1) = o(v^{-1-\delta}).\end{aligned}$$

Therefore $\psi_2(v) = o(1)$. To show that $\psi_1(v) \rightarrow 0$, when $v \rightarrow 0$, we have, integrating by parts and applying Lemma 2,

$$\begin{aligned}\int_v^{2v} \phi_h(u) \frac{d^h}{du^h} \{u^{-1-\beta-\delta} (u-v)^{\beta-1}\} du \\ &= \left[\frac{d^h}{du^h} \{u^{-1-\beta-\delta} (u-v)^{\beta-1}\} \int_v^u \phi_h(t) dt \right]_v^{2v} \\ &\quad - \int_v^{2v} \left[\frac{d^{h+1}}{du^{h+1}} \{u^{-1-\beta-\delta} (u-v)^{\beta-1}\} \int_v^u \phi_h(t) dt \right] du \\ &= \left[\sum_{s=0}^h O \{u^{-1-\beta-\delta-s} (u-v)^{\beta-1-h+s}\} o \{v^\alpha (u-v)^{1+h-\alpha}\} \right]_v^{2v} \\ &\quad + \int_v^{2v} \sum_{s=0}^{h+1} O \{u^{-1-\beta-\delta-s} (u-v)^{\beta-2-h+s}\} o \{v^\alpha (u-v)^{1+h-\alpha}\} du \\ &= o(v^{-1-\delta}) + o(v^{-1-\delta}) = o(v^{-1-\delta}).\end{aligned}$$

Therefore $\psi_1(v) = o(1)$. This completes the proof.

3.

In this section the reciprocal problem is considered and the following theorems are obtained.

THEOREM 4. *If $f(t)$ is integrable in the general Denjoy sense in the interval $(-\pi, \pi)$ and is periodic with period 2π , and if its Fourier series is summable (C, α) to sum s , for $t = x$, then*

$$\phi(t) = \frac{1}{2} \{f(x+t) + f(x-t) - 2s\} \rightarrow 0 \quad (C, \beta)$$

when $t \rightarrow 0$, where $\beta > \alpha + 1$ and $\alpha \geq -1$.

This is a corollary of the following theorem.

THEOREM 5. *If $\sum_{n=1}^{\infty} a_n$ is summable (C, α) to sum s and $\sum_{n=1}^m a_n = o(m^2)$ or, what is the same thing, if $\sum_{n=1}^{\infty} \frac{a_n}{n^2}$ is convergent, then $\beta \sum_{n=1}^{\infty} a_n \gamma_{\beta}(nt) \rightarrow s$ when $t \rightarrow 0$, where $\beta > \alpha + 1$ and $\alpha \geq -1$.*

The analogous theorems for integrals are

THEOREM 6. *If $f(t)$ is integrable in the general Denjoy sense in the interval $(-\pi, \pi)$ and is periodic with period 2π , and if its Fourier integral of finite type is summable (C, α) to s , for $t = x$, then*

$$\phi(t) = \frac{1}{2} \{f(x+t) + f(x-t) - 2s\} \rightarrow 0 \quad (C, \beta)$$

when $t \rightarrow 0$, where $\beta > \alpha + 1$ and $\alpha \geq -1$.

THEOREM 7. *If $a(x)$ is integrable in the general Denjoy sense over every finite interval $(0, \xi)^{\dagger}$, and $\int_0^{\infty} a(x) dx$ is summable (C, α) to s , and $\int_0^{\omega} a(x) dx = o(\omega^2)$ or, what is the same thing, if $\int_0^{\infty} \frac{a(x)}{(1+x)^2} dx$ is convergent, then $\beta \int_0^{\infty} a(x) \gamma_{\beta}(xt) dx \rightarrow s$, when $t \rightarrow 0$, where $\beta > \alpha + 1$ and $\alpha \geq -1$.*

The equivalence of the alternate hypotheses in Theorems 5 and 7 follows from the next two lemmas.

LEMMA 3. *If $\sum_{n=1}^{\infty} a_n$ is summable (C, α) where $\alpha \geq -1$, then a necessary and sufficient condition that the series $\sum_{n=1}^{\infty} \frac{a_n}{n^2}$ should be convergent is that $\sum_{n=1}^m a_n = o(m^2)$.*

[†] The behaviour of $a(x)$ for finite values of x is not the important hypothesis.

This is a consequence of known theorems†. It may also be proved directly by partial summation and application of Lemma 5.

LEMMA 4. If $\int_0^\infty a(x)dx$ is summable (C, a) where $a \geq -1$, then a necessary and sufficient condition that $\int_0^\infty \frac{a(x)}{(1+x)^2} dx$ be convergent is $\int_0^\omega a(x)dx = o(\omega^2)$.

This is the analogue for integrals of Lemma 3 and may be proved by integration by parts.

The order of procedure is first to reduce Theorems 4 and 6 to dependence on Theorems 5 and 7 respectively, to prove Theorem 7 in full, and then go through the steps of the proof of Theorem 5, replacing integrals by series and partial integrations by partial summations. The proof of Theorem 5 is similar to that of the theorem of Hardy and Littlewood which it includes‡.

Writing $D_x^r f(x) = \frac{d^r}{dx^r} f(x)$, $\Delta_n f(n) = f(n) - f(n+1)$,

and $\Delta_n^{r+1} f(n) = \Delta_n^r f(n) - \Delta_n^r f(n+1)$,

we have the following lemma, which is used in the process of transition from integrals to series.

LEMMA 5. If $|D_x^r f(x)| \leq M$ throughout the interval $(n, n+r)$, then $|\Delta_n^r f(n)| \leq M$.

Proof. We have

$$\begin{aligned} \Delta_n f(n) &= f(n) - f(n+1) = \int_{n+1}^n D_x f(x) dx, \\ \Delta_n^2 f(n) &= \int_{n+1}^n D_x f(x) dx - \int_{n+2}^{n+1} D_x f(x) dx = \int_{n+1}^n D_x \{f(x) - f(x+1)\} dx \\ &= \int_{n+1}^n dx_1 \int_{x_1+1}^{x_1} D_{x_2}^2 f(x_2) dx_2, \\ &\dots \dots \dots \dots \dots \dots \dots \dots \dots \\ \Delta_n^r f(n) &= \int_{n+1}^n dx_1 \int_{x_1+1}^{x_1} dx_2 \dots \int_{x_{r-1}+1}^{x_{r-1}} D_{x_r}^r f(x_r) dx_r. \end{aligned}$$

† Hardy and Littlewood, **5**, Theorem 13, 433, and Theorem 14, 435; together with Hardy and Riesz, **12**, Theorem 48, 57.

‡ See footnote ||, p. 145.

Now x_r lies in the interval $(n, n+r)$ and therefore

$$|\Delta_n^r f(n)| \leq \int_{n+1}^n dx_1 \int_{x_1+1}^{x_1} dx_2 \dots \int_{x_{r-1}+1}^{x_{r-1}} M dx_r = M.$$

The lemma is therefore proved.

LEMMA 6. *If $f(t)$ is integrable in the general Denjoy sense in the interval $(-\pi, \pi)$ and is periodic with period 2π , and if its Fourier series has no constant term, then the series obtained by integrating its Fourier series term by term from 0 to x is the Fourier series of $F(x) = \int_0^x f(t) dt$.*

Proof[†]. We have, integrating by parts,

$$\frac{a_n}{n} = \frac{1}{n\pi} \int_{-\pi}^{\pi} f(t) \cos nt = \frac{1}{\pi} \int_{-\pi}^{\pi} F(t) \sin nt dt$$

and
$$-\frac{b_n}{n} = -\frac{1}{n\pi} \int_{-\pi}^{\pi} f(t) \sin nt = \frac{1}{\pi} \int_{-\pi}^{\pi} F(t) \cos nt dt.$$

It follows that the integrated series is the Fourier series of $F(x)$.

Reduction of Theorem 4. We may suppose as usual, without loss of generality, that $x = 0$, $s = 0$, and that $f(t)$ is an even function whose Fourier series has no constant term. We may also restrict ourselves to positive values of t .

Supposing then that $f(t) \sim \Sigma a_n \cos nt$, we have, by Lemma 6, $\phi_1(t) \sim \Sigma a_n n^{-1} \sin nt$. Now $\phi_1(t)$ is continuous and periodic with period 2π , and therefore, by Fejér's theorem[‡], its Fourier series is summable $(C, 1)$ to $\phi_1(t)$ uniformly in any interval. Such a series may be multiplied by an absolutely integrable function and integrated term by term, provided that the series so obtained be convergent. We have then, if $\beta > 1$,

$$\begin{aligned} \phi_\beta(t) &= \frac{1}{\Gamma(\beta-1)} \int_0^t (t-u)^{\beta-2} \phi_1(u) du \\ &= \frac{1}{\Gamma(\beta-1)} \sum_{n=1}^{\infty} \frac{a_n}{n} \int_0^t (t-u)^{\beta-2} C_1(nu) du = \sum_{n=1}^{\infty} \frac{a_n}{n^\beta} C_\beta(nt), \end{aligned}$$

provided that this series is convergent. Therefore

$$\beta \int_0^1 (1-v)^{\beta-1} \phi(tv) dv = \frac{\Gamma(\beta+1) \phi_\beta(t)}{t^\beta} = \beta \sum_{n=1}^{\infty} a_n \gamma_\beta(nt),$$

provided that this series is convergent.

[†] For a suggestion concerning the lemma and for this proof I am indebted to the referee.

[‡] See Hobson, 13, 2, 560.

Now if $-1 \leq a < 1$ and $a+1 < \beta \leq 2$, we have

$$a_n \gamma_\beta(nt) = o(n^a) O\{(nt)^{-\beta}\},$$

and therefore the series is convergent. Further, if $a \geq -1$ and $\beta > 2$, we have $a_n = o(n)$ and

$$\gamma_\beta(nt) = \frac{A}{n^2 t^2} + O\left(\frac{1}{(nt)^3}\right) + O\left(\frac{1}{(nt)^\beta}\right),$$

and therefore, by Lemma 3, the series is convergent.

Next take the case in which $\beta \leq 1$, and suppose that $-1 \leq a < 0$, $a+1 < \beta \leq 1$. Let β' be chosen such that $a+1 < \beta' < \beta$. Then, since $C_{\beta'}(nt)$ is bounded, the series $\sum_{n=1}^{\infty} a_n n^{-\beta'} C_{\beta'}(nt)$ is absolutely and uniformly convergent. Therefore

$$\begin{aligned} \int_0^t \sum_{n=1}^{\infty} \frac{a_n}{n^{\beta'}} C_{\beta'}(nu) du &= \sum_{n=1}^{\infty} \frac{a_n}{n^{\beta'+1}} C_{\beta'+1}(nt) \\ &= \frac{1}{\Gamma(\beta')} \sum_{n=1}^{\infty} \frac{a_n}{n} \int_0^t (t-u)^{\beta'-1} C_1(nu) du \\ &= \frac{1}{\Gamma(\beta')} \int_0^t (t-u)^{\beta'-1} \phi_1(u) du = \int_0^t \phi_{\beta'}(u) du. \end{aligned}$$

Therefore $\phi_{\beta'}(t)$, which we know to exist for almost all values of t †, is almost always equal to $\sum a_n n^{-\beta'} C_{\beta'}(nt)$, and hence

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{a_n}{n^\beta} \gamma_\beta(nt) &= \frac{1}{\Gamma(\beta-\beta')} \int_0^t (t-u)^{\beta-\beta'-1} \sum_{n=1}^{\infty} \frac{a_n}{n^{\beta'}} C_{\beta'}(nu) du \\ &= \frac{1}{\Gamma(\beta-\beta')} \int_0^t (t-u)^{\beta-\beta'-1} \phi_{\beta'}(u) du = \phi_\beta(t). \end{aligned}$$

Thus $\phi_\beta(t)$ exists for all values of t and is equal to $\sum a_n n^{-\beta} C_\beta(nt)$, and so, as above,

$$\beta \int_0^1 (1-v)^{\beta-1} \phi(v) dv = \beta \sum_{n=1}^{\infty} a_n \gamma_\beta(nt).$$

Theorem 4 is therefore reduced to showing that $\beta \sum a_n \gamma_\beta(nt) \rightarrow 0$ when $t \rightarrow 0$.

In the same way Theorem 6 may be reduced to dependence on Theorem 7.

† See Bosanquet, 2.

Proof of Theorem 7. There is no loss of generality in supposing that $s = 0$. For $\lim_{t \rightarrow 0} \beta \gamma_\beta(t) = 1$ and therefore, if $a(x)$ be replaced by $a(x) - s$ in the interval $(0, 1)$, s is replaced by 0. We may also suppose that $\beta \leq h+2$, where h is the greatest integer not greater than a , the proof for larger values of β being similar. We may restrict ourselves to positive values of t .

Suppose first that $a > -1$. Then, writing

$$s^a(\omega) = \frac{1}{\Gamma(a+1)} \int_0^\omega (\omega-x)^a a(x) dx,$$

we have, by the consistency theorem for fractional integrals,

$$s^{h+1}(\omega) = \frac{1}{\Gamma(1+h-a)} \int_0^\omega (\omega-x)^{h-a} s^a(x) dx.$$

Next, by $h+2$ integrations by parts, we have

$$\begin{aligned} \int_0^\omega a(x) \gamma_\beta(xt) dx &= [s^0(x) \gamma_\beta(xt)]_0^\omega - \int_0^\omega s^0(x) D_x \gamma_\beta(xt) dx \\ &\quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ &= s^0(\omega) \gamma_\beta(\omega t) - s^1(\omega) D_\omega \gamma_\beta(\omega t) + \dots + (-1)^{h+1} s^{h+1}(\omega) D_\omega^{h+1} \gamma_\beta(\omega t) \\ &\quad + (-1)^{h+2} \int_0^\omega s^{h+1}(x) D_x^{h+2} \gamma_\beta(xt) dx. \end{aligned}$$

Now, by hypothesis,

$$\begin{aligned} s^r(\omega) &= o(\omega^{r+2}) & (0 \leq r \leq \alpha-2), \\ &= o(\omega^\alpha) & (\alpha-2 \leq r \leq \alpha), \\ &= o(\omega^r) & (r \geq \alpha), \end{aligned}$$

and

$$\begin{aligned} \gamma_\beta^{(r)}(\omega) &= O(\omega^{-r-2}) & (0 \leq r \leq \beta-2), \\ &= O(\omega^{-\beta}) & (\beta-2 \leq r \leq \beta). \end{aligned}$$

Therefore, since, by hypothesis, $\beta-2 \leq h \leq \alpha < h+1 < \beta \leq h+2$,

$$\begin{aligned} s^r(\omega) D_\omega^r \gamma_\beta(\omega t) &= o(\omega^{r+2}) O(\omega^{-r-2}) = o(1) & (0 \leq r \leq \alpha-2), \\ &= o(\omega^\alpha) O(\omega^{-r-2}) = o(1) & (\alpha-2 \leq r \leq \beta-2), \\ &= o(\omega^\alpha) O(\omega^{-\beta}) = o(1) & (\beta-2 \leq r \leq \alpha), \\ &= o(\omega^r) O(\omega^{-\beta}) = o(1) & (\alpha \leq r \leq \beta). \end{aligned}$$

and so

$$\int_0^\omega a(x) \gamma_\beta(xt) dx = o(1) + (-1)^{h+1} \int_0^\omega s^{h+1}(x) D_x^{h+2} \gamma_\beta(xt) dx.$$

Next we have

$$\begin{aligned} \int_0^\omega s^{h+1}(x) D_x^{h+2} \gamma_\beta(xt) dt &= \frac{1}{\Gamma(1+h-a)} \int_0^\omega D_x^{h+2} \gamma_\beta(xt) dx \int_0^x (x-u)^{h-a} s^a(u) du \\ &= \frac{1}{\Gamma(1+h-a)} \int_0^\omega s^a(u) du \int_u^\omega (x-u)^{h-a} D_x^{h+2} \gamma_\beta(xt) dx, \end{aligned}$$

and therefore

$$\int_0^\omega a(x) \gamma_\beta(xt) dx = \frac{(-1)^{h+1}}{\Gamma(1+h-a)} \int_0^\omega s^a(u) du \int_u^\omega (x-u)^{h-a} D_x^{h+2} \gamma_\beta(xt) dx,$$

provided $\int_0^\omega s^a(u) du \int_\omega^\infty (x-u)^{h-a} D_x^{h+2} \gamma_\beta(xt) dx \rightarrow 0$

when $\omega \rightarrow \infty$.

Now, for a fixed t , we have, by the second mean value theorem,

$$\begin{aligned} \left| \int_\omega^\infty (x-u)^{h-a} D_x^{h+2} \gamma_\beta(xt) dx \right| &\leq (\omega-u)^{h-a} \operatorname{Max}_{\omega' > \omega} \left| \int_\omega^{\omega'} D_x^{h+2} \gamma_\beta(xt) dx \right| \\ &= (\omega-u)^{h-a} O(\omega^{-\beta}), \end{aligned}$$

and

$$\omega^{-\beta} \int_0^\omega |s^a(u)| (\omega-u)^{h-a} du = \omega^{-\beta} \int_0^\omega o(u^a) (\omega-u)^{h-a} du = O(\omega^{1+h-\beta}) = o(1),$$

and so our assertion is justified.

Writing

$$\begin{aligned} J(t) &= \int_0^\infty s^a(u) du \int_u^\infty (x-u)^{h-a} D_x^{h+2} \gamma_\beta(xt) dx \\ &= t^{h+2} \int_0^\infty s^a(u) u^{1+h-a} du \int_1^\infty (y-1)^{h-a} \gamma_\beta^{(h+2)}(yut) dy \\ &= t^{h+2} \int_0^\infty s^a(u) u^{1+h-a} I(ut) du, \end{aligned}$$

where

$$I(z) = \int_1^\infty (y-1)^{h-a} \gamma_\beta^{(h+2)}(yz) dy,$$

we have then to show that $J(t) \rightarrow 0$ when $t \rightarrow +0$.

This will be proved if we prove that

$$I(z) = O\{z^{-(1+h-a)}\} \quad (0 < z \leq 1),$$

$$= O\{z^{-\beta-(1+h-a)}\} \quad (z \geq 1).$$

For, writing

$$J(t) = t^{h+2} \int_0^\infty s^a(u) u^{1+h-a} I(ut) du = t^{h+2} \int_0^{1/t} + t^{h+2} \int_{1/t}^\infty = J_1 + J_2,$$

we have

$$J_1 = t^{h+2-(1+h-a)} \int_0^{1/t} o(u^a) u^{1+h-a} O\{u^{-(1+h-a)}\} du = t^{a+1} \int_0^{1/t} o(u^a) du = o(1)$$

when $t \rightarrow +0$, while

$$\begin{aligned} J_2 &= t^{h+2-\beta-(1+h-a)} \int_{1/t}^\infty o(u^a) u^{1+h-a} O\{u^{-\beta-(1+h-a)}\} du \\ &= t^{1+a-\beta} \int_{1/t}^\infty o(u^{a-\beta}) du = o(1) \end{aligned}$$

when $t \rightarrow +0$.

Write

$$I(z) = \int_1^\infty (y-1)^{h-a} \gamma_\beta^{(h+2)}(yz) dy = \int_1^{1+1/z} + \int_{1+1/z}^\infty = I_1 + I_2.$$

Then, if $z \geq 1$, we have, since $h+2 < \beta+1$,

$$\begin{aligned} |I_1| &\leq \int_1^{1+1/z} (y-1)^{h-a} |\gamma_\beta^{(h+2)}(yz)| dy = z^{-\beta} \int_1^{1+1/z} (y-1)^{h-a} O(1) dy \\ &= O\{z^{-\beta-(1+h-a)}\}, \end{aligned}$$

while, if $0 < z \leq 1$, we have, since $(1+1/z)z \leq 2$,

$$\begin{aligned} |I_1| &\leq \int_1^{1+1/z} (y-1)^{h-a} |\gamma_\beta^{(h+2)}(yz)| dy = \int_1^{1+1/z} (y-1)^{h-a} O(1) dy \\ &= O\{z^{-(1+h-a)}\}. \end{aligned}$$

Again, integrating by parts, we have

$$\begin{aligned} I_2 &= \int_{1+1/z}^\infty (y-1)^{h-a} \gamma_\beta^{(h+2)}(yz) dy \\ &= \frac{1}{z} [(y-1)^{h-a} \gamma_\beta^{(h+1)}(yz)]_{1+1/z}^\infty + \frac{a-h}{z} \int_{1+1/z}^\infty (y-1)^{h-a-1} \gamma_\beta^{(h+1)}(yz) dy. \end{aligned}$$

If $z \geq 1$, the first term is

$$z^{-(1+h-\alpha)} O(z^{-\beta}) = O\{z^{-\beta-(1+h-\alpha)}\},$$

and the second is

$$z^{-1-\beta} \int_{1+1/z}^{\infty} (y-1)^{h-\alpha-1} O(1) dy = O\{z^{-\beta-(1+h-\alpha)}\}$$

while, if $0 < z \leq 1$, the first term is

$$z^{-(1+h-\alpha)} O(1) = O\{z^{-(1+h-\alpha)}\},$$

and the second is

$$\frac{1}{z} \int_{1+1/z}^{\infty} (y-1)^{h-\alpha-1} O(1) dy = O\{z^{-(1+h-\alpha)}\}.$$

Thus

$$\begin{aligned} I(z) &= O\{z^{-(1+h-\alpha)}\} \quad (0 < z \leq 1), \\ &= O\{z^{-\beta-(1+h-\alpha)}\} \quad (z \geq 1), \end{aligned}$$

and the theorem is proved when $\alpha > -1$.

If $\alpha = -1$, the hypotheses are $s^0(\omega) = o(1)$, $a(\omega) = o(1/\omega)$. Writing

$$J(t) = \int_0^{\infty} a(x) \gamma_{\beta}(xt) dx = \int_0^{1/t} + \int_{1/t}^{\infty} = J_1 + J_2,$$

we have

$$\begin{aligned} J_1 &= \int_0^{1/t} a(x) \gamma_{\beta}(xt) dx = [s^0(x) \gamma_{\beta}(xt)]_0^{1/t} - t \int_0^{1/t} s^0(x) \gamma'_{\beta}(xt) dx \\ &= o(1) + t \int_0^{1/t} o(1) O(1) dx = o(1) \end{aligned}$$

when $t \rightarrow +0$, while

$$J_2 = \int_{1/t}^{\infty} a(x) \gamma_{\beta}(xt) dx = t^{-\beta} \int_{1/t}^{\infty} o(1/x) O(x^{-\beta}) dx = o(1)$$

when $t \rightarrow +0$. This completes the proof.

Alternative proof of Theorem 7, when α is a non-negative integer. If α is a non-negative integer, we have, by $\alpha+1$ integrations by parts,

$$\int_0^{\infty} a(x) \gamma_{\beta}(xt) dx = (-1)^{h+1} \int_0^{\infty} s^{\alpha}(x) D_x^{\alpha+1} \gamma_{\beta}(xt) dx.$$

Writing

$$J(t) = t^{\alpha+1} \int_0^\infty s^\alpha(x) \gamma_\beta^{(\alpha+1)}(xt) dx = t^{\alpha+1} \int_0^{1/t} + t^{\alpha+1} \int_{1/t}^\infty = J_1 + J_2,$$

we have

$$J_1 = t^{\alpha+1} \int_0^{1/t} s^\alpha(x) \gamma_\beta^{(\alpha+1)}(xt) dx = t^{\alpha+1} \int_0^{1/t} o(x^\alpha) O(1) dx = o(1),$$

when $t \rightarrow +0$, while

$$\begin{aligned} J_2 &= t^{\alpha+1} \int_{1/t}^\infty s^\alpha(x) \gamma_\beta^{(\alpha+1)}(xt) dx \\ &= t^{\alpha+1-\beta} \int_{1/t}^\infty o(x^\alpha) O(x^{-\beta}) dx^* = o(1), \end{aligned}$$

when $t \rightarrow 0$, and this completes the proof.

Proof of Theorem 5. There is no loss of generality in supposing that $s = 0$, for if we replace a_2 by $a_2 - s$, s is replaced by 0. We may also suppose that $\beta \leq h+2$, the proof for larger values being similar, and we may restrict ourselves to positive values of t .

Suppose first that $\alpha > -1$. Then, by the consistency formula for Cesàro summation,

$$s_m^{h+1} = \sum_{n=1}^m s_n^\alpha \frac{\Gamma(m-n+1+h-\alpha)}{\Gamma(m-n+1)\Gamma(1+h-\alpha)}.$$

By $h+2$ partial summations, we have

$$\begin{aligned} \sum_{n=1}^m a_n \gamma_\beta(nt) &= s_m^0 \gamma_\beta(\overline{m+1}t) + \sum_{n=1}^m s_n^0 \Delta_n \gamma_\beta(nt) \\ &\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ &= s_m^0 \gamma_\beta(\overline{m+1}t) + s_m^1 \Delta_m \gamma_\beta(\overline{m+1}t) + \dots + s_m^{h+1} \Delta_m^{h+1} \gamma_\beta(\overline{m+1}t) \\ &\qquad\qquad\qquad + \sum_{n=1}^m s_n^{h+1} \Delta_n^{h+2} \gamma_\beta(nt) \\ &= o(1) + \sum_{n=1}^m s_n^{h+1} \Delta_n^{h+2} \gamma_\beta(nt), \end{aligned}$$

as in the proof of Theorem 7, by application of Lemma 4.

$$\text{Now} \quad \sum_{n=1}^\infty \sum_{\nu=1}^n = \lim_{m \rightarrow \infty} \sum_{n=1}^m \sum_{\nu=1}^n = \lim_{m \rightarrow \infty} \sum_{\nu=1}^m \sum_{n=\nu}^m = \sum_{\nu=1}^\infty \sum_{n=\nu}^\infty,$$

provided that $\overline{\lim}_{m \rightarrow \infty} \left| \sum_{\nu=1}^m \sum_{n=m+1}^{\infty} \right| = 0$. Therefore

$$\begin{aligned} S(t) &= \sum_{n=1}^{\infty} a_n \gamma_{\beta}(nt) = \sum_{n=1}^{\infty} s_n^{h+1} \Delta_n^{h+2} \gamma_{\beta}(nt) \\ &= \frac{1}{\Gamma(1+h-a)} \sum_{n=1}^{\infty} \Delta_n^{h+2} \gamma_{\beta}(nt) \sum_{\nu=1}^n s_{\nu}^a \frac{\Gamma(n-\nu+1+h-a)}{\Gamma(n-\nu+1)} \\ &= \frac{1}{\Gamma(1+h-a)} \sum_{\nu=1}^{\infty} s_{\nu}^a \sum_{n=\nu}^{\infty} \frac{\Gamma(n-\nu+1+h-a)}{\Gamma(n-\nu+1)} \Delta_n^{h+2} \gamma_{\beta}(nt) \\ &= \frac{1}{\Gamma(1+h-a)} \sum_{\nu=1}^{\infty} s_{\nu}^a T_{\nu}, \end{aligned}$$

where

$$T_{\nu} = \sum_{n=\nu}^{\infty} \frac{\Gamma(n-\nu+1+h-a)}{\Gamma(n-\nu+1)} \Delta_n^{h+2} \gamma_{\beta}(nt),$$

provided that $\overline{\lim}_{m \rightarrow \infty} \left| \sum_{\nu=1}^m s_{\nu}^a \sum_{n=m+1}^{\infty} \frac{\Gamma(n-\nu+1+h-a)}{\Gamma(n-\nu+1)} \Delta_n^{h+2} \gamma_{\beta}(nt) \right| = 0$.

Now, for a fixed $t > 0$, we have

$$\begin{aligned} &\left| \sum_{n=m+1}^{\infty} \frac{\Gamma(n-\nu+1+h-a)}{\Gamma(n-\nu+1)} \Delta_n^{h+2} \gamma_{\beta}(nt) \right| \\ &= O \left\{ (m-\nu+1)^{h-a} \text{Max}_{m' > m} \left| \sum_{n=m+1}^{m'} \Delta_n^{h+2} \gamma_{\beta}(nt) \right| \right\} = (m-\nu+1)^{h-a} O(m^{-\beta}) \end{aligned}$$

$$\begin{aligned} \text{and} \quad m^{-\beta} \sum_{\nu=1}^m |s_{\nu}^a| (m-\nu+1)^{h-a} &= m^{-\beta} \sum_{\nu=1}^m o(\nu^a) (m-\nu+1)^{h-a} \\ &= O(m^{1+h-\beta}) = o(1), \end{aligned}$$

and so our assertion is justified.

We have then to show that $S(t) \rightarrow 0$ when $t \rightarrow +0$, and this will be the case if we can prove that

$$\begin{aligned} T_{\nu} &= O(t^{1+a}) \quad (0 < \nu t \leq 1), \\ &= O(t^{1+a-\beta} \nu^{-\beta}) \quad (\nu t > 1), \end{aligned}$$

for, if this be true, we have

$$S(t) = O(t^{1+a}) \sum_{\nu \leq 1/t} o(\nu^a) + O(t^{1+a-\beta}) \sum_{\nu > 1/t} o(\nu^{a-\beta}) = o(1) + o(1) = o(1).$$

Let $\rho = [1/t]$ and write

$$T_{\nu} = \sum_{n=\nu}^{\nu+\rho} + \sum_{n=\nu+\rho+1}^{\infty} = U_{\nu} + V_{\nu}.$$

Then, if $\nu t > 1$, we have

$$\begin{aligned} U_\nu &= \sum_{n=\nu}^{\nu+\rho} \frac{\Gamma(n-\nu+1+h-\alpha)}{\Gamma(n-\nu+1)} \Delta_n^{h+2} \gamma_\beta(nt) \\ &= t^{h+2-\beta} \sum_{n=\nu}^{\nu+\rho} O\{(n-\nu+1)^{h-\alpha}\} O(n^{-\beta}) \\ &= t^{h+2-\beta} O(\rho^{1+h-\alpha}) O(\nu^{-\beta}) = O(t^{1+\alpha-\beta} \nu^{-\beta}), \end{aligned}$$

while, if $0 < \nu t \leq 1$, we have, since $(\nu+\rho)t \leq 2$,

$$U_\nu = t^{h+2} \sum_{n=\nu}^{\nu+\rho} O\{(n-\nu+1)^{h-\alpha}\} O(1) = t^{h+2} O(\rho^{1+h-\alpha}) = O(t^{1+\alpha}).$$

Again, we have, by a partial summation,

$$\begin{aligned} V_\nu &= \sum_{n=\nu+\rho+1}^{\infty} \frac{\Gamma(n-\nu+1+h-\alpha)}{\Gamma(n-\nu+1)} \Delta_n^{h+2} \gamma_\beta(nt) \\ &= \frac{\Gamma(\rho+2+h-\alpha)}{\Gamma(\rho+2)} \Delta_n^{h+1} \gamma_\beta(\nu+\rho-1)t \\ &\quad - \sum_{n=\nu+\rho+1}^{\infty} \Delta_n \frac{\Gamma(n-\nu+1+h-\alpha)}{\Gamma(n-\nu+1)} \Delta_n^{h+1} \gamma_\beta(nt). \end{aligned}$$

If $\nu t > 1$, the first term is

$$O(\rho^{h-\alpha}) t^{h+1} O(1) = O(t^{1+\alpha}),$$

and the second is

$$t^{1+h-\beta} \sum_{n=\nu+\rho+1}^{\infty} O(n-\nu)^{h-\alpha-1} O(n^{-\beta}) = t^{1+h-\beta} O(\rho^{h-\alpha}) O(\nu^{-\beta}) = O(t^{1+\alpha-\beta} \nu^{-\beta}),$$

while, if $0 < \nu t \leq t$, the first term is

$$O(\rho^{h-\alpha}) t^{h+1} O(1) = O(t^{1+\alpha}),$$

and the second is

$$t^{h+1} \sum_{n=\nu+\rho+1}^{\infty} O(n-\nu)^{h-\alpha-1} O(1) = t^{h+1} O(\rho^{h-\alpha}) = O(t^{1+\alpha}).$$

Thus

$$\begin{aligned} T_\nu &= O(t^{1+\alpha}) \quad (0 < \nu t \leq 1) \\ &= O(t^{1+\alpha-\beta} \nu^{-\beta}) \quad (\nu t > 1), \end{aligned}$$

and the theorem is proved when $\alpha > -1$.

The proof for the case $\alpha = -1$ and the alternative proof for the case in which α is a non-negative integer are similar to their analogues for integrals. The theorem is thus completely established.

Finally, I must express my thanks to Prof. G. H. Hardy for his valuable suggestions and criticisms.

List of References.

1. A. F. Anderson, *Studier over Cesàro's summabilitets-metode*. (Copenhagen, 1921.)
2. L. S. Bosanquet, "On Abel's integral equation and fractional integrals", *Proc. London Math. Soc.* (2), 31 (1930), 134-143.
3. M. E. Grimshaw, "A case of distinction between Fourier integrals and Fourier series", *Proc. Camb. Phil. Soc.*, 23 (1927), 755-767.
4. H. Hahn, "Über Fejérs Summierung der Fourierschen Reihe", *Jahresbericht der deutschen Math. Vereinigung*, 25 (1916), 359-366.
5. G. H. Hardy and J. E. Littlewood, "Contributions to the arithmetic theory of series", *Proc. London Math. Soc.* (2), 11 (1913), 411-478.
6. ———, "Abel's theorem and its converse", I, *Proc. London Math. Soc.* (2), 18 (1920), 205-235.
7. ———, "Abel's theorem and its converse", II, *Proc. London Math. Soc.* (2), 22 (1924), 254-269.
8. ———, "Solution of the Cesàro summability problem for power series and Fourier series", *Math. Zeitschrift*, 19 (1924), 67-96.
9. ———, "The allied series of a Fourier series", *Proc. London Math. Soc.* (2), 24 (1926), 211-246.
10. ———, "Notes on the theory of series (III): On the summability of the Fourier series of a nearly continuous function", *Proc. Camb. Phil. Soc.*, 23 (1927), 681-684.
11. ———, "Notes on the theory of series (VII): On Young's convergence criterion for Fourier series", *Proc. London Math. Soc.* (2), 28 (1928), 301-311.
12. G. H. Hardy and M. Riesz, "The general theory of Dirichlet's series", *Camb. Math. Tract*, 18 (1915).
13. E. W. Hobson, *The theory of functions of a real variable* (1926).
14. S. Pollard, "The summation of a Fourier integral of finite type", *Proc. Camb. Phil. Soc.*, 23 (1927), 373-382.
15. ———, "The summation of Denjoy-Fourier series", *Proc. London Math. Soc.* (2), 27 (1928), 209-222.
16. E. C. Titchmarsh, *Proc. London Math. Soc. (Records)* (2), 22, (1924), xxv.
17. W. H. Young, "On infinite integrals involving a generalization of the sine and cosine functions", *Quart. J. of Math.*, 43 (1912), 161-177.
18. W. H. Young, "On the convergence of a Fourier series and of its allied series", *Proc. London Math. Soc.* (2), 10 (1912), 254-272.

ON FUNCTIONS OF BOUNDED DEVIATION

By EINAR HILLE.

[Received 17 August, 1929.—Read 14 November, 1929.]

1. Let $f(t)$ be an integrable function defined in $(0, 2\pi)$ and extended periodically outside of this interval. $f(t)$ is said to be of *bounded deviation* (= *écart fini* in Hadamard's terminology[†]) in $(0, 2\pi)$ if there exists an M independent of n , a , and b such that

$$(1) \quad \left| n \int_a^b f(t) e^{nit} dt \right| \leq M,$$

where n is any integer and $0 \leq a < b \leq 2\pi$. The class of all functions of bounded deviation in $(0, 2\pi)$ will be denoted by BD in the following. Similarly we use the symbol BV for the class of all functions of bounded variation in $(0, 2\pi)$. We have evidently $BD \supseteq BV$.

We further agree that $f(t)$ shall be said to belong to RD if (i) $f(t)$ belongs to BD , (ii) $f(t)$ is continuous, and (iii) there exists an $F(z)$ holomorphic for $|z| < 1$ such that

$$(2) \quad \lim_{r \rightarrow 1} F(re^{it}) = f(t)$$

almost everywhere. G. von Alexits has recently shown[‡] the existence of functions of RD which do not belong to BV . We have consequently

$$(3) \quad BD \supset BV, \quad BD \supset RD, \quad BD - BD.BV \neq 0.$$

Alexits's examples are simple in appearance, but the proofs rest on deep results due to Hardy and Littlewood. In § 2 of the present paper I show that it is possible to find elementary functions $F(z)$ such that

[†] *Journal de Math.* (4), 8 (1892), 101–186. See Hadamard et Mandelbrojt, “La série de Taylor . . .”, *Scientia* 12 (Paris, 1926), 84–85. Further, S. Mandelbrojt, *The Rice Institute Pamphlet*, 14 (1927), 225–352 (292–293). (This and the subsequent references to Mandelbrojt have been added after the paper was accepted for publication.)

[‡] *Math. Zeitschrift*, 30 (1929), 43–46.

$F(e^{it}) = f(t)$ belongs to RD but not to BV . The example also has the advantage of giving an answer to a question on the mapping properties of a function $F(z)$ whose boundary function $F(e^{it})$ belongs to RD but not to BV , which question was raised by Alexits.

In § 3 functions of bounded deviation are investigated from the point of view of the theory of factor sequences which leave invariant the class of a Fourier series. This question is continued in § 4, which also contains examples of power series with gaps whose boundary functions belong to BD but never to BV . The work of this paragraph is closely connected with a theorem due to Hadamard.

2. We put

$$(4) \quad F(z, a) = (1-z)^{-i} [1 - \log(1-z)]^{-a} \quad (a \geq 0)^+.$$

The only singularities of $F(z)$ are 1, $1-e$, and ∞ . When $z \rightarrow 1$ in such a manner that $\arg(1-z)$ is bounded, $F(z) \rightarrow 0$, provided that $a > 0$. It follows that $F(z)$ is continuous in the closed unit-circle. We begin by investigating the mapping properties of $F(z)$. We shall show that $w = F(z)$ maps the interior of the unit-circle on a region whose area is finite if and only if $a > \frac{1}{2}$. It is sufficient to consider the area of the map of the region

$$(5) \quad 0 < \rho \leq |1-z| \leq \frac{1}{2}, \quad |\arg(1-z)| \leq \eta \leq \frac{1}{2}\pi,$$

which is given by the integral

$$(6) \quad \int_{-\eta}^{+\eta} \int_{\rho}^{\frac{1}{2}} |F'(1-re^{i\phi}, a)|^2 r dr d\phi.$$

But

$$F'(z, a) = -(1-z)^{-1-i} [1 - \log(1-z)]^{-a} \{i + a[1 - \log(1-z)]^{-1}\}.$$

Hence, z being in the region (5), we can find two positive constants C_1 and C_2 such that

$$C_1 \frac{1}{r} \left[\log \frac{1}{r} \right]^{-a} < |F'(1-re^{i\phi}, a)| < C_2 \frac{1}{r} \left[\log \frac{1}{r} \right]^{-a}.$$

From this double inequality we see that the integral (6) converges or

+ Similar functions have been considered by Hardy and M. Riesz for other purposes. See Hardy, *Quart. J. of Math.*, 44 (1913), 147-160 (157-159).

diverges when $\rho \rightarrow 0$ simultaneously with the integral

$$\int_{\rho}^{\frac{1}{2}} \left[\log \frac{1}{r} \right]^{-2a} \frac{dr}{r},$$

i.e. according as $a > \frac{1}{2}$ or $\leq \frac{1}{2}$.

We next proceed to show that $F(e^{it}, a)$ is of bounded variation if and only if $a > 1$. We can restrict ourselves to small positive values of t ; small negative values can be discussed in a similar manner. We put $F(e^{it}, a) = Re^{i\theta}$. A simple computation shows that R is monotone for $0 < t < \pi$, and that $R \rightarrow 0$ when $t \rightarrow 0$. Further, there exists a $t(a)$ such that θ is positive and monotone for $0 < t < t(a)$; $\theta \rightarrow +\infty$ when $t \rightarrow 0$. We determine t_n by the conditions $\theta(t_n) = n\pi$, $0 < t_n < t(a)$, $n = 1, 2, \dots$. We can now find a constant C such that

$$F(e^{it_n}, a) = C(-1)^n n^{-a} \left[1 + O\left(\frac{1}{n}\right) \right].$$

$w = F(e^{it}, a)$ describes a spiral in the w -plane with steadily diminishing radius vector when $t \rightarrow 0$; the length of arc of this spiral between two points is equal to the total variation of $F(e^{it}, a)$ between the corresponding values of t . It follows that the variation of $F(e^{it}, a)$ between t_n and t_{n+1} exceeds

$$|F(e^{it_n}, a) - F(e^{it_{n+1}}, a)|,$$

but remains less than π times this quantity. Hence the total variation of $F(e^{it}, a)$ on (t_1, t_n) tends to a finite or infinite limit when $n \rightarrow \infty$ according as the series Σn^{-a} converges or diverges, *i.e.* according as $a > 1$ or ≤ 1 .

In order to show that $F(e^{it}, a)$ is of bounded deviation when $a \geq 0$ it is sufficient to show that

$$n \int_0^a F(e^{it}, a) e^{nit} dt$$

is uniformly bounded with respect to a when $|a| < \frac{1}{2}\pi$ for all integral values of n . We assume a and n to be positive in the proof; the remaining cases can be treated in the same manner. The integral from $t = 0$ to $t = 2/n$ is obviously $O(1/n)$. A somewhat lengthy but elementary calculation shows that

$$F(e^{it}, a) - e^{-\frac{1}{2}\pi} t^{-i} \left(\log \frac{1}{t} \right)^{-a}$$

is an absolutely continuous function and hence, *a fortiori*, is of bounded

deviation. It is consequently sufficient to discuss the integral

$$(7) \quad \int_{2/n}^a t^{-i} \left(\log \frac{1}{t} \right)^{-\alpha} e^{nit} dt.$$

In this integral we substitute $nt = u$, obtaining

$$n^{-1+i} \int_2^{na} u^{-i} \left(\log \frac{n}{u} \right)^{-\alpha} e^{iu} du.$$

The factor $\{\log(n/u)\}^{-\alpha}$ being bounded and steadily increasing when $\alpha \geq 0$ in the interval under consideration, the second mean value theorem shows that it is sufficient for us to prove the integral

$$(8) \quad \int_2^b \exp i(u - \log u) du$$

to be bounded for $b > 2$. A suitable integration by parts gives

$$-i \left[\frac{u}{u-1} \exp i(u - \log u) \right]_2^b - i \int_2^b \exp i(u - \log u) \frac{du}{(u-1)^2},$$

which shows that (8) is uniformly bounded. Consequently, $F(e^{it}, \alpha)$ is of bounded deviation for $\alpha \geq 0$. Thus we have shown that

$F(z, \alpha)$ is holomorphic for $|z| < 1$. It maps the interior of the unit-circle conformally upon a region whose area is finite if and only if $\alpha > \frac{1}{2}$. The boundary function $F(e^{it}, \alpha)$ is of bounded deviation for $\alpha \geq 0$. It is continuous for $\alpha > 0$ and of bounded variation if and only if $\alpha > 1$.

Alexits raised the following question. If $f(t) = F(e^{it})$ belongs to $RD-RD.BV$, does $w = F(z)$ always map $|z| < 1$ on a region of finite area? The function $F(z, \alpha)$, $0 < \alpha \leq \frac{1}{2}$, shows that the area may be infinite.

3. The following problem has been considered by a number of mathematicians during recent years† :—

Let C be a class of trigonometric series having associated sum-functions possessing a property P . Find necessary and sufficient conditions to be satisfied by the sequence $\{\lambda_n\}$ in order that $\sum_{-\infty}^{+\infty} \lambda_n a_n e^{nit}$ shall belong to C whenever $\sum_{-\infty}^{+\infty} a_n e^{nit}$ does so.

† This problem goes back to W. H. Young, *Proc. Royal Soc. London*, 88 (1913), 561-568. For the history of the problem and an important contribution to it, see M. Fekete, *Szeged Acta*, 1 (1922-23), 148-166. Among later contributions we note M. Riesz, *Acta Math.*, 49 (1927), 465-497, and S. Bochner, *Szeged Acta*, 4 (1929), 125-129.

Such a sequence is known as a *factor sequence for the property P*. If P stands for one of the properties integrability, boundedness, continuity, bounded variation, and absolute continuity, $\{\lambda_n\}$ will be a factor sequence if and only if

$$(9) \quad \Lambda(x) \sim \sum'_{-\infty}^{+\infty} \frac{\lambda_n}{n} e^{-nix}$$

represents a function of bounded variation. It is perhaps surprising that this property is sufficient but not necessary if P stands for bounded deviation in $(0, 2\pi)$. That the condition is not necessary will be shown in § 4. In order to establish the sufficiency we begin by proving the

LEMMA. *Put*

$$(10) \quad f(x) \sim \sum_{-\infty}^{+\infty} a_n e^{nix}, \quad s_n(x) = \sum_{\nu=-n}^{+n} a_\nu e^{vi x}, \quad S_n(x) = \sum_{\nu=-n}^{+n} \left(1 - \frac{|\nu|}{n}\right) a_\nu e^{vi x}.$$

A necessary and sufficient condition for $f(x)$ to be of bounded deviation in $(0, 2\pi)$ is the existence of a positive number M , independent of m , a , and b , such that

$$(11) \quad \overline{\lim}_{n \rightarrow \infty} \left| m \int_a^b S_n(x) e^{m i x} dx \right| \leq M,$$

where $m = \pm 1, \pm 2, \dots$, and $0 \leq a < b \leq 2\pi$. The condition remains necessary and sufficient if $S_n(x)$ be replaced by $s_n(x)$.

We begin with $S_n(x)$. If $f(x)$ is of bounded deviation, we conclude from (1) that

$$|2\pi m a_m| = \left| m \int_0^{2\pi} f(x) e^{-m i x} dx \right| \leq M.$$

It follows that $f(x)$ is integrable together with its square. Hence $S_n(x) \rightarrow f(x)$ almost everywhere, and $\int_0^{2\pi} |S_n(x)|^2 dx < A$ for a suitable fixed A . Consequently

$$(12) \quad \lim_{n \rightarrow \infty} m \int_a^b S_n(x) e^{m i x} dx = m \int_a^b f(x) e^{m i x} dx$$

for every integral m and every a and b . Using (1) again we see that (11) is a necessary condition for bounded deviation.

Suppose now that (11) is satisfied by the Fejér sums of a trigonometric series. For $n \geq |m|$ we have

$$(13) \quad \int_0^{2\pi} S_n(x) e^{-m i x} dx = 2\pi \left[1 - \frac{|m|}{n}\right] a_m \rightarrow 2\pi a_m$$

when $n \rightarrow \infty$. Hence, in virtue of (11), $2\pi |ma_n| \leq M$. We then conclude, as above, that $S_n(x)$ tends to a limit $f(x)$ almost everywhere, $f(x)$ being quadratically integrable. It follows that (12) holds; the right-hand side of (12) being less than M in absolute value by (11), we conclude that $f(x)$ is of bounded deviation.

The lemma for $S_n(x)$ is entirely parallel to lemmas given by Fekete in the case of the properties P mentioned above. The proof for $s_n(x)$ consists in replacing $S_n(x)$ by $s_n(x)$ in the proof above, suppressing the middle part of (13). That $s_n(x) \rightarrow f(x)$ almost everywhere follows from a Tauberian theorem due to Hardy: a series summable $(C, 1)$, whose terms are $O(1/n)$, is convergent.

The proof that $\{\lambda_n\}$ is a factor sequence for bounded deviation when $\Lambda(x)$, defined by (9), is of bounded variation, follows classical lines. We put

$$(14) \quad L_n(x) = \sum_{\nu=-n}^{+n} \left(1 - \frac{|\nu|}{n}\right) \lambda_\nu e^{-\nu i x},$$

$$(15) \quad F_n(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x+t) L_n(t) dt = \sum_{\nu=-n}^{+n} \left(1 - \frac{|\nu|}{n}\right) \lambda_\nu a_\nu e^{-\nu i x},$$

assuming $f(x)$ to belong to BD and $\Lambda(x)$ to BV . It is well known that there exists a B such that

$$(16) \quad \int_0^{2\pi} |L_n(t)| dt \leq B$$

for all values of n . We shall show that

$$(17) \quad \overline{\lim}_{n \rightarrow \infty} \left| m \int_a^b F_n(x) e^{m i x} dx \right| \leq \frac{BM}{2\pi},$$

where M is the upper bound of the deviation of $f(x)$ occurring in formula (1). But

$$\begin{aligned} 2\pi m \int_a^b F_n(x) e^{m i x} dx &= m \int_a^b \int_0^{2\pi} f(x+t) e^{m i x} L_n(t) dt dx \\ &= \int_0^{2\pi} L_n(t) \left\{ m \int_a^b f(x+t) e^{m i x} dx \right\} dt, \end{aligned}$$

which is less than BM in absolute value by reason of (1) and (16). Using the lemma we see that

$$F(x) \sim \sum_{-\infty}^{+\infty} \lambda_n a_n e^{n i x}$$

is of bounded deviation. This completes the proof of the sufficiency of the condition on $\Lambda(x)$.

4. That the condition is not necessary follows from a theorem due to Hadamard :—

$f(x)$ is of bounded deviation if (i) $\sum_{-\infty}^{+\infty} |a_n|$ is convergent and
(ii) $a_n = O\{[|n| \log |n|]^{-1}\}$, where $f(x) = \sum_{-\infty}^{+\infty} a_n e^{nix}$.

Hadamard states this theorem with “ o ” instead of “ O ”, but the stronger theorem is easily read off from his proof†. With the aid of this theorem we find that

$\{\lambda_n\}$ is a factor sequence for bounded deviation if (i) $\sum' \left| \frac{\lambda_n}{n} \right|$ is convergent, and (ii) $\lambda_n = O[(\log |n|)^{-1}]$.

It follows from Alexits's examples that $\{\lambda_n\}$ may satisfy this condition without $\Lambda(x)$ being of bounded variation. Conversely, $\Lambda(x)$ may be of bounded variation without the second condition being satisfied. Hence, the two sufficient conditions which we have found are independent of each other and neither is necessary.

It is of interest to note that $\{\lambda_n\}$ is not always a factor sequence for bounded deviation if we assume $\Lambda(x)$ to belong to BD instead of to BV . In order to see this we notice that

$$(1 - e^{ix})^{-i} = \sum_0^{\infty} b_n e^{nix} \quad \text{and} \quad (1 - e^{-ix})^i = \sum_0^{\infty} \bar{b}_n e^{-nix}$$

belong to BD , whereas

$$\sum_1^{\infty} n b_n \bar{b}_n e^{nix} = A \log(1 - e^{ia}) + \sum_1^{\infty} c_n e^{nix}, \quad c_n = O\left(\frac{1}{n^2}\right),$$

does not‡, since the left-hand side of formula (1) becomes $O(\log |n|)$ almost always when we replace $f(t)$ by $\log(1 - e^{it})$.

It is clear that na_n is bounded if $f(x)$ belongs to BD . If $f(x) = (1 - e^{ia})^{-i}$, the limiting values of na_n make up the circumference of a circle in the complex plane. This seems to be the first example of a function belonging to $BD - BD \cdot BV$ such that na_n does not tend to zero with $1/n$. We proceed to give other examples of this nature where, however, na_n takes on only two values 0 and 1.

† Compare Mandelbrojt, *loc. cit.*, 299.

‡ The numerical value of A is $[(1/\pi) \sinh \pi]^{\frac{1}{2}}$. See Hardy, *Quart. J. of Math.*, 44 (1913), 159.

Let p_n be a sequence of positive integers such that (i) $\sum_{n=1}^{\infty} \frac{1}{p_n}$ is convergent, and (ii) $p_{\mu+\nu} \geq p_\mu + p_\nu + k$ for every pair of positive integers μ and ν , k being a fixed integer. If k is negative we require, in addition, that (iii) $p_{\mu+1} > p_\mu$. Then the function $f(x; p) = \sum_{n=1}^{\infty} \frac{e^{p_n i x}}{p_n}$ belongs to the class $RD-RD.BV$.

We first prove that $f(x; p)$ belongs to RD . It is obviously the continuous boundary function of an analytic function holomorphic in the interior of the unit-circle. We have thus merely to show that $f(x; p)$ belongs to BD . We can evaluate

$$D(m, a, b) = m \int_a^b f(x; p) e^{m i x} dx$$

by integration term by term, obtaining

$$D(m, a, b) = m \sum_{n=1}^{\infty *} \frac{e^{(p_n+m)i b} - e^{(p_n+m)i a}}{i p_n (p_n+m)},$$

where the asterisk after the summation sign indicates that, when $m = -p_j$, the corresponding term is to be replaced by its limiting value. It is easily seen that $D(m)$ is uniformly bounded when $m > 0$. If $m = -q$, $q > 0$, it is sufficient to prove that

$$(18) \quad \sum_{n=1}^{\infty'} \frac{q}{p_n |p_n - q|} \leq \sum_1^{\infty} \frac{1}{p_n} + \sum_1^{\infty'} \frac{1}{|p_n - q|}$$

is uniformly bounded. Here the dash indicates that $q \neq p_n$. In order to estimate the second sum on the right of (18), suppose that $q = p_r + s < p_{r+1}$, $s \geq 0$. Since $p_r \geq p_n + p_{r-n} + k$ when $n < r$, we conclude that

$$\sum_{n < r} (q - p_n)^{-1} \leq \sum_{n < r - |k|} [p_{r-n} + s + k]^{-1} + |k|,$$

which is uniformly bounded for all values of q . If $n > r+1$ we have $p_n \geq p_{r+1} + p_{n-r-1} + k$. Hence

$$\begin{aligned} \sum_{n > r} (p_n - q)^{-1} &\leq \sum_{n > r + |k|} [p_{n-r-1} + p_{r+1} - p_r - s + k]^{-1} + |k| + 1 \\ &< \sum_{n > r + |k|} [p_{n-r-1} + k]^{-1} + |k| + 1. \end{aligned}$$

Consequently $f(x; p)$ is always of bounded deviation.

If $f(x; p) = f(x)$ should belong to BV , then $f(-x)$, and hence also $f(x) + f(-x)$ and $f(x) - f(-x)$ must belong to BV . But if two con-

jugate Fourier series belong to BV , then† their Fourier coefficients are $o\{1/n\}$, which is obviously not the case with $f(x; p)$. It follows that $f(x; p)$ belongs to $RD-RD.BV$.

Let us put $p_n = p(n) = n q(n)$. In order to satisfy condition (ii) we have merely to assume the existence of a fixed A such that $q(\mu + \nu) - q(\mu) > A/\mu$ for every pair of positive integers μ and ν . We then take $k = [2A]$ if $A \geq 0$, otherwise the greatest integer less than $2A$. If $q(n)$ is monotone increasing we can take $k = 0$. Thus, in particular,

If $\sum_1^\infty \frac{1}{p_n}$ converges and $\frac{p_n}{n}$ is steadily increasing, $f(x) = \sum_1^\infty \frac{e^{p_n i x}}{p_n}$ belongs to $BD-BD.BV$.

It is not necessary for the p_n to be integers in this statement. We can, of course, vary the assumptions on the p_n in many other ways.

[*Added 8 February, 1930.* After this paper was sent to the Secretaries of the Society I noticed that H. E. Bray had already found an example of a function belonging to BD but not to BV , namely $x \sin(1/x)$ in any interval containing $x = 0^+$. It follows from the discussion in § 2 that $\sin \log |x|$ has the same property.]

† F. and M. Riesz, *Compte rendu du quatrième congrès des Math. Scandinaves à Stockholm* (1916), 27-44 (44).

‡ See Mandelbrojt, *loc. cit.*, 204.

ON EXPANSIONS WHICH ARE FORMALLY REDUCED TO ZERO BY THE OPERATOR $\sinh D - cD$.

By D. M. HIRST.

[Received 2 April, 1929.—Read 25 April, 1929.]

Introduction.

1. A. C. Dixon* has shown that, if a real function $f(x)$ of a real variable x and its first differential coefficient are of limited total fluctuation throughout an interval $(a-1, a+1)$, then $\frac{1}{2}\{f(x+0)+f(x-0)\}$ can be represented at internal points of $(a-1, a+1)$ by the series

$$\sum a_n \exp \lambda_n x,$$

where the exponents λ_n are the roots of the equation $\sinh \lambda - c\lambda = 0$, arranged in order of non-decreasing moduli, and the coefficients are given by the formula

$$a_n = \frac{1}{2(\cosh \lambda_n - c)} \left\{ \int_{a-1}^a f(t) \exp \{-\lambda_n(t+1)\} dt + \int_a^{a+1} f(t) \exp \{-\lambda_n(t-1)\} dt \right\}. \quad (1.1)$$

To establish the convergence of the series at the end point of the interval, he assumes that, in the neighbourhood of the point a , $f(x)$ possesses a second differential coefficient, also of limited total fluctuation. The sum of the series is then

$$\begin{aligned} \frac{1}{2}\{f(a-1+0)+f(a+1-0)\} - cf'(a-0) & \text{ when } x = a-1, \\ \frac{1}{2}\{f(a-1+0)+f(a+1-0)\} + cf'(a+0) & \text{ when } x = a+1. \end{aligned}$$

The number c may have any value, real or complex, except 0 or 1. If $c=0$, the expansion reduces to the Fourier series and the proof breaks

* "Expansions and functions reduced to zero by the operator $\sinh D - cD$ ", *Proc. London Math. Soc.* (2), 21 (1921), 271-290. This paper will be referred to as A. I am indebted to Professor Dixon for suggestions and advice in the extension of his results.

down (A, 286, foot-note). If $c = 1$, the series is still convergent but the sum is modified, because the term $(\sinh \lambda - c\lambda)^{-1}$, which occurs in the contour integrals employed, has a triple pole at the origin instead of a simple one.

In the present paper I extend the class of functions to which the convergence theorem is applicable (§§ 2 and 3), and point out some differences between the behaviour of the Dixon series and the Fourier series of an arbitrary function (§ 9).

Convergence of the expansion.

2. When the coefficients a_n are formed in accordance with (1.1), the series $\Sigma a_n \exp \lambda_n x$ will be called the Dixon expansion associated with $f(x)$. It will be proved that *the Dixon expansion associated with a bounded function $f(x)$ in an interval $(a-1, a+1)$ converges to*

$$\frac{1}{2} \{f(x_1+0) + f(x_1-0)\}$$

at any internal point x_1 of that interval, if $f(x)$ is of limited total fluctuation in a sub-interval, of which x_1 is an internal point and a an end point. To ensure convergence at the end point $a+1$, it is sufficient to assume further that there is a small interval lying immediately on the right of the point $a-1$ in which $f(x)$ is of limited total fluctuation, and another small interval immediately on the right of the point a in which $f(x)$ possesses a differential coefficient, also of limited total fluctuation. There is a similar condition for convergence at $x = a-1$.

If λ_n is a root of the equation $\sinh \lambda - c\lambda = 0$, so also is $-\lambda_n$. Therefore, by Cauchy's theorem, the sum of the series, if convergent, is

$$\frac{1}{4\pi i} \lim_{n \rightarrow \infty} \left\{ \int_{(R_n)} \int_{a-1}^a \frac{f(t) \exp \lambda(t-x+1)}{\sinh \lambda - c\lambda} dt d\lambda + \int_{(R_n)} \int_a^{a+1} \frac{f(t) \exp \lambda(t-x-1)}{\sinh \lambda - c\lambda} dt d\lambda \right\}, \quad (2.1)$$

where n assumes integral values only (odd or even according as the real part of c is positive or negative), the path of integration with respect to t is along the real axis, and the contour R_n for the integration with respect to λ consists of two circular arcs with centre at the origin and radius $\sqrt{\{(n+\frac{1}{2})^2 \pi^2 + (n+\frac{1}{2})\pi\}}$, and the two chords on which the imaginary part of λ is equal to $\pm(n+\frac{1}{2})\pi$. The symbol $\int_{(\infty)}$ will some-

times be used to denote the limit, as $n \rightarrow \infty$, of an integral taken along R_n .

It has been proved by Dixon (A, 286-8) that the integrals

$$\int_{(R_n)} \frac{\exp \lambda t}{\sinh \lambda - c\lambda} d\lambda \quad \text{and} \quad \int_{(R_n)} \frac{\exp \lambda t}{\lambda(\sinh \lambda - c\lambda)} d\lambda \quad (2.2)$$

converge to zero as $n \rightarrow \infty$ for every real value of t in the open interval $(-1, +1)$, the convergence being uniform; and that the first integral diverges, while the second converges to $2\pi it$, when $t = \pm 1$. (2.3)

It is also shown that, if the function $f(t)$ is bounded throughout the range of integration, the integrals

$$\int_{(R_n)} \int_{-u}^v \frac{f(t) \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda \quad \text{and} \quad \int_{(R_n)} \int_{-u}^v \frac{f(t) \exp \lambda t}{\lambda(\sinh \lambda - c\lambda)} dt d\lambda$$

tend uniformly to zero when u, v are real numbers in the open interval $(-1, +1)$; that the second also converges uniformly to zero if u or v is unity; and that the integrals still converge uniformly to zero (for the same values of u and v) if all the elements are taken in absolute value. (2.4)

It will be assumed throughout the work that the function $f(x)$ is real, bounded, and integrable.

The convergence of the integral

$$\int_{(R_n)} \int_v^1 \frac{f(t) \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda \quad (-1 < v < 1)$$

will first be established.

If $f(1-0)$ exists, the number μ can be determined such that $v \leq \mu < 1$ and

$$|f(t) - f(1-0)| < \delta \quad (2.5)$$

for all values of t in the closed interval $(\mu, 1)$, δ being an arbitrarily small positive number. Then

$$\begin{aligned} \int_{(R_n)} \int_v^1 \frac{f(t) \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda &= f(1-0) \int_{(R_n)} \int_{\mu}^1 \frac{\exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda \\ &+ \int_{(R_n)} \int_{\mu}^1 \frac{\{f(t) - f(1-0)\} \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda \\ &+ \int_{(R_n)} \int_v^{\mu} \frac{f(t) \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda. \end{aligned} \quad (2.51)$$

Of the integrals on the right, the third converges to zero by (2.4), because v and μ both lie in the open interval $(-1, +1)$, and the first is equal to

$$\int_{(R_n)} \frac{\exp \lambda - \exp \lambda \mu}{\lambda (\sinh \lambda - c\lambda)} d\lambda,$$

which converges to $2\pi i$ by (2.3).

If $f(t)$ is monotonic in the interval $(\mu, 1)$,

$$\begin{aligned} \int_{\mu}^1 \{f(t) - f(1-0)\} \exp \lambda t dt \\ = \{f(\mu) - f(1-0)\} \left\{ R \int_{\mu}^{\omega_1} \exp \lambda t dt + iI \int_{\mu}^{\omega_2} \exp \lambda t dt \right\} \end{aligned}$$

by the second mean value theorem, R and I denoting respectively the real and imaginary parts of the expressions which follow them, and ω_1, ω_2 denoting real numbers which lie between μ and 1, but which may perhaps approach 1 for certain values of λ . Hence

$$\begin{aligned} \int_{(R_n)} \int_{\mu}^1 \frac{\{f(t) - f(1-0)\} \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda \\ = \{f(\mu) - f(1-0)\} \int_{(R_n)} \left(R \frac{\exp \lambda \omega_1}{\lambda} + iI \frac{\exp \lambda \omega_2}{\lambda} - \frac{\exp \lambda \mu}{\lambda} \right) \frac{d\lambda}{\sinh \lambda - c\lambda}, \end{aligned} \quad (2.52)$$

where ω_1 and ω_2 are functions of λ which are never less than μ or greater than 1.

$$\text{It is known that} \quad \int_{(R_n)} \left| \frac{\exp \lambda t}{\lambda (\sinh \lambda - c\lambda)} d\lambda \right| \quad (2.3)$$

converges to zero if t has any fixed value in the open interval $(-1, +1)$. From the proof of the convergence of the integral

$$\int_{(R_n)} \frac{\exp \lambda}{\lambda (\sinh \lambda - c\lambda)} d\lambda, \quad (2.2)$$

it is obvious that the integral also tends to zero when all the elements are taken in absolute value, and the integration extends over the chords and the circular arc to the left of the imaginary axis. On the arc on which the real part of λ is positive,

$$\frac{\exp \lambda}{\lambda (\sinh \lambda - c\lambda)} = \frac{2}{\lambda} + \frac{\exp(-\lambda) + 2c\lambda}{\lambda (\sinh \lambda - c\lambda)},$$

and therefore, on this part of the curve,

$$\left| \int \frac{\exp \lambda}{\lambda (\sinh \lambda - c\lambda)} d\lambda \right| \leq \left| \int \frac{2}{\lambda} d\lambda \right| + \left| \int \frac{\exp(-\lambda) + 2c\lambda}{\lambda (\sinh \lambda - c\lambda)} d\lambda \right|.$$

The second integral on the right converges to zero by (2.3), the first to 2π . Hence the integral

$$\int_{(R_n)} \left| \frac{\exp \lambda}{\lambda (\sinh \lambda - c\lambda)} d\lambda \right|$$

converges to a limit which is not greater than 2π . (2.53)

$$\text{Now} \quad \left| R \frac{\exp \lambda \omega_1}{\lambda} + iI \frac{\exp \lambda \omega_2}{\lambda} \right| < 2 \left| \frac{\exp \lambda}{\lambda} \right| \quad \text{or} \quad \left| \frac{2}{\lambda} \right|, \quad (2.54)$$

according as the real part of λ is positive or negative. The combination of (2.52) with (2.54), (2.53), (2.3), and (2.5) yields the inequality

$$\left| \lim_{n \rightarrow \infty} \int_{(R_n)} \int_{\mu}^1 \frac{\{f(t) - f(1-0)\} \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda \right| \leq 4\pi |f(\mu) - f(1-0)| < 4\pi\delta. \quad (2.55)$$

Since δ is arbitrarily small, it follows from (2.51) that

$$\int_{(\infty)} \int_v^1 \frac{f(t) \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda = 2\pi i f(1-0) \quad (-1 < v < 1), \quad (2.6)$$

if $f(1-0)$ exists and $f(t)$ is monotonic in an interval immediately on the left of 1. Since any function which is of limited total fluctuation is the sum of two monotonic functions, the integral converges to $2\pi i f(1-0)$ if $f(t)$ is of limited total fluctuation in an interval immediately on the left of 1.

Similarly, if there is an interval immediately on the right of -1 within which $f(t)$ is of limited total fluctuation,

$$\int_{(\infty)} \int_{-1}^v \frac{f(t) \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda = 2\pi i f(-1+0) \quad (-1 < v < 1). \quad (2.7)$$

Again, by the second mean value theorem, if the function $f(t)$ is monotonic in the interval $(-u, v)$,

$$\begin{aligned} \int_{-u}^v f(t) \lambda \exp \lambda t dt &= f(-u+0) R \int_{-u}^{\omega_1} \lambda \exp \lambda t dt + f(v-0) R \int_{\omega_1}^v \lambda \exp \lambda t dt \\ &\quad + i f(-u+0) I \int_{-u}^{\omega_2} \lambda \exp \lambda t dt + i f(v-0) I \int_{\omega_2}^v \lambda \exp \lambda t dt, \end{aligned}$$

where both ω_1 and ω_2 lie between $-u$ and v . Therefore

$$\begin{aligned} & \int_{(R_n)} \int_{-u}^v \frac{f(t) \lambda \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda \\ &= f(-u+0) \int_{(R_n)} \frac{R \exp \lambda \omega_1 + iI \exp \lambda \omega_2 - \exp(-\lambda u)}{\sinh \lambda - c\lambda} d\lambda \\ & \quad + f(v-0) \int_{(R_n)} \frac{\exp \lambda v - R \exp \lambda \omega_1 - iI \exp \lambda \omega_2}{\sinh \lambda - c\lambda} d\lambda, \end{aligned}$$

ω_1 and ω_2 denoting functions of λ whose upper and lower bounds are v and $-u$. If u and v are between -1 and $+1$ (exclusive), each of the integrals tends to zero as $n \rightarrow \infty$, by (2.3). Hence

$$\int_{(\infty)} \int_{-u}^v \frac{f(t) \lambda \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda = 0, \quad (2.8)$$

when $-1 < -u \leq v < +1$, if $f(t)$ is monotonic in the range $(-u, v)$, and consequently also if $f(t)$ is of limited total fluctuation in $(-u, v)$.

As in A, 272, when $v > 0$,

$$\begin{aligned} \psi_1(v) &\equiv \int_{(\infty)} \int_1^{1+v} \frac{f(t) \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda \\ &= \int_{(\infty)} \int_1^{1+v} f(t) \left\{ 2 \exp \lambda(t-1) + \frac{\exp \lambda(t-2) + 2c\lambda \exp \lambda(t-1)}{\sinh \lambda - c\lambda} \right\} dt d\lambda \\ &= \int_{(\infty)} \int_{-1}^{-1+v} \frac{f(t+2) \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda + 2c \int_{(\infty)} \int_0^v \frac{f(t+1) \lambda \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda, \end{aligned}$$

by Cauchy's theorem, since

$$\int_1^{1+v} f(t) \exp \lambda(t-1) dt$$

is an analytic function of λ . Hence

$$(i) \text{ when } 0 < v < 1, \quad \psi_1(v) = 2\pi i f(1+0), \quad (2.9)$$

by (2.7) and (2.8) if $f(t)$ is of limited total fluctuation in the interval $(1, 1+v)$;

(ii) when $v = 1$,

$$\psi_1(v) = 2\pi i \{ f(1+0) - 2cf'(2-0) \}, \quad (2.91)$$

by (2.7) and A, 272, if $f(2-0) = 0$, $f(t)$ is of limited total fluctuation in the interval $(1, 2)$, and $f'(t)$ exists in the interval $(2-\delta, 2)$ and is of limited total fluctuation in it with only a finite number of discontinuities, δ denoting a small positive number.

The conditions for the case $v = 1$ are required to justify the integration by parts by which the formula is obtained.

It may be proved in a similar way that if

$$\psi_2(v) \equiv \int_{(\infty)} \int_{-1-v}^{-1} \frac{f(t) \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda,$$

$$(i) \text{ when } 0 < v < 1, \quad \psi_2(v) = 2\pi i f(-1-0), \quad (2.92)$$

if $f(t)$ is of limited total fluctuation in $(-v-1, -1)$;

(ii) when $v = 1$,

$$\psi_2(v) = 2\pi i \{f(-1-0) + 2cf'(-2+0)\}, \quad (2.93)$$

if $f(-2+0) = 0$, $f(t)$ is of limited total fluctuation in $(-2, -1)$ and $f'(t)$ exists and is of limited total fluctuation with only a finite number of discontinuities in $(-2, -2+\delta)$.

3. If the integrals are convergent,

$$\begin{aligned} \Sigma a_n \exp \lambda_n x &= \frac{1}{4\pi i} \int_{(\infty)} \int_{a-1}^a \frac{f(t) \exp \lambda(t-x+1)}{\sinh \lambda - c\lambda} dt d\lambda \\ &\quad + \frac{1}{4\pi i} \int_{(\infty)} \int_a^{a+1} \frac{f(t) \exp \lambda(t-x-1)}{\sinh \lambda - c\lambda} dt d\lambda \quad (2.1) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{4\pi i} \int_{(\infty)} \int_{a-x}^{a-x+1} \frac{f(t+x-1) \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda \\ &\quad + \frac{1}{4\pi i} \int_{(\infty)} \int_{a-x-1}^{a-x} \frac{f(t+x+1) \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda. \quad (3.1) \end{aligned}$$

It follows that

(i) when $a-1 < x < a$,

$$\Sigma a_n \exp \lambda_n x = \frac{1}{2} \{f(x+0) + f(x-0)\},$$

by (2.6), (2.9), and (2.4), if $f(t)$ is of limited total fluctuation in $(x-\delta, a)$, δ being any small positive number;

(ii) when $x = a$,

$$\Sigma a_n \exp \lambda_n x = \frac{1}{2} \{f(x+0) + f(x-0)\},$$

by (2.6) and (2.7), if $f(t)$ is of limited total fluctuation in $(a-\delta, a+\delta)$;

(iii) when $a < x < a+1$,

$$\Sigma a_n \exp \lambda_n x = \frac{1}{2} \{f(x+0) + f(x-0)\},$$

by (2.4), (2.92), and (2.7), if $f(t)$ is of limited total fluctuation in $(a, x+\delta)$;

(iv) when $x = a-1$,

$$\Sigma a_n \exp \lambda_n x = \frac{1}{2} \{f(a-1+0) - 2cf'(a-0) + f(a+1-0)\},$$

by (2.91) and (2.6), if $f(a-0) = 0$, $f(t)$ is of limited total fluctuation in the intervals $(a-1, a)$ and $(a+1-\delta, a+1)$, and $f'(t)$ exists and is of limited total fluctuation in $(a-\delta, a)$, with only a finite number of discontinuities;

(v) when $x = a+1$,

$$\Sigma a_n \exp \lambda_n x = \frac{1}{2} \{f(a-1+0) + f(a+1-0) + 2cf'(a+0)\},$$

by (2.7) and (2.93), if $f(a+0) = 0$, $f(t)$ is of limited total fluctuation in the intervals $(a-1, a-1+\delta)$ and $(a, a+1)$, and $f'(t)$ exists and is of limited total fluctuation in $(a, a+\delta)$, with only a finite number of discontinuities.

The conditions assumed in order to establish the convergence of the series for any particular value x_1 of x ensure its convergence for all those points lying between a and x_1 at which the limits $f(x+0)$ and $f(x-0)$ exist.

Uniformity of convergence.

4. If δ denote any small positive number, the integral

$$\int_{(R_n)} \int_{-u}^v \frac{f(t) \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda$$

converges uniformly to zero as $n \rightarrow \infty$ for all values of u and v in the closed interval $(-1+\delta, 1-\delta)$, by (2.3). To establish the uniformity of convergence of the series in an interval (a, β) such that

$$a-1+\delta \leq a \leq x \leq \beta \leq a-\delta,$$

it is therefore only necessary to examine the nature of the convergence of the first integral in (3.1). This integral is the sum of the two integrals

$$\int_{(R_n)} \int_{a-x}^1 \frac{f(t+x-1) \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda \quad (4.1)$$

$$\text{and} \quad \int_{(R_n)} \int_1^{a-x+1} \frac{f(t+x-1) \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda. \quad (4.2)$$

The integral (4.1) converges for any particular value of x if $f(t)$ is of limited total fluctuation in an interval $(x-\delta, x)$. Moreover, the number μ , used in obtaining the relation (2.5) on which the convergence depends, is independent of the value of x , provided that $f(t)$ is continuous in the interval $(a-\delta, \beta)$, since continuity in any interval implies uniformity of continuity in that interval. Similarly, the convergence of the integral (4.2), which depends on that of the two integrals

$$\int_{(R_n)} \int_{-1}^{-1+a-x} \frac{f(t+x+1) \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda \quad \text{and} \quad \int_{(R_n)} \int_0^{a-x} \frac{f(t+x) \lambda \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda,$$

is uniform (n approaching ∞ through odd or even integral values only, as described in § 2) if $f(t)$ is continuous in the interval $(a, \beta+\delta)$ and of limited total fluctuation in each interval (x, a) .

Hence the series converges uniformly in any interval (a, β) lying entirely within the interval $(a-1, a)$ when $f(t)$ is continuous in an interval $(a-\delta, \beta+\delta)$ and is of limited total fluctuation in $(a-\delta, a)$.

Similarly, the convergence is uniform in an interval (a, β) lying entirely within the interval $(a, a+1)$ when $f(t)$ is continuous in $(a-\delta, \beta+\delta)$ and of limited total fluctuation in $(a, \beta+\delta)$.

Near the point a the separate integrals in (3.1) are discontinuous, but their sum is still uniformly convergent.

For, if $a-x \geq 0$, the sum of the series is the limit of

$$\begin{aligned} & \frac{1}{4\pi i} \left\{ \int_{(R_n)} \int_{a-x}^1 \frac{f(t+x-1) \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda \right. \\ & \quad + 2 \int_{(R_n)} \int_1^{1+a-x} f(t+x-1) \exp \lambda(t-1) dt d\lambda \\ & \quad + \int_{(R_n)} \int_1^{1+a-x} \frac{f(t+x-1) \exp \lambda(t-2)}{\sinh \lambda - c\lambda} dt d\lambda \\ & \quad + 2c \int_{(R_n)} \int_1^{a-x+1} \frac{f(t+x-1) \lambda \exp \lambda(t-1)}{\sinh \lambda - c\lambda} dt d\lambda \\ & \quad \left. + \int_{(R_n)} \int_{a-x-1}^{a-x} \frac{f(t+x+1) \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda \right\} \\ &= \frac{1}{4\pi i} \left\{ \int_{(R_n)} \int_{a-x}^1 \frac{f(t+x-1) \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda + \int_{(R_n)} \int_{-1}^{a-x-1} \frac{f(t+x+1) \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda \right. \\ & \quad + 2c \int_{(R_n)} \int_0^{a-x} \frac{f(t+x) \lambda \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda \\ & \quad \left. + \int_{(R_n)} \int_{a-x-1}^{a-x} \frac{f(t+x+1) \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda \right\}. \end{aligned}$$

In this expression the parts which are discontinuous at $x = a$ are the second and fourth integrals, which can be combined into the integral

$$\int_{(R_n)} \int_{-1}^{a-x} \frac{f(t+x+1) \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda.$$

The series is therefore uniformly convergent in any interval (α, β) , lying entirely within the interval $(a-1, a+1)$ and containing the point a , when $f(x)$ is continuous and of limited total fluctuation in $(\alpha-\delta, \beta+\delta)$.

Term by term integration of the expansion.

5. It will be proved in the following section that, if the Dixon expansion associated with any bounded, integrable function $f(x)$ is integrated term by term between limits which are interior points of the interval $(a-1, a+1)$, the resulting series converges to the definite integral of $f(x)$. The convergence is uniform when the range of integration lies in $(a-1+\delta, a+1-\delta)$ and $\delta > 0$.

If $m = 1$ or 2 and $f(t)$ is of limited total fluctuation in a small interval whose end points are a and $a + (-)^m \delta$, the series obtained by integration converges also when the upper limit of integration is $a + (-1)^m$, the sum being $(-)^m cf\{a + (-)^m 0\}$ together with the integral of $f(x)$.

The limits of the integrals

$$\int_{(R_n)} \int_{-1-v}^{-1} \frac{f(t) \exp \lambda t}{\lambda (\sinh \lambda - c\lambda)} dt d\lambda$$

and

$$\int_{(R_n)} \int_1^{1+v} \frac{f(t) \exp \lambda t}{\lambda (\sinh \lambda - c\lambda)} dt d\lambda \quad (0 < v \leq 1)$$

will be required.

If $v > 0$,

$$\begin{aligned} & \int_{(R_n)} \int_{-1-v}^{-1} \frac{f(t) \exp \lambda t}{\lambda (\sinh \lambda - c\lambda)} dt d\lambda \\ &= \int_{(R_n)} \int_{-1-v}^{-1} \frac{2f(t) \exp \lambda(t+1)}{\lambda} \\ & \quad \times \frac{\{-\exp \lambda + \exp(-\lambda) + 2c\lambda\} + \{\exp \lambda - 2c\lambda\}}{\exp \lambda - \exp(-\lambda) - 2c\lambda} dt d\lambda \\ &= -2 \int_{(R_n)} \int_{-1-v}^{-1} \frac{f(t) \exp \lambda(t+1)}{\lambda} dt d\lambda \\ & \quad + \int_{(R_n)} \int_{-1-v}^{-1} \frac{f(t) \exp \lambda(t+2)}{\lambda (\sinh \lambda - c\lambda)} dt d\lambda - 2c \int_{(R_n)} \int_{-1-v}^{-1} \frac{f(t) \exp \lambda(t+1)}{\sinh \lambda - c\lambda} dt d\lambda. \end{aligned}$$

The integral
$$\int_{-1-v}^{-1} \frac{f(t) \exp \lambda(t+1)}{\lambda} dt$$

is an analytic function of λ everywhere except at $\lambda = 0$, where it has a pole with residue $\int_{-1-v}^{-1} f(t) dt$. Also

$$\int_{(R_n)} \int_{-1-v}^{-1} \frac{f(t) \exp \lambda(t+2)}{\lambda (\sinh \lambda - c\lambda)} dt d\lambda = \int_{(R_n)} \int_{1-v}^1 \frac{f(t-2) \exp \lambda t}{\lambda (\sinh \lambda - c\lambda)} dt d\lambda,$$

which tends to zero uniformly when $0 \leq v \leq 1$, by (2.4); and

$$\int_{(R_n)} \int_{-1-v}^{-1} \frac{f(t) \exp \lambda(t+1)}{\sinh \lambda - c\lambda} dt d\lambda = \int_{(R_n)} \int_{-v}^0 \frac{f(t-1) \exp \lambda t}{\sinh \lambda - c\lambda} dt d\lambda,$$

which tends to zero, by (2.4), when $0 \leq v < 1$ (uniformly for $v \leq 1-\delta$), but to $2\pi i f(-2+0)$, by (2.7), when $v = 1$, if there is an interval immediately on the right of -2 within which $f(t)$ is of limited total fluctuation.

Thus, when $0 \leq v < 1$,

$$\lim_{n \rightarrow \infty} \int_{(R_n)} \int_{-1-v}^{-1} \frac{f(t) \exp \lambda t}{\lambda (\sinh \lambda - c\lambda)} dt d\lambda = -4\pi i \int_{-1-v}^{-1} f(t) dt, \quad (5.1)$$

the convergence being uniform when $0 \leq v \leq 1-\delta$; and, when $v = 1$,

$$\lim_{n \rightarrow \infty} \int_{(R_n)} \int_{-1-v}^{-1} \frac{f(t) \exp \lambda t}{\lambda (\sinh \lambda - c\lambda)} dt d\lambda = -4\pi i \int_{-1-v}^{-1} f(t) dt - 4c\pi i f(-2+0), \quad (5.11)$$

if there is an interval immediately on the right of -2 within which $f(t)$ is of limited total fluctuation.

Similarly, when $v \geq 0$,

$$\begin{aligned} & \int_{(R_n)} \int_1^{1+v} \frac{f(t) \exp \lambda t}{\lambda (\sinh \lambda - c\lambda)} dt d\lambda \\ &= 2 \int_{(R_n)} \int_1^{1+v} \frac{f(t) \exp \lambda(t-1)}{\lambda} dt d\lambda + \int_{(R_n)} \int_1^{1+v} \frac{f(t) \exp \lambda(t-2)}{\lambda (\sinh \lambda - c\lambda)} dt d\lambda \\ & \quad + 2c \int_{(R_n)} \int_1^{1+v} \frac{f(t) \exp \lambda(t-1)}{\sinh \lambda - c\lambda} dt d\lambda; \end{aligned}$$

therefore, when $0 \leq v < 1$,

$$\lim_{n \rightarrow \infty} \int_{(R_n)} \int_1^{1+v} \frac{f(t) \exp \lambda t}{\lambda (\sinh \lambda - c\lambda)} dt d\lambda = 4\pi i \int_1^{1+v} f(t) dt, \quad (5.2)$$

by (2.4), the convergence being uniform when $0 \leq v \leq 1 - \delta$; and, when $v = 1$,

$$\lim_{n \rightarrow \infty} \int_{(R_n)} \int_1^{1+v} \frac{f(t) \exp \lambda t}{\lambda (\sinh \lambda - c\lambda)} dt d\lambda = 4\pi i \int_1^{1+v} f(t) dt + 4c\pi i f(2-0), \quad (5.21)$$

by (2.4) and (2.6), if there is an interval immediately on the left of 2 within which $f(t)$ is of limited total fluctuation.

6. Now apply these results to the functions

$$\phi_1(x) = \int_{(\infty)} \int_a^{a+1} \int_y^{y+1} \frac{f(y) \exp \lambda(t-x-1)}{\sinh \lambda - c\lambda} dt dy d\lambda$$

and
$$\phi_2(x) = - \int_{(\infty)} \int_{a-1}^a \int_{a-1}^y \frac{f(y) \exp \lambda(t-x+1)}{\sinh \lambda - c\lambda} dt dy d\lambda.$$

$$\begin{aligned} \phi_1(x) &= \int_{(\infty)} \int_a^{a+1} \frac{f(y)}{\sinh \lambda - c\lambda} \cdot \frac{\exp \lambda(a-x) - \exp \lambda(y-x-1)}{\lambda} dy d\lambda \\ &= \int_{(\infty)} \int_a^{a+1} \frac{f(y) \exp \lambda(a-x)}{\lambda (\sinh \lambda - c\lambda)} dy d\lambda - \int_{(\infty)} \int_{a-x-1}^{a-x} \frac{f(y+x+1) \exp \lambda y}{\lambda (\sinh \lambda - c\lambda)} dy d\lambda. \end{aligned}$$

(i) When $x = a-1$,
$$\phi_1(x) = 2\pi i \int_a^{a+1} f(y) dy,$$

by (2.3) and (2.4).

(iia) When $a-1 < x \leq a$,
$$\phi_1(x) = 0,$$

by (2.3) and (2.4), the integral converging uniformly when $a-1+\delta \leq x \leq a$.

(iib) When $a \leq x < a+1$,

$$\begin{aligned} \phi_1(x) &= 4\pi i \int_{a-x-1}^{-1} f(y+x+1) dy \\ &= 4\pi i \int_a^x f(y) dy, \end{aligned}$$

by (2.3), (2.4), and (5.1), the convergence being uniform when $a \leq x \leq a+1-\delta$.

(iii) When $x = a+1$,

$$\begin{aligned} \phi_1(x) &= -2\pi i \int_a^{a+1} f(y) dy + 4\pi i \int_{-2}^{-1} f(y+a+2) dy + 4c\pi i f(x-1+0) \\ &= 2\pi i \int_a^{a+1} f(y) dy + 4c\pi i f(a+0), \end{aligned}$$

by (2.3) and (5.11), if there is an interval immediately on the right of a within which $f(t)$ is of limited total fluctuation.

$$\begin{aligned}\phi_2(x) &= \int_{(\infty)} \int_{a-1}^a \frac{f(y)}{\sinh \lambda - c\lambda} \cdot \frac{\exp \lambda(a-x) - \exp \lambda(y-x+1)}{\lambda} dy d\lambda \\ &= \int_{(\infty)} \int_{a-1}^a \frac{f(y) \exp \lambda(a-x)}{\lambda(\sinh \lambda - c\lambda)} dy d\lambda - \int_{(\infty)} \int_{a-x}^{a-x+1} \frac{f(y+x-1) \exp \lambda y}{\lambda(\sinh \lambda - c\lambda)} dy d\lambda,\end{aligned}$$

and therefore,

(i) when $x = a-1$,

$$\begin{aligned}\phi_2(x) &= 2\pi i \int_{a-1}^a f(y) dy - 4\pi i \int_1^2 f(y+a-2) dy - 4c\pi i f(a-0) \\ &= -2\pi i \int_{a-1}^a f(y) dy - 4c\pi i f(a-0),\end{aligned}$$

by (2.3) and (5.21), if there is an interval immediately on the left of a within which $f(t)$ is of limited total fluctuation;

(iia) when $a-1 < x \leq a$,

$$\begin{aligned}\phi_2(x) &= -4\pi i \int_1^{a-x+1} f(y+x-1) dy \\ &= 4\pi i \int_a^x f(y) dy,\end{aligned}$$

by (2.3), (2.4), and (5.2), the convergence being uniform when $a-1+\delta \leq x \leq a$;

(iib) when $a \leq x < a+1$, $\phi_2(x) = 0$,

by (2.3) and (2.4), the convergence being uniform when $a \leq x \leq a+1-\delta$;

(iii) When $x = a+1$,

$$\phi_2(x) = -2\pi i \int_{a-1}^a f(y) dy,$$

by (2.3) and (2.4).

Hence

(i) when $x = a-1$,

$$\phi_1(x) + \phi_2(x) = 2\pi i \int_a^{a+1} f(y) dy - 2\pi i \int_{a-1}^a f(y) dy - 4c\pi i f(a-0), \quad (6.1)$$

if $f(t)$ is of limited total fluctuation in $(a-\delta, a)$;

(ii) when $a-1 < x < a+1$,

$$\phi_1(x) + \phi_2(x) = 4\pi i \int_a^x f(y) dy, \quad (6.2)$$

the integrals converging uniformly when $a-1+\delta \leq x \leq a+1-\delta$;

(iii) when $x = a+1$,

$$\phi_1(x) + \phi_2(x) = 2\pi i \int_a^{a+1} f(y) dy - 2\pi i \int_{a-1}^a f(y) dy + 4c\pi i f(a+0), \quad (6.3)$$

if $f(t)$ is of limited total fluctuation in $(a, a+\delta)$.

But $\phi_1(x) + \phi_2(x)$ can be expanded by means of Cauchy's theorem. The residue at $-\lambda_n$ ($n \neq 0$) is

$$\begin{aligned} & \int_a^{a+1} \int_y^{a+1} \frac{f(y) \exp\{-\lambda_n(t-x-1)\}}{\cosh \lambda_n - c} dt dy \\ & \quad - \int_{a-1}^a \int_{a-1}^y \frac{f(y) \exp\{-\lambda_n(t-x+1)\}}{\cosh \lambda_n - c} dt dy \\ &= - \frac{\exp \lambda_n x}{\lambda_n (\cosh \lambda_n - c)} \left\{ \int_a^{a+1} f(y) \exp(-\lambda_n a) dy - \int_a^{a+1} f(y) \exp\{-\lambda_n(y-1)\} dy \right. \\ & \quad \left. - \int_{a-1}^a f(y) \exp\{-\lambda_n(y+1)\} dy + \int_{a-1}^a f(y) \exp(-\lambda_n a) dy \right\} \\ &= - \frac{\exp \lambda_n x}{\lambda_n (\cosh \lambda_n - c)} \left\{ \exp(-\lambda_n a) \int_{a-1}^{a+1} f(y) dy - \int_{a-1}^a f(y) \exp\{-\lambda_n(y+1)\} dy \right. \\ & \quad \left. - \int_a^{a+1} f(y) \exp\{-\lambda_n(y-1)\} dy \right\} \\ &= - \frac{\exp \lambda_n (x-a)}{\lambda_n (\cosh \lambda_n - c)} \int_{a-1}^{a+1} f(y) dy + \frac{2a_n}{\lambda_n} \exp \lambda_n x, \end{aligned}$$

where a_n is the coefficient of $\exp \lambda_n x$ in the Dixon expansion associated with $f(x)$.

The residue at the origin is

$$\int_a^{a+1} \int_y^{a+1} \frac{f(y)}{1-c} dt dy - \int_{a-1}^a \int_{a-1}^y \frac{f(y)}{1-c} dt dy.$$

Because $f(y)$ is integrable in the ranges $(a-1, a)$ and $(a, a+1)$, the double integral $\iint f(y)(dy dt)$ exists, and therefore the order of integration may be reversed. The residue at $\lambda = 0$ is therefore

$$\frac{1}{1-c} \left\{ \int_a^{a+1} \int_a^t f(y) dy dt + \int_{a-1}^a \int_a^t f(y) dy dt \right\} = \frac{1}{1-c} \int_{a-1}^{a+1} F(t) dt,$$

if the integral $\int_a^t f(y) dy$ is denoted by $F(t)$.

The expansion of $\phi_1(x) + \phi_2(x)$ is therefore

$$\frac{2\pi i}{1-c} \int_{a-1}^{a+1} F(t) dt - 2\pi i \Sigma' \frac{\exp \lambda_n(x-a)}{\lambda_n(\cosh \lambda_n - c)} \int_{a-1}^{a+1} f(y) dy + 4\pi i \Sigma' \frac{a_n \exp \lambda_n x}{\lambda_n},$$

the prime indicating that the summation extends over all the roots of the equation $\sinh \lambda - c\lambda = 0$ (arranged in the order of non-decreasing moduli) except the root $\lambda = 0$.

The first term, a_0 , in the expansion of $f(x)$ is

$$\frac{1}{2} \int_{a-1}^{a+1} \frac{f(y)}{1-c} dy,$$

and therefore

$$\begin{aligned} \phi_1(x) + \phi_2(x) = & \frac{2\pi i}{1-c} \int_{a-1}^{a+1} F(t) dt - 4\pi i(1-c) a_0 \Sigma' \frac{\exp \lambda_n(x-a)}{\lambda_n(\cosh \lambda_n - c)} \\ & + 4\pi i \Sigma' \frac{a_n \exp \lambda_n x}{\lambda_n}. \quad (6.4) \end{aligned}$$

The function $x-a$ vanishes when $x=a$ and satisfies all the other conditions of the theorem of § 3 throughout the closed interval $(a-1, a+1)$. It follows that its Dixon expansion converges everywhere in this interval, to $-c$ when $x=a-1$, to $+c$ when $x=a+1$, and to $x-a$ for all intermediate values of x . When $\lambda_n \neq 0$, the coefficient of $\exp \lambda_n x$ in its expansion is

$$\begin{aligned} & \frac{1}{2(\cosh \lambda_n - c)} \left(\int_{a-1}^a (t-a) \exp \{-\lambda_n(t+1)\} dt \right. \\ & \quad \left. + \int_a^{a+1} (t-a) \exp \{-\lambda_n(t-1)\} dt \right) \\ &= \frac{1}{2(\cosh \lambda_n - c)} \left(\left[\frac{(t-a) \exp \{-\lambda_n(t+1)\}}{-\lambda_n} + \frac{\exp \{-\lambda_n(t+1)\}}{-\lambda_n^2} \right]_{a-1}^a \right. \\ & \quad \left. + \left[\frac{(t-a) \exp \{-\lambda_n(t-1)\}}{-\lambda_n} + \frac{\exp \{-\lambda_n(t-1)\}}{-\lambda_n^2} \right]_a^{a+1} \right) \\ &= \frac{1}{2(\cosh \lambda_n - c)} \left(\frac{\exp(-\lambda_n a) + \exp(-\lambda_n a)}{-\lambda_n} \right. \\ & \quad \left. + \frac{\exp\{-\lambda_n(a+1)\} - \exp(-\lambda_n a) + \exp(-\lambda_n a) - \exp\{-\lambda_n(a-1)\}}{-\lambda_n^2} \right) \\ &= \frac{\exp(-\lambda_n a)}{\cosh \lambda_n - c} \left(-\frac{1}{\lambda_n} + \frac{\sinh \lambda_n}{\lambda_n^2} \right) \\ &= \frac{\exp(-\lambda_n a)}{\lambda_n(\cosh \lambda_n - c)} (-1+c). \end{aligned}$$

The coefficient of $\exp \lambda_0 x$ is

$$\frac{1}{2(1-c)} \int_{a-1}^{a+1} (t-a) dt = 0.$$

Thus the series

$$\sum' \frac{(c-1) \exp \lambda_n (x-a)}{\lambda_n (\cosh \lambda_n - c)}$$

converges to $-c$ when $x = a-1$, to $x-a$ when $a-1 < x < a+1$, and to $+c$ when $x = a+1$.

$$\text{When } x = a-1, \quad -c = (x-a) + (1-c).$$

$$\text{When } x = a+1, \quad +c = (x-a) + (-1+c).$$

Let the function $h(x)$ be defined in the interval $(a-1, a+1)$ by the formulae :—

$$h(x) = -(1-c) \text{ when } x = a-1;$$

$$h(x) = 0 \text{ when } a-1 < x < a+1;$$

$$h(x) = +(1-c) \text{ when } x = a+1.$$

Then (6.4) may be written

$$\sum' \frac{a_n \exp \lambda_n x}{\lambda_n} + (x-a) a_0 = \frac{\phi_1(x) + \phi_2(x)}{4\pi i} - \frac{1}{2(1-c)} \int_{a-1}^{a+1} F(t) dt + a_0 h(x). \quad (6.5)$$

When $x = a$, this relation becomes

$$\sum' \frac{a_n \exp \lambda_n a}{\lambda_n} = \frac{-1}{2(1-c)} \int_{a-1}^{a+1} F(t) dt, \quad (6.6)$$

since $\phi_1(a) + \phi_2(a) = 0$, by (6.2). It follows, from (6.5) and (6.6), by subtraction, that

$$\sum' \frac{a_n (\exp \lambda_n x - \exp \lambda_n a)}{\lambda_n} + (x-a) a_0 = \frac{1}{4\pi i} \{ \phi_1(x) + \phi_2(x) \} + a_0 h(x). \quad (6.7)$$

When the values obtained for $\phi_1(x) + \phi_2(x)$ in (6.1), (6.2), and (6.3) are substituted in (6.7) it gives :—

(i) when $x = a-1$, if $f(t)$ is of limited total fluctuation in a small interval $(a-\delta, a)$,

$$\begin{aligned} \sum' \frac{a_n (\exp \lambda_n x - \exp \lambda_n a)}{\lambda_n} + (x-a) a_0 &= \frac{1}{2} \left\{ \int_a^{a+1} f(y) dy - \int_{a-1}^a f(y) dy \right\} - cf(a-0) - a_0(1-c) \\ &= \frac{1}{2} \left\{ \int_{a-1}^{a+1} f(y) dy + 2 \int_a^{a-1} f(y) dy \right\} - cf(a-0) - a_0(1-c) \\ &= \int_a^x f(y) dy - cf(a-0); \end{aligned}$$

(ii) when $a-1 < x < a+1$,

$$\sum' \frac{a_n(\exp \lambda_n x - \exp \lambda_n a)}{\lambda_n} + (x-a) a_0 = \int_a^x f(y) dy,$$

the convergence being uniform when $a-1+\delta \leq x \leq a+1-\delta$;

(iii) when $x = a+1$, if $f(t)$ is of limited total fluctuation in $(a, a+\delta)$,

$$\begin{aligned} \sum' \frac{a_n(\exp \lambda_n x - \exp \lambda_n a)}{\lambda_n} + (x-a) a_0 \\ &= \frac{1}{2} \left\{ \int_a^{a+1} f(y) dy - \int_{a-1}^a f(y) dy \right\} + cf(a+0) + a_0(1-c) \\ &= \frac{1}{2} \left\{ 2 \int_a^{a+1} f(y) dy - \int_{a-1}^{a+1} f(y) dy \right\} + cf(a+0) + a_0(1-c) \\ &= \int_a^x f(y) dy + cf(a+0). \end{aligned}$$

$$\text{Now} \quad \sum' \frac{a_n(\exp \lambda_n x - \exp \lambda_n a)}{\lambda_n} + (x-a) a_0$$

is the result of integrating the Dixon expansion associated with the function $f(x)$ term by term between the limits a and x . Thus the theorem stated at the beginning of § 5 has been proved.

These results have been established without any assumptions concerning the convergence of the Dixon expansion of the function $f(x)$.

Term by term differentiation of the expansion.

7. When the function $f(x)$ is not continuous, and $f(x)$ and $f'(x)$ possess a finite number of non-removable discontinuities in the interval $(a-1, a+1)$ and are of limited total fluctuation [$f'(x)$ existing everywhere except at the points of discontinuity of $f(x)$], the result of term by term differentiation of the Dixon expansion associated with $f(x)$ is not identical with the Dixon expansion of $f'(x)$. If, however, $f(a+0)$ or $f(a-0)$ vanishes, there is an interval in which the two series differ by a null function.

$$\text{Let} \quad a_0 + \sum_{n=1}^{\infty} a_n \exp \lambda_n x \quad (7.1)$$

be the Dixon expansion of a function $f(x)$ which possesses a differential coefficient throughout the interval $(a-1, a+1)$ except at a finite number of points.

Then

$$a_0 = \frac{1}{2(1-c)} \int_{a-1}^{a+1} f(t) dt,$$

$$a_n = \frac{1}{2(\cosh \lambda_n - c)} \left\{ \int_{a-1}^a f(t) \exp \{ -\lambda_n(t+1) \} dt \right. \\ \left. + \int_a^{a+1} f(t) \exp \{ -\lambda_n(t-1) \} dt \right\} \quad (\lambda_n \neq 0).$$

Let a'_n denote the coefficient of $\exp \lambda_n x$ in the Dixon expansion of $f'(x)$.

Assume that the intervals $(a-1, a)$ and $(a, a+1)$ can be divided into a finite number of sub-intervals $(a-1, l_m)$, (l_m, l_{m-1}) , ..., (l_1, a) , and (a, p_1) , (p_1, p_2) , ..., $(p_\mu, a+1)$, in which $f(x)$ and $f'(x)$ are continuous and of limited total fluctuation. Then integration by parts may be employed, giving

$$2a_n(\cosh \lambda_n - c) \\ = \left[\frac{f(t) \exp \{ -\lambda_n(t+1) \}}{-\lambda_n} \right]_{a-1}^{l_m} + \left[\frac{f(t) \exp \{ -\lambda_n(t+1) \}}{-\lambda_n} \right]_{l_m}^{l_{m-1}} + \dots \\ + \left[\frac{f(t) \exp \{ -\lambda_n(t+1) \}}{-\lambda_n} \right]_{l_1}^a + \frac{1}{\lambda_n} \int_{a-1}^a f'(t) \exp \{ -\lambda_n(t+1) \} dt \\ + \left[\frac{f(t) \exp \{ -\lambda_n(t-1) \}}{-\lambda_n} \right]_a^{p_1} + \left[\frac{f(t) \exp \{ -\lambda_n(t-1) \}}{-\lambda_n} \right]_{p_1}^{p_2} + \dots \\ + \left[\frac{f(t) \exp \{ -\lambda_n(t-1) \}}{-\lambda_n} \right]_{p_\mu}^{a+1} + \frac{1}{\lambda_n} \int_a^{a+1} f'(t) \exp \{ -\lambda_n(t-1) \} dt.$$

Thus, when $\lambda_n \neq 0$,
$$a_n = -\frac{A_n}{\lambda_n} + \frac{a'_n}{\lambda_n},$$

where

$$A_n = \frac{1}{2(\cosh \lambda_n - c)} \\ \times \left\{ \left[f(t) \exp \{ -\lambda_n(t+1) \} \right]_{a-1}^{l_m} + \dots + \left[f(t) \exp \{ -\lambda_n(t+1) \} \right]_{l_1}^a \right. \\ \left. + \left[f(t) \exp \{ -\lambda_n(t-1) \} \right]_a^{p_1} + \dots + \left[f(t) \exp \{ -\lambda_n(t-1) \} \right]_{p_\mu}^{a+1} \right\}. \quad (7.2)$$

The result of differentiating the series (7.1) term by term is therefore

$$\sum_{n=1}^{\infty} (-A_n + a'_n) \exp \lambda_n x.$$

If $A_n = 0$ for all values of x , which happens when $f(x)$ is everywhere continuous, $f(a) = 0$, and $f(a+1-0) = f(a-1+0)$, the derived series is

$$\sum_{n=1}^{\infty} a'_n \exp \lambda_n x,$$

which converges to $\frac{1}{2}f'(x+0) + \frac{1}{2}f'(x-0) - a'_0$ at all internal points of the interval $(a-1, a+1)$, since these limits exist, and $f'(x)$ is of limited total fluctuation. (7.3)

If $f(a \pm 0) = 0$ but the saltus at each of the points $l_1, l_2, \dots, l_m, p_1, p_2, \dots, p_\mu$ is not zero, from the definition (7.2),

$$\begin{aligned} 2A_n(\cosh \lambda_n - c) &= \sum_{r=1}^m \{f(l_r-0) - f(l_r+0)\} \exp \{-\lambda_n(l_r+1)\} \\ &\quad + \sum_{r=1}^{\mu} \{f(p_r-0) - f(p_r+0)\} \exp \{-\lambda_n(p_r-1)\} \\ &\quad + \{f(a+1-0) - f(a-1+0)\} \exp(-\lambda_n a). \end{aligned}$$

The part of $-\sum_{n=1}^{\infty} A_n \exp \lambda_n x$ due to the saltus at l_r is therefore

$$\begin{aligned} \{f(l_r+0) - f(l_r-0)\} \sum_{n=1}^{\infty} \frac{\exp \lambda_n(x-l_r-1)}{2(\cosh \lambda_n - c)} \\ = \{f(l_r+0) - f(l_r-0)\} \left\{ \frac{1}{2} \sum_{n=0}^{\infty} \frac{\exp \lambda_n(x-l_r-1)}{\cosh \lambda_n - c} - \frac{1}{2(1-c)} \right\}. \end{aligned}$$

Thus

$$\begin{aligned} -\sum_{n=1}^{\infty} A_n \exp \lambda_n x \\ = \sum_{r=1}^m \{f(l_r+0) - f(l_r-0)\} \left\{ \frac{1}{2} \sum_{n=0}^{\infty} \frac{\exp \lambda_n(x-l_r-1)}{\cosh \lambda_n - c} - \frac{1}{2(1-c)} \right\} \\ + \sum_{r=1}^{\mu} \{f(p_r+0) - f(p_r-0)\} \left\{ \frac{1}{2} \sum_{n=0}^{\infty} \frac{\exp \lambda_n(x-p_r+1)}{\cosh \lambda_n - c} - \frac{1}{2(1-c)} \right\} \\ + \{f(a-1+0) - f(a+1-0)\} \left\{ \frac{1}{2} \sum_{n=0}^{\infty} \frac{\exp \lambda_n(x-a)}{\cosh \lambda_n - c} - \frac{1}{2(1-c)} \right\}. \end{aligned}$$

Now

$$\begin{aligned} a'_0 &= \frac{1}{2(1-c)} \int_{a-1}^{a+1} f(t) dt \\ &= \frac{1}{2(1-c)} \{ [f(t)]_{a-1}^{l_m} + [f(t)]_{l_m}^{l_{m-1}} + \dots \\ &\quad + [f(t)]_{l_1}^{a'} + [f(t)]_{a'}^{p_1} + [f(t)]_{p_1}^{p_2} + \dots + [f(t)]_{p_\mu}^{a+1} \} \\ &= \frac{1}{2(1-c)} \left(f(a+1-0) - f(a-1+0) - \sum_{r=1}^m \{f(l_r-0) - f(l_r+0)\} \right. \\ &\quad \left. + \sum_{r=1}^{\mu} \{f(p_r-0) - f(p_r+0)\} \right). \end{aligned}$$

Hence

$$\begin{aligned} & \sum_{n=1}^{\infty} (-A_n \exp \lambda_n x + a'_n \exp \lambda_n x) \\ &= a'_0 + \sum_{n=1}^{\infty} a'_n \exp \lambda_n x + \frac{1}{2} \{f(a-1+0) - f(a+1-0)\} \sum_{n=0}^{\infty} \frac{\exp \lambda_n (x-a)}{\cosh \lambda_n - c} \\ & \quad + \frac{1}{2} \sum_{r=1}^m \{f(l_r+0) - f(l_r-0)\} \sum_{n=0}^{\infty} \frac{\exp \lambda_n (x-l_r-1)}{\cosh \lambda_n - c} \\ & \quad + \frac{1}{2} \sum_{r=1}^{\mu} \{f(p_r+0) - f(p_r-0)\} \sum_{n=0}^{\infty} \frac{\exp \lambda_n (x-p_r+1)}{\cosh \lambda_n - c}. \quad (7.4) \end{aligned}$$

It is shown in A, 276, by considering the first of the integrals (2.2), that the series

$$\sum_{n=0}^{\infty} \frac{\exp \lambda_n x}{\cosh \lambda_n - c}$$

converges to zero for all values of x between -1 and $+1$ exclusive, but diverges for these extreme values. The series of this form on the right in (7.4) converge to zero so long as each of the arguments lies between -1 and $+1$ exclusive, *i.e.* so long as

$$\begin{cases} -1 < x - l_r - 1 < +1 & (r = 1, 2, \dots, m), \\ -1 < x - p_r + 1 < +1 & (r = 1, 2, \dots, \mu), \\ -1 < x - a < +1, \end{cases}$$

i.e. provided that $l_1 < x < p_1$.

When $0 < x < 1$,

$$\sum_n \frac{\exp \lambda_n (x+1)}{\cosh \lambda_n - c} = 2 \sum_n \exp \lambda_n x - \sum_n \frac{\exp \lambda_n (x-1)}{\cosh \lambda_n - c} + 2c \sum_n \frac{\exp \lambda_n x}{\cosh \lambda_n - c}. \quad (7.5)$$

Of these series,

$$\sum_n \frac{\exp \lambda_n (x-1)}{\cosh \lambda_n - c} \quad \text{and} \quad \sum_n \frac{\exp \lambda_n x}{\cosh \lambda_n - c}$$

converge, while $\sum \exp \lambda_n x$ diverges because $\exp \lambda_n x$ does not tend to zero as $n \rightarrow \infty$. Hence the series on the left in (7.5) diverges when $0 < x < 1$. It is already known to diverge when $x = 0$.

Similarly, it may be shown that, if $x_1, x_2, x_3, \dots$ are interior points of the interval $(0, 1)$ and are finite in number and all different from one

another, the series

$$\sum_n \frac{1}{\cosh \lambda_n - c} \{A \exp \lambda_n(1+x_1) + B \exp \lambda_n(1+x_2) + C \exp \lambda_n(1+x_3) + \dots\}$$

diverges unless $A = B = C \dots = 0$.

The relation (7.4) shows therefore that, if $p_1 \leq x \leq a+1$ or $a-1 \leq x \leq l_1$, the series $\sum A_n \exp \lambda_n x$ diverges. Thus the following theorem has been proved.

If the interval $(a-1, a+1)$ can be divided into a finite number of sub-intervals in which $f(x)$ and $f'(x)$ are continuous and of limited total fluctuation, and if $f(a \pm 0) = 0$, the series obtained by term by term differentiation of the Dixon expansion of $f(x)$ differs by a null function from the Dixon expansion of $f'(x)$ at all internal points of the interval (l_1, p_1) , where l_1, p_1 are the two points of non-removable discontinuity of $f(x)$ [i.e. such that $f(x+0) \neq f(x-0)$] which lie nearest to a , one on each side of it. If $l_1 < a$ and x has either of the values l_1, p_1 , or if it lies within one of the intervals $(a-1, l_1)$, $(p_1, a+1)$, the series obtained by differentiation diverges.

If, further, $f(x)$ has no non-removable discontinuity in $(a-1, a+1)$, the derived series is the Dixon expansion of $f'(x)$ throughout the closed interval, provided that $f(a-1+0) = f(a+1-0)$. (7.3)

If $f(a \pm 0) \neq 0$, the extra term occurring in $-\sum A_n \exp \lambda_n x$, from (7.2), is

$$-f(a-0) \sum_{n=1}^{\infty} \frac{\exp \lambda_n(x-a-1)}{2(\cosh \lambda_n - c)} + f(a+0) \sum_{n=1}^{\infty} \frac{\exp \lambda_n(x-a+1)}{2(\cosh \lambda_n - c)}.$$

The first of these series converges only in the range $a < x < a+2$, the second only in the range $a-2 < x < a$.

Thus, if neither $f(a+0)$ nor $f(a-0)$ vanishes, the derived series diverges at all points (except perhaps at $a \pm 1$); if $f(a+0) = 0$, the derived series converges to $\frac{1}{2}\{f'(x+0) + f'(x-0)\}$ at all points in the open interval (a, p_1) ; similarly, if $f(a-0) = 0$, it converges at all points in the open interval (l_1, a) .

Analogue of Parseval's theorem for Fourier series.

8. It is convenient in this section to change the notation, still defining λ_0 as 0, but denoting by $\lambda_1, \lambda_2, \dots, \lambda_n, \dots$ only those roots of the equation $\sinh \lambda - c\lambda = 0$ whose real parts are positive. The remaining roots are $-\lambda_1, -\lambda_2, \dots, -\lambda_n, \dots$.

Let

$$\sum_{n=0}^{\infty} \{a_n \exp \lambda_n x + b_n \exp (-\lambda_n x)\} \quad \text{and} \quad \sum_{n=0}^{\infty} \{a_n \exp \lambda_n x + \beta_n \exp (-\lambda_n x)\},$$

where $b_0 = \beta_0 = 0$, be the Dixon expansions associated in the interval $(a-1, a+1)$ with the bounded functions $f(x)$ and $\phi(x)$ respectively.

Assume that the functions $f(x)$ and $\phi(x)$ satisfy the following conditions:— $f(a \pm 0) = 0$; $\phi(a \pm 0) = 0$; $\phi(x)$ is continuous in two small intervals, one on each side of a , with a as end point; $f(x)$ is continuous and of limited total fluctuation in $(a-1, a+1)$; $f'(x)$ exists in a small interval $(a-\delta', a+\delta')$ and is of limited total fluctuation in it, with only a finite number of discontinuities.

It will be proved that

$$\int_{a-1}^a \phi(x) f(x+1) dx + \int_a^{a+1} \phi(x) f(x-1) dx = 2 \sum_{n=0}^{\infty} (\cosh \lambda_n - c)(a_n \beta_n - b_n a_n).$$

The conditions stated above ensure the uniform convergence of the expansion for $f(x)$ in any closed interval interior to $(a-1, a+1)$ and its convergence at all points of the closed interval $(a-1, a+1)$.

Let K denote the upper bound of

$$|\phi(x)|, \quad |f(x)|, \quad \text{and} \quad \left| \sum_{n=0}^p \{a_n \exp \lambda_n x + b_n \exp (-\lambda_n x)\} \right|$$

($p = 0, 1, 2, \dots$)

in the closed interval $(a-1, a+1)$.

Because $\phi(x)$ is continuous on the left of a , a number δ can be determined such that, when $a-\delta \leq x \leq a$,

$$|\phi(x)| < \frac{\epsilon}{4K},$$

ϵ being an arbitrarily small positive number. Then, for these values of x ,

$$\begin{aligned} |\phi(x)| \cdot \left| f(x+1) - \sum_{n=0}^m \{a_n \exp \lambda_n (x+1) + b_n \exp \{-\lambda_n (x+1)\}\} \right| &< \frac{\epsilon}{4K} \cdot 2K \\ &< \frac{1}{2}\epsilon \quad (8.1) \end{aligned}$$

for all values of m , since the series

$$\sum \{a_n \exp \lambda_n (x+1) + b_n \exp \{-\lambda_n (x+1)\}\}$$

converges for all values of x in the closed interval $(a-2, a)$.

The series

$$\Sigma \left(a_n \exp \lambda_n(x+1) + b_n \exp \{ -\lambda_n(x+1) \} \right)$$

converges uniformly to $f(x+1)$ when $a-1 \leq x \leq a-\delta$. It is therefore possible to determine a number m_0 , such that, for all values of x in this range and for all values of m greater than m_0 ,

$$\left| f(x+1) - \sum_{n=0}^m \left(a_n \exp \lambda_n(x+1) + b_n \exp \{ -\lambda_n(x+1) \} \right) \right| < \frac{\epsilon}{2K}.$$

Hence, when $a-1 \leq x \leq a-\delta$,

$$\begin{aligned} & \left| \phi(x) \cdot \left[f(x+1) - \sum_{n=0}^m \left(a_n \exp \lambda_n(x+1) + b_n \exp \{ -\lambda_n(x+1) \} \right) \right] \right| \\ & < \frac{1}{2}\epsilon \quad \text{if } m > m_0. \end{aligned} \quad (8.2)$$

From (8.1) and (8.2) it follows that

$$\begin{aligned} & \left| \int_{a-1}^a \phi(x) f(x+1) dx - \sum_{n=0}^m \int_{a-1}^a \phi(x) \left(a_n \exp \lambda_n(x+1) \right. \right. \\ & \quad \left. \left. + b_n \exp \{ -\lambda_n(x+1) \} \right) dx \right| < \frac{1}{2}\epsilon. \end{aligned} \quad (8.3)$$

Similarly, because $\phi(x)$ is continuous on the right of a and $\phi(a+0) = 0$, and because

$$\Sigma \left(a_n \exp \lambda_n(x-1) + b_n \exp \{ -\lambda_n(x-1) \} \right)$$

converges uniformly to $f(x-1)$ in any interval $(a+\delta', a+1)$ when δ' is positive, it is possible to determine a number m_1 , such that, when $m > m_1$,

$$\begin{aligned} & \left| \int_a^{a+1} \phi(x) f(x-1) dx - \sum_{n=0}^m \int_a^{a+1} \phi(x) \left(a_n \exp \lambda_n(x-1) \right. \right. \\ & \quad \left. \left. + b_n \exp \{ -\lambda_n(x-1) \} \right) dx \right| < \frac{1}{2}\epsilon. \end{aligned} \quad (8.4)$$

By (8.3) and (8.4), if m is greater than m_0 and m_1 ,

$$\begin{aligned} & \left| \int_{a-1}^a \phi(x) f(x+1) dx + \int_a^{a+1} \phi(x) f(x-1) dx \right. \\ & - \sum_{n=0}^m \left(\int_{a-1}^a \phi(x) a_n \exp \lambda_n(x+1) dx + \int_{a-1}^a \phi(x) b_n \exp \{ -\lambda_n(x+1) \} dx \right. \\ & \quad \left. + \int_a^{a+1} \phi(x) a_n \exp \lambda_n(x-1) dx + \int_a^{a+1} \phi(x) b_n \exp \{ -\lambda_n(x-1) \} dx \right) \Big| < \epsilon. \end{aligned}$$

But

$$\int_{a-1}^a \phi(x) \exp \lambda_n(x+1) dx + \int_a^{a+1} \phi(x) \exp \lambda_n(x-1) dx = 2(\cosh \lambda_n - c) \beta_n$$

and

$$\begin{aligned} & \int_{a-1}^a \phi(x) \exp \{-\lambda_n(x+1)\} dx + \int_a^{a+1} \phi(x) \exp \{-\lambda_n(x-1)\} dx \\ &= 2(\cosh \lambda_n - c) \alpha_n. \end{aligned}$$

Therefore

$$\left| \int_{a-1}^a \phi(x) f(x+1) dx + \int_a^{a+1} \phi(x) f(x-1) dx - \sum_{n=0}^m 2(\cosh \lambda_n - c)(\alpha_n \beta_n + b_n \alpha_n) \right| < \epsilon,$$

when $m > m_0$ and $m > m_1$.

Since ϵ is arbitrarily small, the expression on the left must have the limit zero as $m \rightarrow \infty$, *i.e.*

$$\int_{a-1}^a \phi(x) f(x+1) dx + \int_a^{a+1} \phi(x) f(x-1) dx = 2 \sum_{n=0}^{\infty} (\cosh \lambda_n - c)(\alpha_n \beta_n + b_n \alpha_n). \quad (8.5)$$

The Hurwitz-Liapounoff expansion for the integral $\int \{f(x)\}^2 dx$ may be formally derived from this by putting $c=0$, $\phi(x) = f(x+1)$ when $a-1 < x < a$, and $\phi(x) = f(x-1)$ when $a < x < a+1$.

Comparison with Fourier series.

9. (i) *Convergence.* For the convergence of the Fourier series associated with a bounded, integrable function $f(x)$ at the point x , it is sufficient that $f(x)$ be of limited total fluctuation in the immediate neighbourhood of x . The sufficient condition obtained in § 3 for the Dixon series involves also limited total fluctuation between x and the middle point of the interval. Some such additional condition is to be expected, for without the second mean value theorem it does not appear possible to prove that the n -th term of the expansion tends to zero as $n \rightarrow \infty$. It should be noticed, however, that this condition ensures convergence throughout an interval instead of at a point.

The conditions obtained for convergence at the end points of the interval, involving the vanishing of $f(x \pm 0)$ at the middle point of the

interval and the existence of a differential coefficient near that point, are also more stringent than in the case of the Fourier series.

(ii) *Integration.* It is known* that, if the function $f(x)$ is bounded and integrable, the series obtained by term by term integration of its Fourier expansion converges uniformly to an indefinite integral of $f(x)$. It has been proved in § 6 that the series, obtained by term by term integration of the Dixon expansion between limits lying entirely within the interval in which the series is associated with $f(x)$, converges uniformly to the definite integral of $f(x)$. At the end points, the new series is convergent if $f(x)$ is of limited total fluctuation in a small sub-interval containing the middle point of the interval in its interior. The convergence at the end points is certainly not uniform if the function does not vanish at the middle point, since the sum is discontinuous.

(iii) *Differentiation.* If the function $f(x)$ is continuous and if $f(x)$ and $f'(x)$ are bounded and integrable in the range $(a-1, a+1)$ and $f(a+1-0) = f(a-1+0)$, then the Fourier series associated with $f'(x)$ in that interval (whether it converges or not) is obtained† from term by term differentiation of the Fourier expansion of $f(x)$.

If the interval $(a-1, a+1)$ can be divided into a finite number of sub-intervals within which $f(x)$ and $f'(x)$ are continuous and of limited total fluctuation, the series obtained by term by term differentiation of the Dixon expansion of $f(x)$ is identical with the Dixon expansion of $f'(x)$ only when $f(x \pm 0) = 0$, $f(a+1-0) = f(a-1+0)$, and $f(x)$ possesses no non-removable discontinuities. In this case, the Dixon expansion of $f'(x)$ is convergent, by the theorem of §§ 2 and 3.

If $f(x)$ possesses non-removable discontinuities, but one at least of the limits $f(a \pm 0)$ vanishes, there is a sub-interval, containing a as an internal or end point, in which the two series differ by a null function and converge to $\frac{1}{2}\{f'(x+0) + f'(x-0)\}$ (§ 7).

(iv) *Parseval's theorem.* A theorem similar to Parseval's theorem for Fourier series exists, but the terms of the series (8.5), which is equal to the integral of the product of two real functions, are complex.

(v) *Existence of a null series with non-vanishing coefficients.* If a trigonometric series converges to zero everywhere with the exception of

* See Hilb and M. Riesz, "Neuere Untersuchungen über trigonometrische Reihen", *Encyk. der Math. Wiss.*, II 3, 11 C 10, 1214.

† See, for example, Hobson's *Theory of functions of a real variable*.

a reducible set of points, all its coefficients vanish*. No null series in $\exp \lambda_n x$, converging at all points of a closed interval $(a-1, a+1)$, has been found; but the series

$$\sum \frac{\exp \lambda_n x}{\cosh \lambda_n - c}$$

converges to zero at all internal points of the interval $(-1, +1)$.

It is obvious that a null series, if it exists, cannot be uniformly convergent throughout the closed interval; for, if it were uniformly convergent, the series could be integrated term by term and the coefficients shown to be the Dixon coefficients of a function vanishing everywhere in the range of integration.

* G. Cantor, *Journal für Math.*, 72 (1870), 130-138, 139-142, and 73 (1871), 294-296, and *Math. Annalen*, 5 (1872), 123-132.

SOME DEFINITE INTEGRALS INVOLVING BESSEL FUNCTIONS

By W. N. BAILEY.

[Received 11 September, 1929.—Read 14 November, 1929.]

1. Some definite integrals have been given by Bateman and Kapteyn, of which the following are typical* :

$$(1.1) \quad \int_0^z J_\mu(t) J_\nu(z-t) \frac{dt}{t} = \frac{J_{\mu+\nu}(z)}{\mu},$$

$$(1.2) \quad \int_0^z \frac{J_\mu(t) J_\nu(z-t)}{t(z-t)} dt = \left(\frac{1}{\mu} + \frac{1}{\nu} \right) \frac{J_{\mu+\nu}(z)}{z},$$

$$(1.3) \quad \int_0^z \cos(z-t) J_0(t) dt = z J_0(z),$$

$$(1.4) \quad \int_0^z \sin(z-t) J_1(t) dt = \sin z - z J_0(z).$$

In a recent paper† I have given the generalizations of (1.3) and (1.4) when the Bessel functions in the integrand are replaced by $J_\mu(t)$. I now find that these results are capable of further generalization. I give here these generalizations, and evaluate some other definite integrals of similar type.

2. From the formula‡

$$(2.1) \quad \left(\frac{1}{2}t\right)^\lambda J_\mu(t) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\lambda + \mu + n) (\lambda)_n}{n! \Gamma(\mu + n + 1)} (\lambda + \mu + 2n) J_{\lambda + \mu + 2n}(t),$$

* See G. N. Watson, *Theory of Bessel functions*, §§ 12.2 and 12.21.

† W. N. Bailey, "Some integrals of Kapteyn's type involving Bessel functions", *Proc. London Math. Soc.* (2), 30 (1930), 422-424. See also L. J. Mordell, "Note on Kapteyn's and Bateman's integrals involving Bessel functions", *Journal London Math. Soc.*, 5 (1930), 203-208.

‡ G. N. Watson, *loc. cit.*, § 5.21. When written in the above form the formula is still true when λ is a negative integer.

we have, using (1.1),

$$\begin{aligned}
 (2.2) \quad & \int_0^z t^{\lambda-1} J_\mu(t) J_\nu(z-t) dt \\
 &= 2^\lambda \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\lambda+\mu+n) (\lambda)_n}{n! \Gamma(\mu+n+1)} (\lambda+\mu+2n) \int_0^z J_{\lambda+\mu+2n}(t) J_\nu(z-t) \frac{dt}{t} \\
 &= 2^\lambda \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\lambda+\mu+n) (\lambda)_n}{n! \Gamma(\mu+n+1)} J_{\lambda+\mu+\nu+2n}(z)
 \end{aligned}$$

if $R(\lambda+\mu) > (0)$, $R(\nu) > -1$.

This integral contains as particular cases a number of known results. For $\lambda = 0$ we simply get (1.1), while for $\lambda = 1$ we get*

$$(2.21) \quad \int_0^z J_\mu(t) J_\nu(z-t) dt = 2 \sum_{n=0}^{\infty} (-1)^n J_{\mu+\nu+2n+1}(z).$$

Evidently, when λ is a negative integer, the right-hand side of (2.2) reduces to a finite sum of Bessel functions. Thus, for example,

$$(2.22) \quad \int_0^z J_\mu(t) J_\nu(z-t) \frac{dt}{t^2} = \frac{1}{2\mu} \left[\frac{1}{\mu-1} J_{\mu+\nu-1}(z) + \frac{1}{\mu+1} J_{\mu+\nu+1}(z) \right].$$

If in (2.2) we take $\mu = \pm \frac{1}{2}$, we obtain generalizations of Kapteyn's trigonometrical integrals (1.3) and (1.4). In particular, for $\mu = \frac{1}{2}$, $\lambda = \frac{3}{2}$, we deduce that

$$\begin{aligned}
 (2.3) \quad & \int_0^z \sin(z-t) J_\nu(t) dt \\
 &= 4 \sum_{n=0}^{\infty} (-1)^n (n+1) J_{\nu+2n+2}(z) \\
 &= 2 \sum_{n=0}^{\infty} (-1)^n (\nu+2n+2) J_{\nu+2n+2}(z) - 2\nu \sum_{n=0}^{\infty} (-1)^n J_{\nu+2n+2}(z) \\
 &= z J_{\nu+1}(z) - 2\nu \sum_{n=0}^{\infty} (-1)^n J_{\nu+2n+2}(z),
 \end{aligned}$$

and for $\mu = -\frac{1}{2}$, $\lambda = \frac{3}{2}$, we obtain

$$\begin{aligned}
 (2.4) \quad & \int_0^z \cos(z-t) J_\nu(t) dt = 4 \sum_{n=0}^{\infty} (-1)^n (n+\frac{1}{2}) J_{\nu+2n+1}(z) \\
 &= 2 \sum_{n=0}^{\infty} (-1)^n (\nu+2n+1) J_{\nu+2n+1}(z) - 2\nu \sum_{n=0}^{\infty} (-1)^n J_{\nu+2n+1}(z) \\
 &= z J_\nu(z) - 2\nu \sum_{n=0}^{\infty} (-1)^n J_{\nu+2n+1}(z).
 \end{aligned}$$

* G. N. Watson, *loc. cit.*, § 12.2.

These are the generalizations of Kapteyn's integrals previously obtained by another method*. The integrals can be expressed in finite form when ν is a positive integer by using the Neumann series for $\sin z$ and $\cos z$.

3. We now consider the more general integral

$$I \equiv \int_0^z t^{\lambda-1} (z-t)^{\kappa-1} J_\mu(t) J_\nu(z-t) dt.$$

Using (2.1) we have

$$\begin{aligned} & \left(\frac{1}{2}t\right)^\lambda \left\{\frac{1}{2}(z-t)\right\}^\kappa J_\mu(t) J_\nu(z-t) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+n} \Gamma(\lambda+\mu+n) (\lambda)_n \Gamma(\kappa+\nu+m) (\kappa)_m}{n! \Gamma(\mu+n+1) m! \Gamma(\nu+m+1)} \\ & \quad \times (\lambda+\mu+2n) (\kappa+\nu+2m) J_{\lambda+\mu+2n}(t) J_{\kappa+\nu+2m}(z-t). \end{aligned}$$

From (1.2) it follows that

$$\begin{aligned} (3.1) \quad I &= \frac{2^{\kappa+\lambda}}{z} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+n} \Gamma(\lambda+\mu+n) (\lambda)_n \Gamma(\kappa+\nu+m) (\kappa)_m}{n! \Gamma(\mu+n+1) m! \Gamma(\nu+m+1)} \\ & \quad \times (\kappa+\lambda+\mu+\nu+2n+2m) J_{\kappa+\lambda+\mu+\nu+2n+2m}(z). \end{aligned}$$

If κ and λ are zero or negative integers, the double series on the right of (3.1) reduces to a finite sum of Bessel functions. For example, if $\kappa = \lambda = 0$ we get (1.2). If $\lambda = 0$, $\kappa = -1$, we are led to

$$(3.2) \quad \int_0^z \frac{J_\mu(t) J_\nu(z-t)}{t(z-t)^2} dt = \frac{1}{2\mu\nu z} \left[\frac{\mu+\nu-1}{\nu-1} J_{\mu+\nu-1}(z) + \frac{\mu+\nu+1}{\nu+1} J_{\mu+\nu+1}(z) \right],$$

and if $\kappa = \lambda = 1$,

$$\begin{aligned} (3.3) \quad \int_0^z \frac{J_\mu(t) J_\nu(z-t)}{t^2(z-t)^2} dt &= \frac{1}{2\mu\nu z^2} \left[\frac{(\mu+\nu-1)(\mu+\nu-2)}{(\mu-1)(\nu-1)} J_{\mu+\nu-1}(z) \right. \\ & \quad \left. + \frac{(\mu+\nu+1)(\mu+\nu+2)}{(\mu+1)(\nu+1)} J_{\mu+\nu+1}(z) \right]. \end{aligned}$$

If λ is zero (3.1) reduces to

$$\begin{aligned} (3.4) \quad \int_0^z (z-t)^{\kappa-1} J_\mu(t) J_\nu(z-t) \frac{dt}{t} \\ = \frac{2^\kappa}{\mu z} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(\kappa+\nu+m) (\kappa)_m}{m! \Gamma(\nu+m+1)} (\kappa+\mu+\nu+2m) J_{\kappa+\mu+\nu+2m}(z), \end{aligned}$$

and this is obviously a generalization of (1.1) and (1.2).

* W. N. Bailey, *loc. cit.*

4. We now arrange the double series in (3.1) as a simple series by writing $m = t - n$, and we find that

$$(4.1) \quad \int_0^z t^{\lambda-1} (z-t)^{\kappa-1} J_\mu(t) J_\nu(z-t) dt \\ = \frac{2^{\kappa+\lambda} \Gamma(\lambda+\mu)}{z \Gamma(\mu+1)} \sum_{t=0}^{\infty} \frac{(-1)^t \Gamma(\kappa+\nu+t) (\kappa)_t}{t! \Gamma(\nu+t+1)} (\kappa+\lambda+\mu+\nu+2t) \\ \times {}_4F_3 \left[\begin{matrix} \lambda, \lambda+\mu, -\nu-t, -t \\ \mu+1, 1-\kappa-\nu-t, 1-\kappa-t \end{matrix} \right] J_{\kappa+\lambda+\mu+\nu+2t}(z).$$

In certain circumstances there is a simplification. Thus, if $\lambda = \mu + 1$, $\kappa = \nu + 1$, the series ${}_4F_3$ reduces to ${}_2F_1$, which can be summed by Gauss's formula, and we find, after using (2.1), that

$$(4.2) \quad \int_0^z t^\mu (z-t)^\nu J_\mu(t) J_\nu(z-t) dt = \frac{\Gamma(\mu+\frac{1}{2}) \Gamma(\nu+\frac{1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{1}{2}) \Gamma(\mu+\nu+1)} z^{\mu+\nu+\frac{1}{2}} J_{\mu+\nu+\frac{1}{2}}(z),$$

which is equivalent to a known result*.

Again, if we put $\lambda = \mu + 1$, $\kappa = -\mu$ in (4.1), the ${}_4F_3$ reduces to ${}_3F_2$, which can be summed by Saalschütz's formula, and we are led to

$$(4.3) \quad \int_0^z t^\mu (z-t)^{-\mu-1} J_\mu(t) J_\nu(z-t) dt = \frac{2^\mu \Gamma(\mu+\frac{1}{2}) \Gamma(\nu-\mu)}{\Gamma(\frac{1}{2}) \Gamma(\mu+\nu+1)} z^\mu J_\nu(z),$$

where $R(\nu) > R(\mu) > -\frac{1}{2}$. This formula is equivalent to

$$(4.31) \quad \int_0^{\frac{1}{2}\pi} J_\mu(z \sin^2 \theta) J_\nu(z \cos^2 \theta) \tan^{2\mu+1} \theta d\theta \\ = \frac{2^{\mu-1} \Gamma(\mu+\frac{1}{2}) \Gamma(\nu-\mu)}{\Gamma(\frac{1}{2}) \Gamma(\mu+\nu+1)} z^\mu J_\nu(z).$$

The particular case of (4.3) when $\mu = \frac{1}{2}$ is worth noting. It is

$$(4.32) \quad \int_0^z t^{-\frac{3}{2}} J_\nu(t) \sin(z-t) dt = \frac{z^{\frac{1}{2}} J_\nu(z)}{\nu^2 - \frac{1}{4}},$$

provided that $R(\nu) > \frac{1}{2}$.

From (4.3) in conjunction with (2.1) we have

$$(4.4) \quad \int_0^z t^\mu (z-t)^{\lambda-\mu-1} J_\mu(t) J_\nu(z-t) dt \\ = 2^\lambda \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\lambda+\nu+n) (\lambda)_n}{n! \Gamma(\nu+n+1)} (\lambda+\nu+2n) \\ \times \int_0^z t^\mu (z-t)^{-\mu-1} J_\mu(t) J_{\lambda+\nu+2n}(z-t) dt \\ = \frac{2^{\lambda+\mu} z^\mu \Gamma(\mu+\frac{1}{2})}{\Gamma(\frac{1}{2})} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\lambda+\nu+n) (\lambda)_n \Gamma(\lambda+\nu-\mu+2n)}{n! \Gamma(\nu+n+1) \Gamma(\lambda+\mu+\nu+2n+1)} \\ \times (\lambda+\nu+2n) J_{\lambda+\nu+2n}(z).$$

* G. N. Watson, *loc. cit.*, § 12.22 (1).

This integral can be obtained in finite form if λ is a negative integer. If $\lambda = \mu + \nu + 1$ we easily derive (4.2) by using (2.1). The particular case of (4.4) when $\mu = \frac{1}{2}$ is

$$(4.41) \quad \int_0^z t^{\lambda-\frac{1}{2}} \sin(z-t) J_\nu(t) dt \\ = 2^\lambda z^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\lambda + \nu + n) (\lambda)_n}{n! \Gamma(\nu + n + 1)} \frac{(\lambda + \nu + 2n) J_{\lambda + \nu + 2n}(z)}{(\lambda + \nu + 2n + \frac{1}{2})(\lambda + \nu + 2n - \frac{1}{2})}.$$

For example,

$$(4.42) \quad \int_0^z t^{-\frac{1}{2}} \sin(z-t) J_\nu(t) dt = \frac{z^{\frac{1}{2}}}{2\nu} \left[\frac{J_{\nu-1}(z)}{(\nu - \frac{1}{2})(\nu - \frac{3}{2})} + \frac{J_{\nu+1}(z)}{(\nu + \frac{1}{2})(\nu + \frac{3}{2})} \right].$$

The formula (4.41) is a generalization of a formula obtained in another paper.* It can be put in another form by an artifice employed in that paper. By using (2.1) we find that

$$\int_0^z t^{\lambda-\frac{1}{2}} \sin(z-t) J_\nu(t) dt \\ = 2^{\lambda+\alpha} z^{\frac{1}{2}-\alpha} \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^n \Gamma(\lambda + \nu + n) (\lambda)_n}{n! \Gamma(\nu + n + 1)} \frac{(\lambda + \nu + 2n)}{(\lambda + \nu + 2n + \frac{1}{2})(\lambda + \nu + 2n - \frac{1}{2})} \\ \times \frac{(-1)^s \Gamma(\alpha + \lambda + \nu + 2n + s) (\alpha)_s (\alpha + \lambda + \nu + 2n + 2s)}{s! \Gamma(\lambda + \nu + 2n + s + 1)} J_{\alpha + \lambda + \nu + 2n + 2s}(z).$$

Putting $s = t - n$, we get on the right, after some reduction,

$$2^{\lambda+\alpha} z^{\frac{1}{2}-\alpha} \sum_{t=0}^{\infty} \frac{(-1)^t \Gamma(\lambda + \nu) \Gamma(\alpha + \lambda + \nu + t) (\alpha)_t}{t! \Gamma(\nu + 1) \Gamma(\lambda + \nu + t + 1)} \\ \times \frac{(\alpha + \lambda + \nu + 2t) (\lambda + \nu)}{(\lambda + \nu + \frac{1}{2})(\lambda + \nu - \frac{1}{2})} J_{\alpha + \lambda + \nu + 2t}(z) \\ \times {}_7F_6 \left[\begin{matrix} \lambda + \nu, 1 + \frac{1}{2}(\lambda + \nu), \lambda, \alpha + \lambda + \nu + t, \frac{1}{2}(\lambda + \nu) - \frac{1}{4}, \\ \frac{1}{2}(\lambda + \nu) + \frac{1}{4}, -t; \end{matrix} \right] \\ \times \frac{\frac{1}{2}(\lambda + \nu), \nu + 1, 1 - \alpha - t, \frac{1}{2}(\lambda + \nu) + \frac{5}{4},}{\frac{1}{2}(\lambda + \nu) + \frac{3}{4}, 1 + \lambda + \nu + t}.$$

The ${}_7F_6$ can be summed by Dougall's theorem† if $\alpha = 1 - \lambda$, and we are led to the formula

$$(4.5) \quad \int_0^z t^\lambda \sin(z-t) J_\nu(t) dt \\ = \frac{2z^{\lambda+1} \Gamma(\nu + \lambda + 1)}{\Gamma(\nu - \lambda)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\nu - \lambda + 2n)}{\Gamma(\nu + \lambda + 2n + 3)} (\nu + 2n + 1) J_{\nu + 2n + 1}(z),$$

provided that $R(\lambda + \nu) > -1$.

* W. N. Bailey, *loc. cit.*, formula (5).

† J. Dougall, "On Vandermonde's theorem, and some more general expansions", *Proc. Edinburgh Math. Soc.*, 25 (1907), 114-132 (6).

Similarly from (4.4) we find that

$$\begin{aligned}
 (4.43) \quad & \int_0^z t^\mu (z-t)^{\lambda-\mu-1} J_\mu(t) J_\nu(z-t) dt \\
 &= \frac{2^{2\mu+\frac{1}{2}} z^{\lambda-\frac{1}{2}} \Gamma(\lambda+\nu-\mu)}{\Gamma(\frac{1}{2}) \Gamma(\mu+\nu-\lambda+1)} \\
 &\quad \times \sum_{t=0}^{\infty} \frac{(-1)^t \Gamma(\mu+\nu+t+\frac{1}{2}) \Gamma(\mu+t+\frac{1}{2}) \Gamma(\mu+\nu-\lambda+2t+1)}{t! \Gamma(\lambda+\mu+\nu+2t+1) \Gamma(\nu+t+1)} \\
 &\quad \times (\mu+\nu+2t+\frac{1}{2}) J_{\mu+\nu+2t+\frac{1}{2}}(z).
 \end{aligned}$$

If $\mu+\nu-\lambda+1$ is zero or a negative integer, this integral can be found in finite form, and (4.2) is an obvious particular case.

Differentiating (4.5) with regard to z , we obtain, after a little reduction,

$$\begin{aligned}
 (4.6) \quad & \int_0^z t^\lambda \cos(z-t) J_\nu(t) dt \\
 &= \frac{z^{\lambda+1} J_\nu(z)}{\lambda+\nu+1} + \frac{2\Gamma(\nu+\lambda+1)}{\Gamma(\nu-\lambda)} z^{\lambda+1} \sum_{n=1}^{\infty} \frac{(-1)^n \Gamma(\nu-\lambda+2n-1)}{\Gamma(\nu+\lambda+2n+2)} (\nu+2n) J_{\nu+2n}(z),
 \end{aligned}$$

where again $R(\lambda+\nu) > -1$.

If $\lambda = -1$, (4.5) and (4.6) reduce to known formulae*. When $\lambda = -2$, the resulting formulae are not without interest. They are

$$(4.51) \quad \int_0^z \sin(z-t) J_\nu(t) \frac{dt}{t^2} = \frac{2}{\nu(\nu^2-1)z} \sum_{n=0}^{\infty} (-1)^n (\nu+2n+1)^2 J_{\nu+2n+1}(z),$$

$$(4.61) \quad \int_0^z \cos(z-t) J_\nu(t) \frac{dt}{t^2} = \frac{J_\nu(z)}{z(\nu-1)} + \frac{2}{\nu(\nu^2-1)z} \sum_{n=1}^{\infty} (-1)^n (\nu+2n)^2 J_{\nu+2n}(z).$$

These integrals can thus be obtained in finite form when $\nu-1$ is a positive integer, by using the Neumann series for $z \sin z$ and $z \cos z$.

The integrals (4.5) and (4.6) can themselves be obtained in finite form when $\lambda+\nu$ is a positive integer or zero, and λ is not a positive integer or zero, by using the Neumann series for $z^{-\lambda-1} \sin z$ and $z^{-\lambda-1} \cos z$.

If $\lambda-\nu$ is a positive integer or zero, the series in (4.5) and (4.6) terminate.

5. More general though more complicated results can be obtained by using Bateman's expansion† instead of (2.1). Thus from (1.1), in con-

* G. N. Watson, *loc. cit.*, § 12.21 (4) and (5).

† G. N. Watson, *loc. cit.*, § 11.6.

junction with Bateman's expansion, we derive the result

$$\begin{aligned}
 (5.1) \quad & \int_0^z J_\lambda(z-t) J_\mu(t \cos \phi \cos \Phi) J_\nu(t \sin \phi \sin \Phi) dt \\
 &= \frac{2 \cos^\mu \phi \cos^\mu \Phi \sin^\nu \phi \sin^\nu \Phi}{\{\Gamma(\nu+1)\}^2} \\
 &\times \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\mu+\nu+n+1) \Gamma(\nu+n+1)}{n! \Gamma(\mu+n+1)} J_{\lambda+\mu+\nu+2n+1}(z) \\
 &\times {}_2F_1(-n, \mu+\nu+n+1; \nu+1; \sin^2 \phi) {}_2F_1(-n, \mu+\nu+n+1; \nu+1; \sin^2 \Phi).
 \end{aligned}$$

Similarly from (4.3) we obtain

$$\begin{aligned}
 (5.2) \quad & \int_0^z t^{-\lambda}(z-t)^\lambda J_\lambda(z-t) J_\mu(t \cos \phi \cos \Phi) J_\nu(t \sin \phi \sin \Phi) dt \\
 &= \cos^\mu \phi \cos^\mu \Phi \sin^\nu \phi \sin^\nu \Phi \frac{2^{\lambda+1} \Gamma(\lambda+\frac{1}{2}) z^\lambda}{\Gamma(\frac{1}{2}) \{\Gamma(\nu+1)\}^2} \\
 &\times \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\mu+\nu+n+1) \Gamma(\nu+n+1) \Gamma(\mu+\nu-\lambda+2n+1)}{n! \Gamma(\mu+n+1) \Gamma(\lambda+\mu+\nu+2n+2)} \\
 &\times (\mu+\nu+2n+1) J_{\mu+\nu+2n+1}(z) \\
 &\times {}_2F_1(-n, \mu+\nu+n+1; \nu+1; \sin^2 \phi) {}_2F_1(-n, \mu+\nu+n+1; \nu+1; \sin^2 \Phi).
 \end{aligned}$$

These formulae include as particular cases some of the results already obtained. Thus, from (5.1) when $\phi \rightarrow 0$, $\Phi \rightarrow 0$, we obtain (2.2), and when $\phi \rightarrow 0$, $\Phi \rightarrow 0$, $\lambda = \frac{1}{2}$, (5.2) gives (4.41).

6. Similar methods applied to the expansion*

$$(6.1) \quad J_\nu(z) = \frac{\Gamma(\mu+1)}{\Gamma(\nu-\mu)} \sum_{m=0}^{\infty} \frac{\Gamma(\nu-\mu+m) \frac{1}{2}(z)^{\nu-\mu+m}}{\Gamma(\nu+m+1) m!} J_{\mu+m}(z)$$

lead to results of a different type. Thus from (4.2) used in conjunction with (6.1) we derive

$$\begin{aligned}
 (6.2) \quad & \int_0^z t^{2\mu-\nu}(z-t)^\lambda J_\nu(t) J_\lambda(z-t) dt = \frac{\Gamma(\mu+1) \Gamma(\lambda+\frac{1}{2})}{2^{\nu-\mu+\frac{1}{2}} \Gamma(\frac{1}{2}) \Gamma(\nu-\mu)} \\
 &\times \sum_{m=0}^{\infty} \frac{\Gamma(\nu-\mu+m) \Gamma(\mu+m+\frac{1}{2})}{2^m m! \Gamma(\nu+m+1) \Gamma(\mu+m+\lambda+1)} z^{\mu+m+\lambda+\frac{1}{2}} J_{\mu+m+\lambda+\frac{1}{2}}(z).
 \end{aligned}$$

* G. N. Watson, *loc. cit.*, § 5.23.

The right-hand side of (6.2) can be summed by (6.1) when $\mu = \nu + \frac{1}{2}$, and we get

$$(6.21) \quad \int_0^z t^{\nu+1} (z-t)^\lambda J_\nu(t) J_\lambda(z-t) dt = \frac{\Gamma(\nu + \frac{3}{2}) \Gamma(\lambda + \frac{1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{1}{2}) \Gamma(\nu + \lambda + 2)} z^{\nu + \lambda + \frac{3}{2}} J_{\nu + \lambda + \frac{1}{2}}(z),$$

which is a particular case of (4.43).

If $\nu = 2\mu + \lambda + 1$, we can again sum by (6.1), but we are simply led to (4.3). If $\lambda = \frac{1}{2}$, (6.2) gives in a different form the value of the integral in (4.5).

Similarly from the multiplication theorem*

$$(6.3) \quad J_\mu(\lambda z) = \lambda^\mu \sum_{m=0}^{\infty} (-1)^m \frac{(\lambda^2 - 1)^m (\frac{1}{2}z)^m}{m!} J_{\mu+m}(z)$$

we obtain, with (4.2),

$$(6.4) \quad \int_0^z t^\mu (z-t)^\nu J_\mu(\lambda t) J_\nu(z-t) dt \\ = \frac{\lambda^\mu \Gamma(\nu + \frac{1}{2})}{\Gamma(\frac{1}{2})} \sum_{m=0}^{\infty} \frac{(-1)^m (\lambda^2 - 1)^m \Gamma(\mu + m + \frac{1}{2})}{2^{m+\frac{1}{2}} m! \Gamma(\mu + m + \nu + 1)} z^{\mu + \nu + m + \frac{1}{2}} J_{\mu + \nu + m + \frac{1}{2}}(z),$$

and this is evidently a generalization of (4.2).

7. Finally consider the integral

$$I \equiv z \int_0^{\frac{1}{2}\pi} J_\mu(z \sin 2\phi) J_\nu(z \sin 2\phi) \sin^k \phi \cos^k \phi d\phi$$

which is suggested by some results obtained in another paper†. By using the power series for the product of the Bessel functions and substituting the Neumann series for the various powers of z (taking $2z$ as the variable), we find that

$$(7.1) \quad I = \frac{\Gamma\{\frac{1}{2}(\kappa + \mu + \nu + 1)\} \Gamma\{\frac{1}{2}(\lambda + \mu + \nu + 1)\}}{2\Gamma(\mu + 1) \Gamma(\nu + 1) \Gamma\{\frac{1}{2}(\kappa + \lambda) + \mu + \nu + 1\}} \\ \times \sum_{t=0}^{\infty} \frac{\Gamma(\mu + \nu + t + 1)}{t!} (\mu + \nu + 2t + 1) J_{\mu + \nu + 2t + 1}(2z) \\ \times {}_6F_5 \left[\begin{matrix} \frac{1}{2}(\mu + \nu + 1), \frac{1}{2}(\mu + \nu) + 1, \frac{1}{2}(\kappa + \mu + \nu + 1), \frac{1}{2}(\lambda + \mu + \nu + 1), \\ \mu + \nu + t + 1, -t; \\ \mu + \nu + 1, \mu + 1, \nu + 1, \frac{1}{4}(\kappa + \lambda) + \frac{1}{2}(\mu + \nu + 1), \\ \frac{1}{4}(\kappa + \lambda) + \frac{1}{2}(\mu + \nu) + 1; \end{matrix} \right].$$

* G. N. Watson, *loc. cit.*, § 5.22 (16).

† W. N. Bailey, "Some definite integrals allied to an integral of Jacobi", *Proc. Lond. Math. Soc.* (2), 30 (1930), 415-421, § 5.

The ${}_6F_5$ is Saalschützian, and can consequently be summed whenever it reduces to a ${}_3F_2$. In this way we can derive a number of results of which it will suffice to give the following:

$$(7.2) \quad z \int_0^{\frac{1}{2}\pi} J_\mu(z \sin 2\phi) J_\nu(z \sin 2\phi) \tan^{\mu-\nu+1} \phi \, d\phi \\ = \frac{\Gamma(\nu)}{2\Gamma(\mu+1)} \sum_{n=0}^{\infty} \frac{\Gamma(\mu+n+1)}{\Gamma(\nu+n+1)} (\mu+\nu+2n+1) J_{\mu+\nu+2n+1}(2z).$$

$$(7.3) \quad z \int_0^{\frac{1}{2}\pi} J_\nu^2(z \sin 2\phi) \sin^{2\nu+1} \phi \cos^{2\nu+1} \phi \, d\phi \\ = \frac{\Gamma(2\nu+1)}{2^{2\nu+1} \{\Gamma(\nu+1)\}^2} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2}) \Gamma(2\nu+n+1)}{n! \Gamma(2\nu+n+\frac{3}{2})} (2\nu+2n+1) J_{2\nu+2n+1}(2z).$$

The formula (7.2) is a generalization of the result (5.7) of the paper referred to at the beginning of this paragraph. (7.3) may be written in the form

$$(7.31) \quad \int_0^z \frac{t^{2\nu+1} J_\nu^2(t)}{\sqrt{(z^2-t^2)}} \, dt \\ = \frac{z^{2\nu} \Gamma(2\nu+1)}{2^{2\nu+1} \{\Gamma(\nu+1)\}^2} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2}) \Gamma(2\nu+n+1)}{n! \Gamma(2\nu+n+\frac{3}{2})} (2\nu+2n+1) J_{2\nu+2n+1}(2z).$$

THE APPLICATION OF OPERATIONAL METHODS TO SOME ELECTRICAL PROBLEMS IN DIFFUSION

By T. J. I'A. BROMWICH.

[Received 27 May, 1929—Read 13 June, 1929.]

Those who have studied Heaviside's work on the decay of electric currents in conductors will not be surprised to find that operational methods yield results more rapidly than the usual analysis (on the lines of Fourier) which has been applied to these problems. But it is important to bring out the further point that the symbolic solutions may often suggest alternative series, which are effective in some interval of time when the Fourier series converge so slowly as to be of little use for numerical work. In one well known example (arising out of his original investigations on the Atlantic cable) Lord Kelvin obtained the alternative series, and this series was shown by Cayley to be deducible from the Fourier series by a transformation of theta functions*. That this relation could be derived from operational methods must have been familiar to Heaviside, although I have found no explicit reference to the matter in his published work: but my attention was recently directed to the matter by a paper of Prof. Lamb's† (on a "Paradox in viscous fluid motion"). In deducing Prof. Lamb's results by Heaviside's methods, I took occasion to remark that the theta-function transformation is generally derived from the same complex integral which arises in the interpretation of Heaviside's solution (by my own methods); and I pointed out further that there were other cognate problems in which the operational methods might be expected to produce similar results, *not attainable by means of any known formulae of theta-functions.*

The following example has been selected from a variety of others, mainly because the numerical results of the Fourier series had been calculated and were easily available. The two series given in (17) below were

* Lord Kelvin, *Math. and Phys. Papers*, 2 (1884), 48-60.

† *Journal London Math. Soc.*, 2 (1927), 109; in this paper Prof. Lamb uses the relation found by Kelvin and Cayley as the basis of his transformations.

found originally by Prof. Lamb*; and I have compared his results (as tabulated) with those found from (20) and (21) below, for the values $t/\tau = 0.02, 0.04, 0.06, 0.08, 0.10, 0.20$. The agreement was in every case as close as could be predicted from the estimate of errors made below; and thus the method adopted has received a careful test of a numerical character.

In connexion with various lectures (1924–28) I have obtained similar series for other problems on the decay of electric currents†: but I have found no other tabulated values with which to compare my results. Thus these series, although good examples of Heaviside's methods, did not prove of value for the purpose of numerical tests by comparison with the results of Fourier analysis‡.

A convenient summary of the theory of operational methods will be found in the *Tract* by Dr. H. Jeffreys, *Operational methods in mathematical physics* (Cambridge, 1927).

The decay of currents in a metal cylinder.

It is supposed that the cylinder is originally placed in a uniform magnetic field, parallel to the generators: the field is suddenly removed, and our problem is to discuss the mode of decay of the currents set up by this removal§.

In the subsequent state, the magnetic force (H) will remain in the same direction, and will be accompanied by a system of electric currents flowing in circles, coaxial with the given cylinder; thus the electric force (E) will also be in circles round the axis of the cylinder. Let us denote the polar coordinates (cylindrical) by ρ , ϕ , and let σ be the specific resistance, and μ the permeability of the metal (both measured in electro-magnetic units). Then the equations of Maxwell become

$$(1) \quad \frac{4\pi}{\sigma} E = -\frac{\partial H}{\partial \rho}, \quad -\mu \frac{\partial H}{\partial t} = \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho E).$$

* *Proc. London Math. Soc.* (1), 15 (1884), 143; Sir J. J. Thomson's *Recent Researches* (Cambridge, 1893), Arts. 303–305.

† Such as those solved in Arts. 306, 307 (for longitudinal currents in a cylinder), and Arts. 321–323 (for a sphere) of Sir J. J. Thomson's *Recent Researches*.

‡ Some other numerical tests are given in my paper, *Proc. Camb. Phil. Soc.*, 20 (1921), 411–427.

§ The problem was originally proposed and solved by Prof. H. Lamb [*Proc. London Math. Soc.* (1), 15 (1884), 143], and the results will be found in Sir J. J. Thomson's *Recent Researches* (Cambridge, 1893), Arts. 303, 305.

In order to solve by the operational method, we write*

$$(2) \quad p \text{ for } \frac{\partial}{\partial t} \quad \text{and} \quad p(H-H_0) \text{ for } \frac{\partial H}{\partial t},$$

where H_0 (a constant) denotes the strength of the initial field; there is no change in E , since $\partial E/\partial t$ does not occur in (1), and the initial value of E is zero. Having made the substitutions (2), we are to proceed to solve as if p were a constant; and then the results can be interpreted by using certain standard rules.

The equations (1), (2) lead to the auxiliary equations

$$(3) \quad \frac{4\pi}{\sigma} E = -\frac{\partial H}{\partial \rho}, \quad -\mu p(H-H_0) = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E).$$

Thus we have the equation for H

$$(4) \quad \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial H}{\partial \rho} \right) = q^2(H-H_0),$$

where

$$(5) \quad q^2 = 4\pi\mu p/\sigma.$$

Since p is to be treated as a constant, the same is true of q ; and in the interpretation by means of complex integrals (used in my paper just quoted[†]), the symbol λ , which replaces p , is a complex number with a positive real part. In dealing with q either sign might be attached to the symbol[‡]; but it is most convenient to regard its real part as being also positive.

The solution of (4) and (3) can now be written in the form

$$(6) \quad H = H_0 - AI_0(q\rho); \quad E = (\sigma q/4\pi) AI_1(q\rho),$$

where I_0, I_1 denote the modified Bessel functions defined by

$$(7) \quad \begin{cases} I_0(x) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots, \\ I_1(x) = I'_0(x) = \frac{x}{2} + \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} + \dots \end{cases}$$

* See § 1.4 of the *Tract* by Dr. H. Jeffreys (quoted above). The rule there called by the name of Heaviside was given (for the first time, so far as I know) in the *general* case in a paper of my own, *Proc. London Math. Soc.* (2), 15 (1916), 401-448. Heaviside had worked out previously a number of special cases (similar to the present example) in which the initial state could be regarded as due to the application of a uniform force (electric or magnetic) to the systems considered.

† See also Ch. II of the *Tract* of Dr. H. Jeffreys.

‡ In actual fact, the series in (6) and in later formulae, when written out at length, will be found to contain only q^2 ; but in making a proper estimate of the asymptotic series used in (16), it is necessary to discriminate between the two values of q .

The corresponding solution outside the cylinder, where, initially, the field H_0 is zero, so that no changes of the type (2) are needed, will be given by

$$(8) \quad H = BK_0(p\rho/c), \quad E = BcK_1(p\rho/c),$$

the functions K_0, K_1 being the associated functions which tend to zero at infinity*; and c denotes the velocity of propagation of electric waves in the sphere outside the cylinder.

The surface-condition at $\rho = a$ is given by the continuity of H and E (since both forces are tangential to the surface); that is, we can at once eliminate B from (8) by making H/E continuous at $\rho = a$. Now it will be found that†

$$\frac{K_0(pa/c)}{K_1(pa/c)} = \frac{pa}{c} \left\{ \log \left(\frac{2c}{pa} \right) - C \right\},$$

approximately, when pa/c is small; and for the ordinary metals and rates of decay (with a bar 2 cm. in diameter) the value of pa/c is of the order 10^{-6} , so that K_0/K_1 is of the order 10^{-5} (or less); it follows that the surface condition at $\rho = a$ may be taken to be

$$H = 0.$$

Thus the solution (6) takes the form

$$(9) \quad H = H_0 \left\{ 1 - \frac{I_0(q\rho)}{I_0(qa)} \right\}, \quad E = \frac{\sigma q}{4\pi} H_0 \frac{I_1(q\rho)}{I_0(qa)}.$$

This operational solution can be converted into the forms given by Lamb if we use the "rule of partial fractions"†; the roots of the denominator are known to be purely imaginary, and so it is convenient to write

$$(10) \quad qa = iz \quad \text{or} \quad p = -\frac{\sigma z^2}{4\pi\mu a^2},$$

or, say§,

$$(11) \quad p = -z^2/\tau, \quad \text{with} \quad \tau = 4\pi\mu a^2/\sigma.$$

Then (9) becomes

$$(12) \quad H = H_0 \left\{ 1 - \frac{J_0(z\rho/a)}{J_0(z)} \right\}, \quad E = -\frac{\sigma z}{4\pi a} H_0 \frac{J_1(z\rho/a)}{J_0(z)},$$

* See the article just quoted; here C denotes Euler's constant = 0.5772.

† See, for instance, Art. 117 of my book on *Infinite series* (2nd ed.); it is a consequence of the fact that p may be regarded as having a *positive* real part, that the functions K_0, K_1 are to be used here.

‡ See pp. 11, 19 of Dr. Jeffrey's *Tract*.

§ For a copper bar 2 cm. in diameter ($\mu = 1$, $\sigma = 1600$ in C.G.S. units) τ is about $\frac{1}{138}$ of a second.

and the general formula of Heaviside gives

$$(13) \quad \begin{cases} H = H_0 \Sigma \frac{J_0(z\rho/a)}{-J'_0(z)} \frac{dp}{p dz} e^{pt} \\ E = \frac{\sigma H_0}{4\pi a} \Sigma \frac{J_1(z\rho/a)}{-J'_0(z)} \frac{dp}{p dz} e^{pt}, \end{cases} \quad \text{with } p = -z^2/\tau.$$

There are no terms independent of t in (13), because both the expressions in (9) are zero when $z = 0$ (*i.e.* for $p = 0$); and the summation extends to all positive* roots of $J_0(z) = 0$.

$$\text{Now } p = -\frac{z^2}{\tau}, \quad \frac{1}{p} \frac{dp}{dz} = \frac{2}{z}, \quad \text{and} \quad -J'_0(z) = J_1(z),$$

so that (13) is equivalent to the two forms†

$$(14) \quad \begin{cases} H = 2H_0 \Sigma \frac{J_0(z\rho/a)}{zJ_1(z)} e^{-z^2 t/\tau}, \\ E = \frac{\sigma H_0}{2\pi a} \Sigma \frac{J_1(z\rho/a)}{zJ_1(z)} e^{-z^2 t/\tau}, \end{cases}$$

where the summations refer to all positive roots of $J_0(z) = 0$.

The convergence of these series is very rapid as soon as $t > \frac{1}{5}\tau$; this will be evident from the following table of values‡ of z , z^2 , and $zJ_1(z)$.

Roots of $J_0(z) = 0$.

	z	z^2	$J_1(z)$	$zJ_1(z)$
1	2.405	5.78	0.5191	1.248
2	5.520	30.47	-0.3403	-1.878
3	8.654	74.89	0.2715	2.349
4	11.792	139.0	-0.2325	-2.741
5	14.931	223.2	0.2065	3.084
6	18.071	326.6	-0.1877	-3.392
7	21.212	449.9	0.1733	3.675

But the convergence of (14) is far from satisfactory when t/τ is smaller; thus, for $t/\tau = \frac{1}{50} = 0.02$, the seventh term contains an exponential approximately equal to $e^{-9} = \frac{1}{8} \times 10^{-3}$ (say). To deal easily with this range of

* The Heaviside formula refers to all values of p which make the denominators zero; and we obtain these by taking all positive roots of $J_0(z) = 0$.

† For $t/\tau = \frac{1}{5}$, the third exponential is roughly equal to $e^{-15} = \frac{1}{8} \times 10^{-6}$, nearly.

‡ These agree (except for notation) with the results found by Prof. Lamb, and reproduced in Arts. 304, 305 of the *Recent Researches*.

values of t , it is more convenient to introduce the asymptotic series* for $I_0(qa)$, $I_1(qa)$, etc., in the original formula (9). Instead of writing out the complete formulae, which are rather complicated, we shall give some derived results, which can be compared directly with those tabulated in Art. 305 of the *Recent recherches*.

The magnitudes tabulated there are the total induction (**B**) across the section of the bar and the total electromotive force (**F**) round the circumference. That is,

$$(15) \quad \mathbf{B} = \int_0^a 2\pi(\mu H) \rho d\rho, \quad \mathbf{F} = 2\pi a E_a.$$

Now it is easy to verify that

$$\int_0^a I_0(q\rho) \rho d\rho = \frac{a}{q} I_1(qa),$$

so that (9) and (15) give the formulae

$$(16) \quad \mathbf{B} = \pi a^2 \mu H_0 \left\{ 1 - \frac{2}{qa} \frac{I_1(qa)}{I_0(qa)} \right\}, \quad \mathbf{F} = \sigma H_0 \left\{ \frac{qa}{2} \frac{I_1(qa)}{I_0(qa)} \right\}.$$

Proceeding as before, we can express (16) by formulae similar to (14), thus

$$(17) \quad \frac{\mathbf{B}}{\mathbf{B}_0} = \Sigma \frac{4}{z^2} e^{-z^2 t/\tau}, \quad \frac{\mathbf{F}}{\sigma H_0} = \Sigma e^{-z^2 t/\tau},$$

from which (it would seem) the table given by Prof. Lamb was constructed.

Two series which are more convenient when $t/\tau < \frac{1}{50}$ can be obtained from the asymptotic series for $I_0(x)$ and $I_1(x)$; it will be seen from the articles which are quoted above that each of these functions involves two asymptotic series, the first starting with $e^x/\sqrt{(2\pi x)}$ and the second with $e^{-x}/\sqrt{(2\pi x)}$; and, to obtain reasonably simple formulae, the second series must be negligible in comparison with the first.

To satisfy this condition, we must regard $e^{-2\xi}$ as small, where ξ is the real part of x ; thus if we are prepared to neglect 10^{-4} , we must suppose that $\xi \geq 4.6$ roughly.

So far we have spoken as if x were an ordinary complex number; but in (17) the argument is qa , and some interpretation becomes necessary. The investigations given in Ch. VII of Dr. Jeffreys' *Tract* can be modified without much difficulty to prove that, when t/τ is small, e^{-2qa} is of the order of magnitude $e^{-t/\tau}$; so that, if $t/\tau \leq \frac{1}{10}$, we can proceed with safety (if the results are required only to four significant figures); for three significant figures $t/\tau \leq \frac{1}{5}$ will be sufficient.

* See the article quoted above (in my *Infinite series*).

In these circumstances, an expression given by Heaviside* may be used, namely,

$$(18) \quad \frac{I_1(x)}{I_0(x)} = 1 - \frac{1}{2x} - \frac{1}{8x^2} - \frac{1}{8x^3} - \frac{25}{128x^4} - \frac{13}{32x^5} - \frac{1073}{1024x^6} - \dots$$

If we substitute the series (18) in (16), we can evaluate the solution by means of the rule†

$$p^{-n} = t^n / \Gamma(1+n),$$

from which we see that

$$(19) \quad \begin{cases} qa = \frac{1}{(\tau p)^{-\frac{1}{2}}} = \left(\frac{\tau}{t}\right)^{\frac{1}{2}} \frac{1}{\Gamma(\frac{1}{2})} = \sqrt{\left(\frac{\tau}{\pi t}\right)}, \\ \frac{1}{qa} = \frac{1}{(\tau p)^{\frac{1}{2}}} = \left(\frac{t}{\tau}\right)^{\frac{1}{2}} \frac{1}{\Gamma(\frac{3}{2})} = 2\sqrt{\left(\frac{t}{\pi \tau}\right)}, \\ \frac{1}{(qa)^3} = \frac{1}{(\tau p)^{\frac{3}{2}}} = \left(\frac{t}{\tau}\right)^{\frac{3}{2}} \frac{1}{\Gamma(\frac{5}{2})} = \frac{4t}{3\tau} \sqrt{\left(\frac{t}{\pi \tau}\right)}, \\ \frac{1}{(qa)^5} = \frac{1}{(\tau p)^{\frac{5}{2}}} = \left(\frac{t}{\tau}\right)^{\frac{5}{2}} \frac{1}{\Gamma(\frac{7}{2})} = \frac{8t^2}{15\tau^2} \sqrt{\left(\frac{t}{\pi \tau}\right)}, \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \end{cases}$$

while the even powers of qa reduce to ordinary integrations with respect to t .

Thus we find that the formulae (16)–(19) give the series

$$(20) \quad \frac{\mathbf{B}}{\mathbf{B}_0} = 1 + \frac{t}{\tau} + \frac{1}{8} \left(\frac{t}{\tau}\right)^2 + \frac{13}{96} \left(\frac{t}{\tau}\right)^3 + \dots \\ - 4 \sqrt{\left(\frac{t}{\pi \tau}\right)} \left\{ 1 - \frac{1}{12} \frac{t}{\tau} - \frac{5}{96} \left(\frac{t}{\tau}\right)^2 - \dots \right\}$$

and

$$(21) \quad \frac{\mathbf{F}}{\sigma H_0} = \frac{1}{2} \sqrt{\left(\frac{\tau}{\pi t}\right)} \left\{ 1 - \frac{1}{4} \frac{t}{\tau} - \frac{25}{96} \left(\frac{t}{\tau}\right)^2 - \dots \right\} \\ - \left\{ \frac{1}{4} + \frac{1}{16} \frac{t}{\tau} + \frac{13}{128} \left(\frac{t}{\tau}\right)^2 + \dots \right\}.$$

* *Electromagnetic theory*, vol. 3, p. 371; but the last coefficient appears there as $\frac{69}{64} = \frac{1104}{1024}$; that this numerator should be 1073 has been checked in two independent ways. See also *Phil. Mag.* (7), 6 (1928), 84, formula (37).

† See p. 25 of Dr. Jeffreys' *Tract*; the result is obvious by repeated integration when n is a positive integer, and this rule forms the natural extension to other values of n ; the cases $n = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots$, are examined in my paper, *Proc. Camb. Phil. Soc.*, 20 (1921), 411–427.

As a verification, take $t/\tau = 0.02 = \frac{1}{50}$, the least value tabulated by Prof. Lamb; the above series become

$$\begin{aligned}\frac{B}{B_0} &= 1 + 0.02 + 0.00005 + \dots - (0.3192)(1 - 0.00167 - 0.00002 - \dots) \\ &= 0.7014,\end{aligned}$$

and

$$\begin{aligned}\frac{F}{\sigma H_0} &= 1.995(1 - 0.005 - 0.00010 - \dots) \\ &\quad - (0.25 + 0.00125 + 0.00004 + \dots) \\ &= 1.733.\end{aligned}$$

These results agree precisely with those given by Prof. Lamb; to obtain this degree of accuracy from (17), about six terms in each of the two series would be needed. That is the same number as we have used above; but each term has to be calculated separately, at the cost of far more labour than the simple powers of (t/τ) used here.

For a value such as $t/\tau = \frac{1}{100} = 0.01$, the series (20) and (21) will converge even more rapidly than in the example just given: but in the second series of (17) even the seventh term exceeds 0.01, while the whole sum, estimated from (21), is about 2.6. Thus it would be almost impossible to obtain more than three significant figures correctly by using (17).

It is therefore evident that these asymptotic figures supplement the usual Fourier series by providing convenient means of calculation for the precise values* of t/τ when the former series converge so slowly as to be almost valueless.

* The interval of time (in the numerical example of a copper bar 2 cm. in diameter) is very minute, according to ordinary standards of time; but, for certain purposes, it is the variation in *this* interval that is most instructive to study.

ELECTROMAGNETIC INDUCTION IN A CONDUCTING SPHERE.

By A. T. PRICE.

[Received 30 May, 1929.—Read 13 June, 1929.]

1. *Introduction.*

The determination of the electric currents induced in a conducting sphere by a varying aperiodic external magnetic field which satisfies prescribed initial conditions is of importance in the theory of terrestrial magnetic storms. A discussion of such a problem, based on Lamb's* memoir on the possible current systems in a sphere, was given by Chapman and Whitehead†, but their work requires correction owing to a wrong assumption made therein as to the nature of the free motions of electricity in a sphere. The corrected solution is now given‡. A particular case leads to a simple relation between the zeros of Bessel functions.

2. *Statement of the problem.*

The radius of the sphere is taken to be qa cms. and its conductivity κ e.m.u. The magnetic permeability is assumed to be unity inside and outside the sphere. The induced current-system and its magnetic field are first determined for the case of an external magnetic field whose potential is expressible as a single zonal harmonic, and which rises from zero to a given value according to a simple exponential law. This is then easily generalized for the case of any field expressible in the form (36), which appears to be sufficiently general for the discussion of the above-mentioned terrestrial phenomena.

3. *The free motions of electricity in the sphere.*

In order to satisfy the prescribed initial conditions, it is necessary to consider the nature of the free motions of electricity in the sphere, *i.e.* those motions which can exist without any extraneous inducing field. Lamb has shown (*loc. cit.*) that they can be expressed in terms of series of exponentials of the form e^{-lt} , and that they fall into two distinct classes.

* H. Lamb, *Phil. Trans. Royal Soc. (A)*, 180 (1889), 467.

† S. Chapman and T. T. Whitehead, *Proc. International Math. Congress* (Toronto, 1924).

‡ The application of the solution to the theory of magnetic storms will be dealt with elsewhere in a joint paper by S. Chapman and the author.

In the first class the currents are always perpendicular to the radii of the sphere, and possess a magnetic field outside as well as inside the sphere. The values of l , corresponding to each separate spherical harmonic of degree n in the expression for the potential of the magnetic field outside the sphere, are the roots of the equation

$$(1) \quad R_{n-1}(4\pi\kappa l q^2 a^2) = 0,$$

where

$$(2) \quad R_n(x^2) = 1 - \frac{x^2}{2(2n+3)} + \frac{x^4}{2 \cdot 4(2n+3)(2n+5)} \cdots = \frac{\Gamma(n+\frac{3}{2})}{(\frac{1}{2}x)^{n+\frac{1}{2}}} J_{n+\frac{1}{2}}(x).$$

In the second class the currents are not in general perpendicular to the radius vector, and possess no magnetic field outside the sphere. The values of l corresponding to separate terms in the vector potential of the currents inside the sphere are the roots of

$$(3) \quad R_n(4\pi\kappa l q^2 a^2) = 0.$$

This type of motion is not involved in the solution of the problem at present under consideration, for changes in the magnetic field outside the sphere cannot generate or affect such currents. This is not quite true if the velocity of propagation of electromagnetic waves is taken as finite; the necessary modification is, however, unimportant if l is small compared with $c/(qa)$, which is approximately 47 for a sphere of the earth's size. Now e^{-47t} decays to one-tenth of its initial value in 0.05 seconds, so that only those free motions would be affected for which the rates of decay are exceedingly rapid.

4. The determination of the induced field.

We proceed now to determine the induced magnetic field corresponding to an inducing field of which the potential is given by

$$(4) \quad E = E_n(1 - e^{-\lambda t}) P_n,$$

where E_n and λ are given constants and P_n is the solid zonal harmonic of degree n .

The magnetic field of the induced currents will have a potential outside the sphere, which can be expressed in the form

$$(5) \quad C = \left\{ C_n e^{-\lambda t} + \sum_{k=1}^{\infty} C_{n,k} e^{-l_{n,k} t} \right\} P_{-n-1},$$

where the first term corresponds to the forced motions and the remaining terms to the free motions of electricity in the sphere.

The vector potential, $\mathbf{A}$, of the total magnetic field inside the sphere will have its r - and θ -components zero, and the ϕ -component will be of

the form

(6)

$$c = \left\{ c_n R_n(4\pi\kappa\lambda r^2) e^{-\lambda t} + \sum_{k=1}^{\infty} c_{n,k} R_n(4\pi\kappa l_{n,k} r^2) e^{-l_{n,k} t} + \frac{E_n}{n+1} \right\} \frac{r^n dP_n(\cos \theta)}{a^{n-1} d\theta},$$

the constant term $\left[\frac{E_n}{n+1} \frac{r^n dP_n(\cos \theta)}{a^{n-1} d\theta} \right]$ in this expression corresponding to the constant term $[E_n P_n]$ in the potential of the inducing magnetic field.

The boundary conditions, viz. that the normal magnetic induction and the tangential magnetic force are continuous at the surface of the sphere, lead to the following relations between the coefficients in these expressions.

For the forced motions,

$$(7) \quad C_n = nq^{2n+1} \{ R_{n-1}(4\pi\kappa\lambda q^2 a^2) - R_n(4\pi\kappa\lambda q^2 a^2) \} c_n,$$

$$(8) \quad (n+1) R_{n-1}(4\pi\kappa\lambda q^2 a^2) c_n = -E_n.$$

For the free motions,

$$(9) \quad R_{n-1}(4\pi\kappa l_{n,k} q^2 a^2) = 0,$$

$$(10) \quad C_{n,k} = -nq^{2n+1} R_n(4\pi\kappa l_{n,k} q^2 a^2) c_{n,k}.$$

The initial conditions are that the inducing field and the induced currents are both zero; hence

$$(11) \quad C_n + \sum_{k=1}^{\infty} C_{n,k} = 0,$$

$$(12) \quad c_n R_n(4\pi\kappa\lambda r^2) + \sum_{k=1}^{\infty} c_{n,k} R_n(4\pi\kappa l_{n,k} r^2) + \frac{E_n}{n+1} = 0.$$

When we substitute from equations (7), (8), and (10) in the last two equations, these become respectively

$$(13) \quad -\frac{n}{n+1} q^{2n+1} \left\{ 1 - \frac{R_n(4\pi\kappa\lambda q^2 a^2)}{R_{n-1}(4\pi\kappa\lambda q^2 a^2)} \right\} E_n \\ - \sum_{k=1}^{\infty} nq^{2n+1} R_n(4\pi\kappa l_{n,k} q^2 a^2) c_{n,k} = 0,$$

$$(14) \quad \left\{ 1 - \frac{R_n(4\pi\kappa\lambda r^2)}{R_{n-1}(4\pi\kappa\lambda q^2 a^2)} \right\} \frac{E_n}{n+1} + \sum_{k=1}^{\infty} R_n(4\pi\kappa l_{n,k} r^2) c_{n,k} = 0,$$

and it is obvious that (13) is a special case of (14) when $r = qa$.

5 The coefficients c_n , C_n corresponding to the forced motions are determined by (7) and (8), and the "moduli", $l_{n,k}$, of the free motions are determined by the transcendental equation (9). When the $l_{n,k}$'s are known, equation (14) suffices to determine the $c_{n,k}$'s, and equation (10) then gives the $C_{n,k}$'s.

The determination of the coefficients $c_{n,k}$ from equation (14) may be effected in two ways. The equation may be expanded in powers of r and the coefficients of like powers equated, giving an infinite set of simultaneous linear equations in the $c_{n,k}$'s, which may then be solved. Alternately, these coefficients may be calculated by the aid of the integral formulae given by Dini for the expansion of an arbitrary function in terms of Bessel functions of a certain type. The first and more elementary method is here given, because it leads more directly than the second to the simplest form of these coefficients.

Writing

$$(15) \quad y_{n,k} = c_{n,k}/c_n = -(n+1) R_{n-1}(4\pi\kappa\lambda q^2 a^2) c_{n,k}/E_n,$$

$$(16) \quad x_{n,k}^2 = 4\pi\kappa l_{n,k} q^2 a^2,$$

$$(17) \quad \beta^2 = 4\pi\kappa\lambda q^2 a^2,$$

equation (14) becomes

$$(18) \quad \sum_{k=1}^{\infty} y_{n,k} R_n \left(x_{n,k}^2 \frac{r^2}{q^2 a^2} \right) = R_{n-1}(\beta^2) - R_n \left(\beta^2 \frac{r^2}{q^2 a^2} \right).$$

Expanding each side of this equation in powers of r , and equating coefficients of like powers, we have

$$(19) \quad \begin{cases} y_{n,1} + y_{n,2} + y_{n,3} + \dots = R_{n-1}(\beta^2) - 1, \\ x_{n,1}^2 y_{n,1} + x_{n,2}^2 y_{n,2} + x_{n,3}^2 y_{n,3} + \dots = -\beta^2, \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ x_{n,1}^{2j} y_{n,1} + x_{n,2}^{2j} y_{n,2} + x_{n,3}^{2j} y_{n,3} + \dots = -\beta^{2j} \end{cases} \quad (j = 1, 2, \dots, \infty).$$

Solving these equations by the usual method of determinants, we have, for example,

$$(20) \quad y_{n,1} = \lim_{j \rightarrow \infty} \left\{ \frac{\Delta_{2,j} - \Delta_{3,j}}{\Delta_{1,j}} \right\},$$

where

$$\Delta_{1,j} = \begin{vmatrix} 1 & 1 & \dots \\ x_{n,1}^2 & x_{n,2}^2 & \dots \\ \dots & \dots & \dots \\ x_{n,1}^{2j} & x_{n,2}^{2j} & \dots \end{vmatrix}; \quad \Delta_{2,j} = \begin{vmatrix} R_{n-1}(\beta^2) & 1 & 1 & \dots \\ 0 & x_{n,2}^2 & x_{n,3}^2 & \dots \\ \dots & \dots & \dots & \dots \\ 0 & x_{n,2}^{2j} & x_{n,3}^{2j} & \dots \end{vmatrix};$$

$$\Delta_{3,j} = \begin{vmatrix} 1 & 1 & 1 & \dots \\ \beta^2 & x_{n,2}^2 & x_{n,3}^2 & \dots \\ \dots & \dots & \dots & \dots \\ \beta^{2j} & x_{n,2}^{2j} & x_{n,3}^{2j} & \dots \end{vmatrix},$$

whence

$$(21) \quad \frac{\Delta_{2,j}}{\Delta_{1,j}} = \frac{R_{n-1}(\beta^2)(x_{n,2}^2 x_{n,3}^2 \dots x_{n,j}^2)(x_{n,2}^2 - x_{n,3}^2)(x_{n,2}^2 - x_{n,4}^2) \dots (x_{n,3}^2 - x_{n,4}^2) \dots}{(x_{n,1}^2 - x_{n,2}^2)(x_{n,1}^2 - x_{n,3}^2) \dots (x_{n,2}^2 - x_{n,3}^2) \dots (x_{n,j-1}^2 - x_{n,j}^2)} \cdot \frac{(x_{n,j-1}^2 - x_{n,j}^2)}{(x_{n,1}^2 - x_{n,2}^2)(x_{n,1}^2 - x_{n,3}^2) \dots (x_{n,1}^2 - x_{n,j}^2)}.$$

Now $R_{n-1}(\beta^2)$ is an integral function of β , having an infinite number of simple zeros at the points $\pm x_{n,k}$, where $|x_{n,k}|$ tends to infinity as k tends to infinity, and it may be expressed in the form*

$$(22) \quad R_{n-1}(\beta^2) = \prod_{j=1}^{\infty} \left(1 - \frac{\beta^2}{x_{n,j}^2}\right).$$

Substituting this value in equation (21), and taking the limit as j tends to infinity, we obtain

$$(23) \quad \lim_{j \rightarrow \infty} \frac{\Delta_{2,j}}{\Delta_{1,j}} = \frac{x_{n,1}^2 - \beta^2}{x_{n,1}^2} \prod_{i=2}^{\infty} \left(\frac{x_{n,i}^2 - \beta^2}{x_{n,i}^2 - x_{n,1}^2}\right).$$

Similarly

$$(24) \quad \lim_{j \rightarrow \infty} \left\{ -\frac{\Delta_{3,j}}{\Delta_{1,j}} \right\} = -\prod_{j=2}^{\infty} \left(\frac{x_{n,j}^2 - \beta^2}{x_{n,j}^2 - x_{n,1}^2}\right),$$

and therefore

$$(25) \quad y_{n,1} = \frac{x_{n,1}^2 - \beta^2 - x_{n,1}^2}{x_{n,1}^2} \prod_{j=2}^{\infty} \left(\frac{x_{n,j}^2 - \beta^2}{x_{n,j}^2 - x_{n,1}^2}\right) = \frac{-\beta^2 \prod_{j=2}^{\infty} (1 - \beta^2/x_{n,j}^2)}{x_{n,1}^2 \prod_{j=2}^{\infty} (1 - x_{n,1}^2/x_{n,j}^2)}.$$

In the same way we have, for the general coefficient,

$$(26) \quad y_{n,k} = -\frac{\beta^2}{x_{n,k}^2} \prod_{j=1}^{\infty} \left(1 - \frac{\beta^2}{x_{n,j}^2}\right) \bigg/ \prod_{j=1}^{\infty} \left(1 - \frac{x_{n,k}^2}{x_{n,j}^2}\right),$$

where the dash means that the factor for which $j = k$ is to be omitted from the product.

To express this coefficient in terms of the R -functions, we have, by l'Hospital's rule,

$$(27) \quad \begin{aligned} y_{n,k} &= -\frac{\beta^2}{x_{n,k}^2} \cdot \frac{R_{n-1}(\beta^2)}{(1 - \beta^2/x_{n,k}^2)} \bigg/ \left[\lim_{x \rightarrow x_{n,k}} \frac{R_{n-1}(x^2)}{(1 - x^2/x_{n,k}^2)} \right] \\ &= -\frac{\beta^2}{x_{n,k}^2} \cdot \frac{R_{n-1}(\beta^2)}{(1 - \beta^2/x_{n,k}^2)} \bigg/ \left[\frac{x_{n,k}^2}{2(2n+1)} R_n(x_{n,k}^2) \right] \\ &= -2(2n+1) \frac{\beta^2}{x_{n,k}^2 (x_{n,k}^2 - \beta^2)} \cdot \frac{R_{n-1}(\beta^2)}{R_n(x_{n,k}^2)}. \end{aligned}$$

* Graf und Gubler, *Einleitung in die Theorie der Besselschen Funktionen*, 1 (1898), 123-130.

Hence, from (15), we have

$$(28) \quad c_{n,k} = 2(2n+1) \frac{\beta^2}{x_{n,k}^2(x_{n,k}^2 - \beta^2)} \cdot \frac{1}{R_n(x_{n,k}^2)} \cdot \frac{E_n}{n+1},$$

and therefore, from (10),

$$(29) \quad C_{n,k} = -2(2n+1) \frac{\beta^2}{x_{n,k}^2(x_{n,k}^2 - \beta^2)} \cdot \frac{nq^{2n+1}}{n+1} \cdot E_n.$$

6. All the unknowns in the expressions (5) and (6) have now been found, so that the magnetic field both inside and outside the sphere has been completely determined. It is interesting to check the result by means of equation (13), which, it will be noted, is not equivalent to any single one of the equations (19) used to determine the coefficients. On substituting these coefficients in (13), we get

$$(30) \quad \frac{R_n(\beta^2)}{R_{n-1}(\beta^2)} - 1 - 2(2n+1) \sum_{k=1}^{\infty} \frac{\beta^2}{x_{n,k}^2(x_{n,k}^2 - \beta^2)} = 0.$$

To show that this is true, differentiate equation (22) logarithmically; and then

$$\frac{R'_{n-1}(\beta^2)}{R_{n-1}(\beta^2)} = -2\beta \sum_{k=1}^{\infty} \frac{1}{x_{n,k}^2 - \beta^2},$$

i.e.

$$(31) \quad \frac{R_n(\beta^2)}{R_{n-1}(\beta^2)} = 2(2n+1) \sum_{k=1}^{\infty} \frac{1}{x_{n,k}^2 - \beta^2} \\ = 2(2n+1) \left\{ \sum_{k=1}^{\infty} \frac{\beta^2}{(x_{n,k}^2 - \beta^2)x_{n,k}^2} + \sum_{k=1}^{\infty} \frac{1}{x_{n,k}^2} \right\},$$

each series being absolutely convergent except when $\beta = x_{n,k}$. But the last sum on the right of (31) is equal to minus the coefficient of x^2 in the expansion in ascending powers of x of $R_{n-1}(x^2)$, *i.e.* it is equal to $\{2(2n+1)\}^{-1}$. The required expansion (30) then follows immediately.

7. The induced current system.

To find the distribution of the induced currents, we note that, inside the sphere,

$$(32) \quad \nabla^2 \mathbf{A} = -4\pi \mathbf{i}$$

$$(33) \quad \text{and} \quad (\nabla^2 + 4\pi\kappa l) \mathbf{A}_l = 0,$$

where $\mathbf{A}_l$ is the portion of the vector potential corresponding to each separate exponential term $e^{-l\tau}$ in equation (6). Hence

$$(34) \quad \mathbf{i} = -\frac{1}{4\pi} \nabla^2 \mathbf{A} = \kappa \sum l \mathbf{A}_l,$$

the summation extending over $l = \lambda, l_{n,k}$ ($k = 1, 2 \dots \infty$).

Thus the r - and θ -components of the current-intensity are zero (*i.e.* the lines of flow are circles, having the axis of harmonics as their common axis), and the ϕ -component is given, in e.m.u., by

$$(35) \quad i_{\phi} = \kappa \left\{ c_n R_n (4\pi\kappa\lambda r^2) \lambda e^{-\lambda t} + \sum_{k=1}^{\infty} c_{n,k} R_n (4\pi\kappa l_{n,k} r^2) \cdot l_{n,k} e^{-l_{n,k} t} \right\} \\ \times \frac{r^n dP_n(\cos \theta)}{a^{n-1} d\theta}.$$

8. Generalization of the solution.

The above result can be immediately generalized to give the solution for any external inducing field of which the potential can be expressed in the form

$$(36) \quad \Omega = \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} (1 - e^{-\lambda_{m,n} t}) E_{m,n} \right\} P_n.$$

For since the effects of each separate harmonic are independent, the solution may be obtained for each harmonic separately, and the results superposed. Moreover, since each term $(1 - e^{-\lambda_{m,n} t}) E_{m,n}$ in (36) is zero when $t = 0$, it is clear that the initial conditions will be correctly satisfied if the effects of each of these terms are separately calculated and the results superposed. Thus the solution for an inducing field of the form (36) is simply the algebraic sum of solutions, for different m 's and n 's, of the type already obtained.

9. The special case analogous to resonance.

An interesting special case of the solution, which is also particularly suitable for numerical computations, arises when λ becomes equal to one of the moduli ($l_{n,k}$ say) of the free motions. The coefficients C_n and $C_{n,k}$ in the expression^{*} for the potential of the magnetic field of the induced currents each become infinite, but the corresponding terms

$$C_n e^{-\lambda t} + C_{n,k} e^{-l_{n,k} t},$$

when taken together, tend to a finite limit as λ tends to $l_{n,k}$. We obtain, in fact, after a number of reductions,

$$(37) \quad C = \left\{ \frac{l_{n,k} t e^{-l_{n,k} t}}{x_{n,k}^2} + \frac{(2n+1)(2n+5) - 2x_{n,k'}^2}{4(2n+1)x_{n,k'}^2} e^{-l_{n,k'} t} \right. \\ \left. - x_{n,k'}^2 \sum_{k=1}^{\infty} \frac{e^{-l_{n,k} t}}{x_{n,k}^2 (x_{n,k}^2 - x_{n,k'}^2)} \right\} \frac{2n(2n+1)}{n+1} q^{2n+1} E_n P_{-n-1}.$$

While in a sense this case corresponds *mathematically* to that of resonance in a vibrating system, there is here, of course, no *physical* phenomenon analogous to resonance, because the expression (37) does not in general give a value which is a maximum, or in any sense exceptional, of the magnetic potential, regarded as a function of λ .

10. A relation between the zeros of Bessel functions.

Since, by hypothesis, $C = 0$ when $t = 0$, on putting $t = 0$ in the expression (37) for C , we obtain the relation

$$(38) \quad \frac{(2n+1)(2n+5)-2x_{n,k'}^2}{4(2n+1)x_{n,k'}^2} = x_{n,k'}^2 \sum_{k=1}^{\infty} \frac{1}{x_{n,k}^2(x_{n,k}^2 - x_{n,k'}^2)},$$

from which, by the aid of (31), we derive

$$(39) \quad \frac{2n+1}{x_{n,k'}^2} = \sum_{k=1}^{\infty} \frac{1}{x_{n,k}^2 - x_{n,k'}^2}.$$

This is a special case of the more general relation

$$(40) \quad \frac{\nu+1}{2j^2} = \sum_{k=1}^{\infty} \frac{1}{j_k^2 - j^2},$$

where j_k denotes any zero of the Bessel function $J_{\nu}(z)$, ν being any real number except a negative integer, j being any one of the zeros j_k and the summation extending over all the zeros except j .

To establish this relation, we first obtain the expansion

$$(41) \quad \frac{J_{\nu+1}(z)}{J_{\nu}(z)} = \sum_{k=1}^{\infty} \frac{2z}{j_k^2 - z^2}$$

from Mittag-Leffler's theorem, whence

$$(42) \quad \frac{(z^2 - j^2) J_{\nu+1}(z) + 2z J_{\nu}(z)}{2z(z^2 - j^2) J_{\nu}(z)} = \sum_{k=1}^{\infty} \frac{1}{j_k^2 - z^2}.$$

The expansion on the right is uniformly convergent near $z = j$, and the limit, when $z \rightarrow j$, of the function on the left may be evaluated by a double application of l'Hospital's rule. After a number of transformations by the aid of the recurrence formulae for Bessel functions, we obtain the relation (40), which appears to be new and of some mathematical interest.

SOME THEOREMS IN THE TENSOR CALCULUS

By H. S. RUSE.

[Received 17 July, 1929.—Read 14 November, 1929.]

Notation and definitions.

1. Let $(x^1, x^2, \dots, x^p)$ be generalized coordinates in a p -dimensional Euclidean manifold* specified by the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (1)$$

due regard being had for the summation convention.

For brevity, we denote the point whose coordinates are $(x^1, x^2, \dots, x^p)$ by (x^i) .

Let $P, (x^i)$, be a variable point and $Q, (q^i)$, be a fixed point of the space. We exclude from our discussion points P, Q which are singular, *e.g.* points at which any of the functions $g_{\mu\nu}$ become infinite. We shall call Q the base-point of the manifold.

Let Ω denote the square of the geodesic distance from Q to P . Thus we define Ω by the relation

$$\sqrt{\Omega} = \int_{QP} ds, \quad (2)$$

the integral being taken along the arc of the geodesic (assumed unique) from Q to P .

Then it is evident from the definition that Ω is a scalar function of $x^1, x^2, \dots, x^p$, and it is a well known fact that it satisfies the partial differential equation

$$g^{\mu\nu} \Omega_\mu \Omega_\nu = 4\Omega, \quad (3)$$

where Ω_μ, Ω_ν are covariant derivatives of Ω . This is indeed easily proved by the methods of the Calculus of Variations.

We notice that when the coordinates are rectangular Cartesian, we have

$$ds^2 = (dx^1)^2 + (dx^2)^2 + \dots + (dx^p)^2,$$

* *I.e.* one for which the Riemann tensor is everywhere zero.

† See, *e.g.*, Hadamard, *Lectures on Cauchy's problem* (1923), 90, § 59.

so that $g_{\mu\nu} = \delta_\nu^\mu$ (where $\delta_\nu^\mu = 0$ or 1 according as $\mu \neq \nu$ or $\mu = \nu$) and the function Ω is simply

$$\Omega = \sum_{r=1}^n (x^r - q^r)^2 = \sum_{\mu, \nu} \delta_\nu^\mu (x^\mu - q^\mu)(x^\nu - q^\nu). \quad (4)$$

It is the properties of the function Ω , defined by equation (2), that we propose to investigate.

2. THEOREM. *The scalar Ω satisfies the partial differential equations*

$$\Omega_{\mu\nu} = 2g_{\mu\nu}, \quad (5)$$

where $\Omega_{\mu\nu}$ is the second covariant derivative of Ω .

In other words, the fundamental tensor $g_{\mu\nu}$ is, in a Euclidean space, the second covariant derivative of the scalar $\frac{1}{2}\Omega$; which statement is really meaningless, since we cannot form covariant derivatives without an *a priori* knowledge of the value of $g_{\mu\nu}$.*

By way of proof, we demonstrate the truth of the theorem for two particular cases, and infer its general truth from the tensor character of equation (5). That the theorem does in fact hold for all Euclidean spaces may, however, easily be proved by a transference to normal coordinates.

Consider then the case in which the coordinates are rectangular Cartesian. Since the Christoffel symbols are then all zero, we have simply

$$\begin{aligned} \Omega_{\mu\nu} &= \frac{\partial^2 \Omega}{\partial x^\mu \partial x^\nu} = 2\delta_\nu^\mu \quad \{\text{by equation (4)}\} \\ &= 2g_{\mu\nu}. \end{aligned}$$

Again, if the metric is given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

$$(p = 3, \quad x^1 = r, \quad x^2 = \theta, \quad x^3 = \phi),$$

then $g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = r^2 \sin^2 \theta, \quad g_{\mu\nu} = 0 \quad (\mu \neq \nu),$

and the only non-vanishing Christoffel symbols are

$$\{22, 1\} = -r, \quad \{33, 1\} = -r \sin^2 \theta, \quad \{33, 2\} = -\sin \theta \cos \theta,$$

$$\{12, 2\} = \{21, 2\} = \{13, 3\} = \{31, 3\} = \frac{1}{r},$$

$$\{23, 3\} = \{32, 3\} = \cot \theta.$$

* I owe this remark to Dr. E. T. Copson.

Also, the square of the geodesic distance between the points (r, θ, ϕ) and (a, α, β) is given by

$$\Omega = r^2 + a^2 - 2ar \{ \cos \theta \cos \alpha + \sin \theta \sin \alpha \cos(\phi - \beta) \}.$$

Substituting these values in the formula

$$\Omega_{\mu\nu} = \frac{\partial^2 \Omega}{\partial x^\mu \partial x^\nu} - \{ \mu\nu, \alpha \} \frac{\partial \Omega}{\partial x^\alpha},$$

we at once get $\Omega_{11} = 2 = 2g_{11}$, $\Omega_{22} = 2r^2 = 2g_{22}$, and so on.

COROLLARY. In equation (5), raising the suffix ν , we get

$$\Omega_\mu^\nu = 2g_\mu^\nu = 2\delta_\mu^\nu, \quad (6)$$

and, putting $\nu = \mu$ (and summing),

$$\Omega_\mu^\mu = 2p. \quad (7)$$

Taylor's series; invariant form.

3. We can now establish the following theorem:—

Let $V(x^i)$ be any scalar which possesses covariant derivatives $V_\lambda, V_{\lambda\mu}, V_{\lambda\mu\nu}, \dots$, in a given region $\mathfrak{K}$ of the continuum, and let $V(q^i)$ be its value at the point (q^i) in $\mathfrak{K}$. Then, if (x^i) be also in $\mathfrak{K}$ and Ω be the square of the geodesic distance between (q^i) and (x^i) ,

$$V(q^i) = V(x^i) - \frac{\Omega^\lambda V_\lambda}{2.1!} + \frac{\Omega^\lambda \Omega^\mu V_{\lambda\mu}}{2^2.2!} - \frac{\Omega^\lambda \Omega^\mu \Omega^\nu V_{\lambda\mu\nu}}{2^3.3!} + \dots, \quad (8)$$

where $\Omega^\lambda, \Omega^\mu, \dots$, are contravariant derivatives of Ω , and where the summation convention is observed throughout.

We remark that, although we shall not concern ourselves with questions of rigour, the series on the right-hand side is assumed to be either terminating, convergent, or to possess a suitable “remainder after n terms.”

To prove the theorem, we observe that the series certainly represents some scalar function of $x^1, x^2, \dots, x^p$, say F . Forming the first covariant derivative $\partial F / \partial x^\kappa$ by term-by-term covariant differentiation (which we assume legitimate), it at once becomes clear that $\partial F / \partial x^\kappa = 0$; provided that we utilize equation (6) and remember that covariant differentiations are commutative in a Euclidean space. Hence it follows that F is independent of the x 's, and is unaltered if in the series it represents we put $(x^i) = (q^i)$. We at once get

$$F = V(q^i),$$

since a consideration of the Cartesian case renders evident the fact that the first contravariant derivative of Ω is zero when $(x^i) = (q^i)$. And this proves the theorem.

We notice that when the coordinates are orthogonal Cartesian, we have, by equation (4),

$$\Omega^\lambda = g^{\lambda\alpha} \frac{\partial \Omega}{\partial x^\alpha} = 2(x^\lambda - q^\lambda), \quad (9)$$

while $V_\lambda, V_{\lambda\mu}, \dots$, are simply $\partial V / \partial x^\lambda, \partial^2 V / \partial x^\lambda \partial x^\mu, \dots$. Substitution in equation (8) gives

$$V(q^i) = V(x^i) + \frac{1}{1!} \sum_{\lambda=1}^p (q^\lambda - x^\lambda) \frac{\partial V}{\partial x^\lambda} + \frac{1}{2!} \sum_{\mu, \nu} \delta_\nu^\mu (q^\mu - x^\mu)(q^\nu - x^\nu) \frac{\partial^2 V}{\partial x^\mu \partial x^\nu} + \dots,$$

which is the classical Taylor's theorem.

The theorem may readily be verified to give a true result in a few simple cases; for example, the case in which

$$ds^2 = dr^2 + r^2 d\theta^2, \quad \Omega = r^2 + a^2 - 2ar \cos(\theta - \alpha),$$

and

$$V = r^2 - 2kr \cos \theta + k^2.$$

Homogeneous tensors.

4. Let $T_{\beta_1 \beta_2 \dots \beta_s}^{\alpha_1 \alpha_2 \dots \alpha_r}$ be a tensor with r contravariant and s covariant suffixes, which we denote briefly by T_β^α . In the following the word "tensor" is taken to include scalars as a particular case.

Then we define T_β^α to be a homogeneous tensor of degree n with respect to the base-point (q^i) if it satisfies the partial differential equations

$$\Omega^\mu T_{\beta\mu}^\alpha = 2n T_\beta^\alpha, \quad (10)$$

where $\Omega^\mu = g^{\mu\lambda} (\partial \Omega / \partial x^\lambda)$ and $T_{\beta\mu}^\alpha$ is the first covariant derivative of T_β^α .

We notice from equation (9) that in the orthogonal Cartesian case, with the base-point (q^i) taken at the origin, equation (10) reduces to

$$\sum_{\mu=1}^p x^\mu \frac{\partial}{\partial x^\mu} T_\beta^\alpha = n T_\beta^\alpha,$$

so that each component of the tensor is homogeneous in the ordinary sense.

THEOREM I. *If T_β^α is a homogeneous tensor of degree n , then*

$$\Omega^\mu \Omega^\nu T_{\beta\mu\nu}^\alpha = 2n(2n-2) T_\beta^\alpha, \quad (11)$$

the suffixes μ, ν denoting covariant or contravariant differentiations.

$$\begin{aligned}
 \text{For} \quad \Omega^\mu \Omega^\nu T_{\beta\mu\nu}^\alpha &= \Omega^\nu (\Omega^\mu T_{\beta\mu}^\alpha)_\nu - \Omega^\nu \Omega_\nu^\mu T_{\beta\mu}^\alpha \\
 &= \Omega^\nu (\Omega^\mu T_{\beta\mu}^\alpha)_\nu - \frac{1}{2} (\Omega^\nu \Omega_\nu)^\mu T_{\beta\mu}^\alpha \\
 &= \Omega^\nu \cdot 2n T_{\beta\nu}^\alpha - 2\Omega^\mu T_{\beta\mu}^\alpha, \quad \text{by equations (10) and (3),} \\
 &= 2n(2n-2) T_{\beta}^\alpha, \quad \text{by equation (10).}
 \end{aligned}$$

Proceeding thus, we could prove that

$$\Omega^{\mu_1} \Omega^{\mu_2} \dots \Omega^{\mu_r} T_{\beta\mu_1\mu_2\dots\mu_r}^\alpha = 2n(2n-2)\dots(2n-2r+2) T_\beta^\alpha,$$

which, in the orthogonal Cartesian case, reduces to Euler's theorem on homogeneous functions, namely,

$$\left(x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + \dots + x^p \frac{\partial}{\partial x^p} \right)^r T_\beta^\alpha = n(n-1)\dots(n-r+1) T_\beta^\alpha,$$

the base-point being taken at the origin.

THEOREM II. *The first covariant derivative $T_{\beta\gamma}^\alpha$ of a homogeneous tensor T_β^α of degree n is a homogeneous tensor of degree $n-1$.*

For since, in a Euclidean space, covariant differentiations are commutative, we have

$$\begin{aligned}
 \Omega^\mu T_{\beta\gamma\mu}^\alpha &= \Omega^\mu T_{\beta\mu\gamma}^\alpha = (\Omega^\mu T_{\beta\mu}^\alpha)_\gamma - \Omega_\gamma^\mu T_{\beta\mu}^\alpha \\
 &= (2n T_{\beta\gamma}^\alpha)_\gamma - 2\delta_\gamma^\mu T_{\beta\mu}^\alpha, \quad \text{by equations (10) and (6),} \\
 &= 2(n-1) T_{\beta\gamma}^\alpha,
 \end{aligned}$$

and the theorem follows from the definition.

Additional theorems on homogeneous tensors.

The following theorems result at once from the definition.

A. If T_β^α , U_β^α are two homogeneous tensors of the same kind, both of degree n , then $T_\beta^\alpha + U_\beta^\alpha$ is homogeneous of degree n .

B. If T_β^α is a homogeneous tensor of degree n , and V_κ^ϵ a homogeneous tensor of degree m , then the product $T_\beta^\alpha V_\kappa^\epsilon$ is a homogeneous tensor of degree $(m+n)$.

C. Ω is itself a homogeneous scalar of degree 2. {For, by equation (3), $\Omega^\lambda \Omega_\lambda = 2n\Omega$, where $n = 2$.}

D. The fundamental tensors $g_{\mu\nu}$, $g^{\mu\nu}$ are homogeneous of degree zero.

From A, B, and D it therefore follows that the operations of raising and lowering suffixes, and of "contraction", may be performed without

altering the homogeneity or degree of a homogeneous tensor. For these operations really consist of multiplications (by the fundamental tensor), and of summations of like tensors.

An application to the solution of a set of partial differential equations.

Let Δ denote the operation of forming the second covariant derivative of a tensor, of raising the second suffix, and contracting. Thus

$$\Delta T_{\beta}^{\alpha} = g^{\mu\nu} T_{\beta\mu\nu}^{\alpha} = T_{\beta\mu}^{\alpha\mu}.$$

Let $\Delta^2, \Delta^3, \dots$ denote repetitions of this operation.

Then, if $\Phi_{\beta}^{\alpha} \equiv \Phi_{\beta_1 \beta_2 \dots \beta_s}^{\alpha_1 \alpha_2 \dots \alpha_r}$ be a tensor which satisfies the partial differential equations

$$\Delta \Phi_{\beta}^{\alpha} = 0, \quad (12)$$

we have the theorem :

A solution of the partial differential equations (12) is given by

$$\begin{aligned} \Phi_{\beta}^{\alpha} = & \left[1 - \frac{\Omega \Delta N_{\beta}^{\alpha}}{2(2n+p-4)} + \frac{\Omega^2 \Delta^2 N_{\beta}^{\alpha}}{2.4.(2n+p-4)(2n+p-6)} - \dots \right. \\ & \left. + \frac{(-1)^m \Omega^m \Delta^m N_{\beta}^{\alpha}}{2.4 \dots (2m).(2n+p-4) \dots (2n+p-2m-2)} + \dots \right], \quad (13) \end{aligned}$$

where $\Omega^2, \Omega^3, \dots, \Omega^m, \dots$ denote the 2nd, 3rd, ..., m -th, ... powers of Ω , and N_{β}^{α} is a homogeneous tensor of degree n of the same kind as Φ_{β}^{α} . The series on the right-hand side is assumed to be either terminating or convergent.

The theorem is capable of immediate proof by direct substitution of the series in equation (12), term-by-term differentiation being assumed legitimate. We omit the proof on account of its length, but remark that the theorem is an obvious generalization of one due to E. W. Hobson on the solution of Laplace's equation*.

In conclusion, we note that if $p = 4$, and Φ_{β}^{α} is a four-vector ϕ_{μ} , we obtain a solution of the electromagnetic equations $\Delta \phi_{\mu} = 0$ in the Theory of Relativity†, provided that the tensor N_{β}^{α} be so chosen that we have in addition $g^{\mu\nu} \phi_{\mu\nu} = 0$.

* *Proc. London Math. Soc.* (1), 24 (1892), 55, 66.

† Eddington, *Mathematical theory of relativity* (1924), § 74.

DOUBLY ORTHOGONAL SYSTEMS OF FUNCTIONS

PART I.—ENUMERABLE SYSTEMS.

By G. TEMPLE.

[Received 4 June, 1929.—Read 13 June, 1929.]

1. *Introduction.*

It is well known that there is a remarkable distinction between the general theory of complete systems of orthogonal functions and the theory of those special systems with which mathematical physics is most familiar. This distinction relates to the convergence of a series of the form

$$a_1 \phi_1(x) + a_2 \phi_2(x) + \dots + a_n \phi_n(x) \dots, \quad (\text{F})$$

in which the coefficients $\{a_n\}$ of the normal orthogonal functions $\{\phi_n(x)\}$ are all derived from the same function $f(x)$ according to the law

$$a_n = \int_a^b f(x) \phi_n(x) dx.$$

In the case of Fourier series it has been shown by Riemann and Lebesgue that the convergence at any point is entirely controlled by the behaviour of $f(x)$ in the immediate vicinity of that point. The corresponding theorem for all series of Sturm-Liouville functions has been given by Haar and Mercer. But, when the orthogonal functions are not restricted to have a special form, the convergence of the series (F) depends upon the convergence of the series $\sum a_n^2 (\log n)^2$, *i.e.* upon the behaviour of $f(x)$ throughout the whole of the interval (a, b) .

The object of this paper is to lay bare the fundamental reason for this distinction by developing the theory of a very general class of series (F) which enjoy the same property as a Fourier series relative to convergence at a point. It transpires that the properties of these series depend upon

the properties of the special series (H) obtained by making the generating function $f(x)$ equal to Heaviside's unit function, the "gross" properties of the series (F) depending upon the weak convergence of the series (H), and the "finer" properties of the series (F) upon the bounded convergence of the series (H). The Young-Lebesgue method of integration is implied throughout the paper.

2. Weak convergence.

The general theory of strong and weak convergence is due to F. Riesz*. The special theory which applies particularly to systems of orthogonal functions may be summarised as follows:

A succession of functions $\{f_n(x)\}$ is said to converge *weakly* to the limit-function $f(x)$, in the interval (a, b) , if

(1) all the functions $f_n^2(x)$ are integrable over (a, b) ;

(2) all the integrals $\int_a^b f_n^2(x) dx$ are less than a fixed number κ^2 independent of n ;

(3) the sequence of indefinite integrals

$$g_n(x) = \int_a^x f_n(t) dt$$

converges to the limit function $g(x) = \int_a^x f(t) dt$ when x is in (a, b) .

It follows at once from Schwartz's inequality that (if $\kappa > 0$)

$$|g_n(x_2) - g_n(x_1)| \leq \kappa |x_2 - x_1|^{\frac{1}{2}},$$

and

$$|g_n(x)| \leq \kappa (x - a)^{\frac{1}{2}}.$$

Hence the sequence $\{g_n(x)\}$ is equicontinuous and bounded in any finite part of the interval (a, b) . It follows from a theorem due to Arzelà that there exists a subsequence which converges uniformly to the limit function $g(x)$ in any finite interval, *i.e.* that the sequence $\{g_n(x)\}$ has simple uniform convergence. It may now be proved that the limit function $f(x)$ is of integrable square in (a, b) .

The fundamental property of a weakly convergent succession $\{f_n(x)\}$ is that, if $\phi(x)$ is any function of integrable square in (a, b) , the sequence

* *Math. Ann.*, 69 (1910) 449-497.

of indefinite integrals $\int_a^x \phi(t) f_n(t) dt$ converges to the limit function $\int_a^x \phi(t) f(t) dt$ when x is in (a, b) —the domain of the weak convergence of $\{f_n(x)\}$.

If $f(x)$ and $\phi(x)$ are two limit functions for the same succession of weakly convergent functions, it follows that

$$\int_a^x [f(t) - \phi(t)] dt = 0$$

for all values of x in (a, b) . Hence $f(x)$ and $\phi(x)$ can only differ in value at a set of points of measure zero, *i.e.* they are equivalent functions.

3. Doubly orthogonal systems of functions.

A pair of ordered sets of numbers $(\xi_1, \xi_2, \dots, \xi_n)$ and $(\eta_1, \eta_2, \dots, \eta_n)$ is said to be orthogonal if

$$\xi_1 \eta_1 + \xi_2 \eta_2 + \dots + \xi_n \eta_n = 0,$$

and the set $(\xi_1, \xi_2, \dots, \xi_n)$ is said to be normalized if

$$\xi_1^2 + \xi_2^2 + \dots + \xi_n^2 = 1.$$

A pair of functions $u(x)$ and $v(x)$ is said to be orthogonal over (a, b) if

$$\int_a^b u(x) v(x) dx = 0,$$

and the function $u(x)$ is said to be normalized over (a, b) if

$$\int_a^b u^2(x) dx = 1.$$

Both definitions of orthogonality and normality may be applied to a succession of functions $\{\phi_n(x)\}$. The classical definition of orthogonality considers a pair of functions $\phi_\kappa(x)$ and $\phi_\lambda(x)$, obtained by giving two different values, κ and λ , to the parameter n , and requires these functions to be orthogonal over (a, b) . But we may also consider the pair of ordered sets of numbers $[\phi_1(\xi), \phi_2(\xi), \dots]$ and $[\phi_1(\eta), \phi_2(\eta), \dots]$, obtained by giving two different values, ξ and η , to the variable x , and impose a suitable restriction on the series $\phi_1(\xi) \phi_1(\eta) + \phi_2(\xi) \phi_2(\eta) + \dots$. The restriction suggested by Weyl and Hellinger is that the series

$$\int_I \phi_1(\xi) d\xi \int_{II} \phi_1(\eta) d\eta + \int_I \phi_2(\xi) d\xi \int_{II} \phi_2(\eta) d\eta + \dots \quad (W)$$

should converge to zero if the closed connected intervals I and II both lie in (a, b) and have at most one common point.

Again, the classical definition of normalization considers a function $\phi_n^2(x)$, and makes the integral $\int_a^b \phi_n^2(x) dx$ independent of n . But we may also consider the series (W) when the intervals I and II coincide, and impose the condition that the series

$$\int_I \phi_1(\xi) d\xi \int_I \phi_1(\eta) d\eta + \int_I \phi_2(\xi) d\xi \int_I \phi_2(\eta) d\eta + \dots$$

converges to a sum independent of the position of the interval I in (a, b) .

The new conditions of orthogonality and normality introduced above require the convergence of the series

$$G(s, t) = \sum_n \int_a^s \phi_n(\xi) d\xi \int_a^t \phi_n(\eta) d\eta.$$

It is clear that $G(s, t) = G(t, s)$ and that $G(s, t)$ depends only upon the smaller of the numbers s and t . Hence we may write

$$G(s, t) = \begin{cases} G(s) & \text{if } s \leq t, \\ G(t) & \text{if } s \geq t. \end{cases}$$

Moreover the series $\sum_n \int_s^{s+t} \phi_n(\xi) d\xi \int_s^{s+t} \phi_n(\eta) d\eta$ has the sum $G(s+t) - G(s)$ if $a < s, t$ and $s+t < b$. But, since this sum is to be independent of s , it can only be a function of t , say $\Gamma(t)$. Hence

$$G(s+t) = G(s) + \Gamma(t).$$

It is now clear that $\Gamma(t) = G(t)$, whence

$$G(s+t) = G(s) + G(t).$$

Since $G(s)$ is bounded, it must be a linear function of s , and there is evidently no loss of generality in assuming that

$$G(s) = s - a.$$

A system of functions, $\{\phi_n(x)\}$, which satisfies both the classical conditions of orthogonality and normality, together with the new conditions given above, may be said to be "doubly orthogonal." The main property of such a doubly orthogonal system is that the series of multiples

of these functions derived from Heaviside's unit function as generating function—the series (H)—always has weak convergence to its generating function as limit.

4. The expansion of Heaviside's unit function.

Heaviside's unit function, $H(x)$, is defined by the equations,

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{1}{2} & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Let $\{h_n(s)\}$ denote the set of coefficients derived from Heaviside's function $H(s-x)$, according to the law

$$\begin{aligned} h_n(s) &= \int_a^b H(s-x) \phi_n(x) dx \\ &= \int_a^s \phi_n(x) dx. \end{aligned}$$

Then the series derived from $H(s-x)$ as generating function is

$$\sum_n h_n(s) \phi_n(x). \quad (\text{H})$$

If $H_n(x, s)$ denotes the n -th partial sum of this series it follows from the orthogonal properties of the functions $\{\phi_n(x)\}$ that

$$\int_a^b H_n^2(x, s) dx = h_1^2(s) + h_2^2(s) + \dots + h_n^2(s).$$

But if the system $\{\phi_n(x)\}$ is doubly orthogonal, the series $\sum_n h_n^2(s)$ converges to the limit $G(s) = s - a$. Hence

$$\int_a^b H_n^2(x, s) dx \leq s - a,$$

i.e. the succession $\{H_n(x, s)\}$ satisfies the first and second conditions for weak convergence.

Moreover, since

$$g_n(t, s) = \int_a^t H_n(x, s) ds = h_1(s) h_1(t) + h_2(s) h_2(t) + \dots + h_n(s) h_n(t),$$

it also follows from the double orthogonality of the system $\{\phi_n(x)\}$ that the sequence $\{g_n(t, s)\}$ converges to the limit $G(s, t)$. Now $G(s, t)$ equals the

smaller of the numbers $(s-a)$ and $(t-a)$, *i.e.*

$$G(s, t) = \int_a^t H(s-x) dx.$$

Therefore
$$\int_a^t H_n(x, s) ds \rightarrow \int_a^t H(s-x) dx, \quad \text{as } n \rightarrow \infty.$$

Accordingly, the series (H), derived from $H(s-x)$ as generating function, necessarily has weak convergence to the limiting function $H(s-x)$.

5. The gross properties of doubly orthogonal systems.

I. The generalization of Fatou's theorem for Fourier series.

If $f(x)$ is any function of integrable square in (a, b) , and $\{a_n\}$ is the set of coefficients derived from $f(x)$ according to the law

$$a_n = \int_a^b f(x) \phi_n(x) dx,$$

the associated series

$$a_1 \phi_1(x) + a_2 \phi_2(x) + \dots + a_n \phi_n(x) + \dots \quad (\text{F})$$

converges weakly to $f(x)$ in (a, b) .

For, if $f_n(x)$ denotes the n -th partial sum,

$$a_1 \phi_1(x) + a_2 \phi_2(x) + \dots + a_n \phi_n(x),$$

it follows at once from the fundamental property of the weakly convergent succession $\{H_n(x, s)\}$ that

$$\begin{aligned} \int_a^s f_n(x) dx &= a_1 h_1(s) + a_2 h_2(s) + \dots + a_n h_n(s) \\ &= \int_a^b f(x) H_n(x, s) dx \\ &\rightarrow \int_a^b f(x) H(s-x) dx \quad (\text{as } n \rightarrow \infty) \\ &= \int_a^s f(x) dx. \end{aligned}$$

Moreover

$$\int_a^b f_n^2(x) dx = a_1^2 + a_2^2 + \dots + a_n^2,$$

and
$$\int_a^b [f(x) - f_n(x)]^2 dx = \int_a^b f^2(x) dx - (a_1^2 + a_2^2 + \dots + a_n^2).$$

Hence the series $a_1^2 + a_2^2 + \dots + a_n^2 + \dots$

is convergent and none of the integrals $\int_a^b f_n^2(x) dx$ can exceed its sum.

It follows that $\{f_n(x)\}$ converges weakly to $f(x)$ in (a, b) .

II. The generalization of Parseval's theorem.

The series $a_1^2 + a_2^2 + \dots + a_n^2 + \dots$ converges to the sum $\int_a^b f^2(x) dx$.

For, using the fundamental property of weakly convergent succession $\{f_n(x)\}$, it follows that

$$\begin{aligned} \int_a^b f^2(x) dx &= \lim_{n \rightarrow \infty} \int_a^b f(x) f_n(x) dx \\ &= \lim_{n \rightarrow \infty} (a_1^2 + a_2^2 + \dots + a_n^2). \end{aligned}$$

III. The system of functions $\{\phi_n(x)\}$ is complete.

Let $\nu(x)$ denote any function of summable square in (a, b) for which all the integrals $\int_a^b \nu(x) \phi_n(x) dx$ are zero. Then $f(x) + \nu(x)$ is also a limit function for the weakly convergent succession $\{f_n(x)\}$. Hence $\nu(x)$ is zero except at a set of points of measure zero, i.e. the system of functions $\{\phi_n(x)\}$ is complete.

6. The bounded convergence of the series (H).

In order to make further progress with the theory of doubly orthogonal functions it is necessary to make further assumptions regarding the series (H), and, in particular, in order to discuss the convergence of the series (F), derived from any generating function $f(x)$, it is necessary to assume that the series (H) has bounded convergence.

Hitherto the only restriction we have imposed on the system $\{\phi_n(x)\}$, in addition to the classical conditions for orthogonality and normality, has been the convergence of the series

$$G(s, t) = \sum_n \int_a^s \phi_n(\xi) d\xi \int_a^t \phi_n(\eta) d\eta$$

to the smaller of the numbers $(s-a)$ and $(t-a)$. We now impose the

more stringent condition that the series

$$\sum_n \phi_n(\xi) \int_a^t \phi_n(\eta) d\eta = \sum_n h_n(t) \phi_n(\xi)$$

should converge *boundedly* to the sum $H(t-\xi)$ when $a < \xi < b$, and to the sums $H(t-a-0)$, $H(t-b+0)$ when $\xi = a, b$ and $t > a$. It is clear that, if this new condition is satisfied, then the system $\{\phi_n(x)\}$ is certainly doubly convergent.

If $Z_n(s, t)$ denotes the n -th partial sum of the series (Z), $\sum_n \phi_n(s) \phi_n(t)$, it follows that

$$\int_a^\beta Z_n(s, t) dt = H_n(s, \beta) - H_n(s, a)$$

using $H_n(s, t)$ as before to denote the n -th partial sum of the series (H). Hence $\int_a^\beta Z_n(s, t) dt$ is less in absolute value than some fixed number M independent of a, β, s , and n .

7. The nature of the conditions for convergence at a point.

Consider the series (F) obtained from a function $f(x)$ which is integrable together with the products $\{f(x) \phi_n(x)\}$ over (a, b) . Such a function will be called a "generating function." Let a_n denote the coefficient $\int_a^b f(x) \phi_n(x) dx$, and let $f_n(x)$ denote the n -th partial sum-function

$$a_1 \phi_1(x) + a_2 \phi_2(x) + \dots + a_n \phi_n(x) = \int_a^b f(t) Z_n(x, t) dt$$

of the series (F).

If the series (F) is of "Riemannian type" its convergence at any point $x = s$ will depend only upon the behaviour of $f(x)$ in the immediate vicinity of the point $x = s$. A necessary condition for this is that to every value of s in (a, b) there should correspond two non-negative numbers, ϵ and η , such that

$$a \leq s - \epsilon < s + \eta \leq b,$$

and

$$\lim_{n \rightarrow \infty} \int_{s+\eta}^b f(t) Z_n(s, t) dt = 0,$$

$$\lim_{n \rightarrow \infty} \int_a^{s-\epsilon} f(t) Z_n(s, t) dt = 0.$$

It follows from the converse of Hobson's general convergence theorem* that, if the series (F) is to be of Riemannian type for all integrable generating functions $f(x)$, it is necessary and sufficient that $Z_n(s, t)$ should satisfy the following condition: to every integral value of n (with the possible exception of a certain finite number of values) and for all values of t in $(a, s-\epsilon)$ and $(s+\eta, b)$, $Z_n(s, t)$ is equivalent to a function of s which does not exceed in absolute value a fixed number K , independent of t and n . (It is, of course, implied throughout that the conditions for the double orthogonality of the system $\{\phi_n(x)\}$ are satisfied).

But even if this condition is not satisfied, the series (F) may still be of Riemannian type for certain classes of generating functions $f(x)$. In particular it will be shown that, if there exists an integrable function $\psi(x)$ such that

$$|Z_n(s, t)| < \psi(t)$$

for all integral values of $n > n_0$ (some fixed integer), and for all values of t such that $|s-t| > \delta$ (some positive number), then the series (F) is of Riemannian type for all generating functions $f(x)$ such that $f(x)\psi(x)$ is absolutely integrable over (a, b) .

Let E denote the set of points at which $\psi(x) \geq A$, and F the complementary set in (a, b) . Then A can be chosen so that

$$\int_E \psi(x) |f(x)| dx < \epsilon.$$

Also, since $f(x)$ is integrable in (a, b) , there exists a function $g(x)$ constant in stretches and such that

$$\int_a^b |f(x) - g(x)| dx < \epsilon/A.$$

Hence, if $a < s - \epsilon < s$, and e, f are the parts of E, F which lie in $(a, s - \epsilon)$,

$$\begin{aligned} & \left| \int_{(e+f)} f(t) Z_n(s, t) dt - \int_{(f)} g(t) Z_n(s, t) dt \right| \\ &= \left| \int_{(e)} f(t) Z_n(s, t) dt + \int_{(f)} [f(t) - g(t)] Z_n(s, t) dt \right| \\ &\leq \int_{(e)} |f(t)| \psi(t) dt + A \int_{(f)} |f(t) - g(t)| dt \quad (\text{if } n > n_0) \\ &\leq 2\epsilon. \end{aligned}$$

* Hobson, "The theory of functions of a real variable," 2nd Ed., 2 (1926), § 288.

Therefore

$$\lim_{n \rightarrow \infty} \int_a^{s-\epsilon} f(t) Z_n(s, t) dt = \lim_{n \rightarrow \infty} \int_{(f)} g(t) Z_n(s, t) dt = 0,$$

since $g(t)$ is stretch-wise constant and

$$\int_a^x Z_n(s, t) dt \rightarrow H(x-s) = 0$$

if $x < s$.

$$\text{Similarly} \quad \lim_{n \rightarrow \infty} \int_{s+\eta}^b f(t) Z_n(s, t) dt = 0,$$

$$\text{whence} \quad \lim_{n \rightarrow \infty} f_n(s) = \lim_{n \rightarrow \infty} \int_{s-\epsilon}^{s+\eta} f(t) Z_n(s, t) dt.$$

Hence the convergence of the series derived from the generating function $f(x)$ at any point $x = s$ depends only upon the behaviour of $f(x)$ in an interval $(s-\epsilon, s+\eta)$ in the immediate neighbourhood of $x = s$.

8. Jordan's criterion for convergence.

The only generally useful criterion for the convergence of a series (F) of Riemannian type at any point $x = s$, which holds independently of the form of the functions $\{\phi_n(x)\}$, is that the generating function $f(x)$ should have bounded variation in some interval $(s-\delta, s+\delta)$.

To show the sufficiency of this condition, write

$$f(x) = f(s+0) + P(x) - Q(x) \quad (s \leq x \leq b),$$

where $P(x)$, $Q(x)$ are monotone increasing functions which tend to zero as x tends to s . Then, if $a \leq s < s+\eta \leq b$,

$$\int_s^{s+\eta} \{f(t) - f(s+0)\} Z_n(s, t) dt = \int_s^{s+\eta} \{P(t) - Q(t)\} Z_n(s, t) dt.$$

Now the integral on the right-hand side is less in absolute value than $M\{P(s+\eta) + Q(s+\eta)\}$, where M is the upper boundary of $\left| \int_a^\beta Z_n(s, t) dt \right|$ for all intervals (a, β) included in $(s, s+\eta)$. Hence η can be chosen so that

$$\left| \int_s^{s+\eta} f(t) Z_n(s, t) dt - f(s+0) \{H_n(s, s+\eta) - H_n(s, s)\} \right| < \delta,$$

where δ is any positive number independent of η , s , and n .

Therefore

$$\begin{aligned}\lim_{\eta \rightarrow 0} \lim_{n \rightarrow \infty} \int_s^{s+\eta} f(t) Z_n(s, t) dt &= \lim_{\eta \rightarrow 0} f(s+0) \{H(\eta) - H(0)\} \\ &= \frac{1}{2}f(s+0) \quad \text{if } s > a, \\ &\text{or } \lim_{\eta \rightarrow 0} f(a+0) \{H(\eta) - 0\} \\ &= f(a+0) \quad \text{if } s = a.\end{aligned}$$

Similarly it can be shown that, if $a < s - \epsilon < s \leq b$,

$$\begin{aligned}\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{s-\epsilon}^s f(t) Z_n(s, t) dt &= \frac{1}{2}f(s-0) \quad \text{if } s < b, \\ &\text{or } f(b-0) \quad \text{if } s = b.\end{aligned}$$

Hence

$$\lim_{n \rightarrow \infty} f_n(s) = \begin{cases} f(a+0) & \text{if } s = a, \\ \frac{1}{2}f(s-0) + \frac{1}{2}f(s+0) & \text{if } a < s < b, \\ f(b-0) & \text{if } s = b. \end{cases}$$

9. Conclusion.

The whole theory of the expansion of an arbitrary analytic function $f(z)$ of the complex variable z in a series of analytic functions depends upon the possibility of expanding the elementary function $(z-\xi)^{-1}$ in a series of the prescribed form. Once the expansion of $(z-\xi)^{-1}$ has been determined, the expansion of $f(z)$ follows from Cauchy's theorem.

The present paper attempts to establish a corresponding theory for the expansion of functions $f(x)$ of a real variable x in series of orthogonal functions by making everything depend upon the nature of the series (H) derived from Heaviside's unit function $H(s-x)$ and the null-series (Z) obtained by differentiating the series (H) term by term. The main results obtained may be restated as follows.

It is assumed

- (I) that the system of functions $\{\phi_n(x)\}$ satisfies the classical conditions of orthogonality and normality;
- (II) that the series (H) converges boundedly to the limit $H(s-x)$ when $a < x < b$, and to the limits $H(s-a-0)$, $H(s-b+0)$ when $x = a, b$ and $s > a$;
- (III) that if $Z_n(s, t)$ is the n -th partial sum of the series (Z), then $|Z_n(s, t)|$ is less than an integrable function $\psi(t)$ if $|s-t| > \delta$;

- (IV) that the generating function $f(x)$ is such that $|f(x)|\psi(x)$ is integrable over (a, b) .

It follows

- (I) that the system $\{\phi_n(x)\}$ is doubly orthogonal ;
 (II) that the series (F) is of Riemannian type ;
 (III) that, if $f(x)$ has bounded variation in the neighbourhood of $x = s$, the series (F) converges at the point $x = s$ to the sum

$$\begin{aligned} f(a+0) & \text{ if } s = a, \\ \frac{1}{2}f(s-0) + \frac{1}{2}f(s+0) & \text{ if } a < s < b, \\ f(b-0) & \text{ if } s = b. \end{aligned}$$

The following advantages may be claimed for this method of treatment. In the first place, when we are dealing with the systems of orthogonal functions which occur in theoretical physics, the discussion of the properties of the special series (H) and (Z), and the verification of the assumptions made above, is usually the first and simplest problem to arise. Secondly, the preceding method may be extended at once to continuous systems of functions and to the problems of the representation of functions as definite integrals. This question will be treated in Part II of this paper.

DOUBLY ORTHOGONAL SYSTEMS OF FUNCTIONS

PART II.—CONTINUOUS SYSTEMS

By G. TEMPLE.

[Received 25 September, 1929.—Read 14 November, 1929.]

1. *Introduction.*

The theory of integral transforms due to Plancherel* depends upon the representation of the function $\phi(x, \xi)$, which replaces $(2/\pi)^{\frac{1}{2}} \cos x\xi$ in the general theory, in the form of a series $\sum \phi_n(x) \psi_n(\xi)$, where the functions $\{\phi_n(x)\}$ and $\{\psi_n(\xi)\}$ form two normal and orthogonal systems for the interval $(0, \infty)$. Titchmarsh† has given a simpler and more direct treatment of Fourier transforms based on the properties of Fourier series. Recently F. Riesz‡ has discussed a similar topic by introducing a convergence factor $\kappa(\sigma)$ into the integral

$$\iiint \kappa(\sigma) \phi(s, \sigma) \phi(t, \sigma) f(s) g(t) d\sigma ds dt.$$

The object of this paper is to give a treatment of integral transforms which is based only upon the double orthogonality of the system $\{\phi(x, \xi)\}$ and which is independent of (1) the specific form of the basic function $\phi(x, \xi)$, (2) the theory of any allied series of orthogonal functions, and (3) the theory of summability. This investigation, carried out in § 4, seems to be complete as far as functions of summable square are concerned.

The corresponding problems for summable functions are made to depend upon the properties of the integral transform of Heaviside's unit function $H(c-x)$, and of the integral representation of this function.

* *Rend. di Palermo*, 30 (1910), 289.

† *Proc. London Math. Soc.* (2), 23 (1924), 279.

‡ *Acta Szeged* (3), (1928), quoted by Shin-ichi Izumi in *Proc. Imp. Acad. Tokyo*, 5 (1929), 113.

The final results, §§ 5 and 6, are analogous to those obtained for enumerable systems of orthogonal functions in Part I of this paper.

2. Strong convergence.

For the sake of completeness a summary is given here of the theory of strong convergence—the main instrument of research in the theory of integral transforms. Strong convergence “with exponent $p = 2$ ” only is considered.

A sequence of functions $\{f_n(x)\}$ of summable square in $(0, \infty)$ is said to converge strongly to a limit function $f(x)$ if

$$\lim_{n \rightarrow \infty} \int_0^\infty \{f(x) - f_n(x)\}^2 dx = 0. \quad (2.1)$$

It is clear that if the sequences $\{f_n(x)\}$ and $\{g_n(x)\}$ converge strongly to $f(x)$ and $g(x)$ respectively, then the sequence $\{f_n(x) + g_n(x)\}$ converges strongly to $f(x) + g(x)$. This follows from the inequality

$$\int_0^\infty |f + g - f_n - g_n|^2 dx \leq 2 \int_0^\infty |f - f_n|^2 dx + 2 \int_0^\infty |g - g_n|^2 dx.$$

The limit function $f(x)$ of a strongly convergent sequence $\{f_n(x)\}$ is also of summable square in $(0, \infty)$, for the inequality

$$\left| \left[\int_0^a f^2 dx \right]^{\frac{1}{2}} - \left[\int_0^a f_n^2 dx \right]^{\frac{1}{2}} \right| \leq \left| \int_0^a |f - f_n|^2 dx \right|^{\frac{1}{2}} \quad (2.2)$$

holds for all values of a . The same inequality shows that a sequence of functions $\{f_n(x)\}$, which are related to a function of summable square $f(x)$ by (2.1), are all of summable square in $(0, \infty)$, with the possible exception of a finite number.

The most useful properties of a strongly convergent sequence are summarised in the equations

$$\int_0^\infty f^2(x) dx = \lim_{n \rightarrow \infty} \int_0^\infty f_n^2(x) dx, \quad (2.3)$$

$$\text{and} \quad \int_0^\infty f(x) g(x) dx = \lim_{n \rightarrow \infty} \int_0^\infty f_n(x) g(x) dx, \quad (2.4)$$

where $g(x)$ is any function of summable square in $(0, \infty)$.

The necessary and sufficient condition for the strong convergence of a sequence $\{f_n(x)\}$ is that

$$\lim_{m, n \rightarrow \infty} \int_0^\infty |f_m(x) - f_n(x)|^2 dx = 0. \quad (2.5)$$

The limit function of such a sequence may often be conveniently determined by making use of the relation (2.4) and writing

$$g(x) = 1 \text{ if } x \leq t, \quad g(x) = 0 \text{ if } x > t.$$

Then

$$F(t) = \int_0^t f(t) dt = \lim_{n \rightarrow \infty} \int_0^t f_n(x) dx.$$

Since

$$\int_0^t |f_n(x)| dx \leq t^{\frac{1}{2}} \left[\int_0^\infty f_n^2(x) dx \right]^{\frac{1}{2}},$$

the function $F(t)$ is of bounded variation, and the relation

$$f(t) = \frac{d}{dt} \left[\lim_{n \rightarrow \infty} \int_0^t f_n(x) dx \right] \quad (2.6)$$

holds almost everywhere.

3. Doubly orthogonal systems of functions.

Let $\phi(x, \xi)$ be a function of the two variables x and ξ defined for all positive values of x and ξ , and of summable square over any finite part of this domain. We define the first and second subsidiary functions by the equations

$$\eta(x, \xi) = \int_0^\xi \phi(x, \sigma) d\sigma, \quad y(x, \xi) = \int_0^x \phi(s, \xi) ds. \quad (3.1)$$

The single function $\phi(x, \xi)$ may be regarded as a continuous system of functions of x with ξ as a distinguishing parameter, or as a continuous system of functions of ξ with x as a distinguishing parameter. This double aspect of the basic function $\phi(x, \xi)$ is obscured in the most familiar examples by the symmetry of $\phi(x, \xi)$ in the variables x and ξ .

If we regard $\phi(x, \xi)$ as a system of functions of x , the condition for orthogonality is that the integral

$$\int_0^\infty [\eta(x, \beta_2) - \eta(x, \beta_1)] [\eta(x, \alpha_2) - \eta(x, \alpha_1)] dx$$

should be zero when the intervals (α_1, α_2) and (β_1, β_2) have at most one common point. The condition for normality is that the integral

$$\int_0^\infty [\eta(x, \beta) - \eta(x, \alpha)]^2 dx$$

should have a value independent of the position of the interval (α, β) in $(0, \infty)$.

It is clear that, if $\lambda \neq \mu$, the integral

$$\omega(\lambda, \mu) = \int_0^\infty \eta(x, \lambda) \eta(x, \mu) dx$$

has a value which depends only on the smaller of the parameters λ and μ . Hence we may write

$$\omega(\lambda, \mu) = \begin{cases} \omega(\lambda) & \text{if } \lambda \leq \mu, \\ \omega(\mu) & \text{if } \mu \leq \lambda, \end{cases}$$

where

$$\omega(\lambda) = \int_0^\infty \eta^2(x, \lambda) dx.$$

Now, since
$$\int_0^\infty [\eta(x, \mu) - \eta(x, \lambda)]^2 dx = \omega(\mu) - \omega(\lambda),$$

it follows from the condition of normality that $\omega(\mu) - \omega(\lambda)$ depends only upon $\mu - \lambda$. Hence $\omega(\lambda)$ is a linear function of λ , and there is no loss of generality in writing $\omega(\lambda) \equiv \lambda$.

The conditions that the system of functions $\{\phi(x, \xi)\}$ should be orthogonal and normal in x may now be summarized in the single equation

$$\omega(\lambda, \mu) = \int_0^\infty \eta(x, \lambda) \eta(x, \mu) dx = \begin{cases} \lambda & \text{if } \lambda \leq \mu, \\ \mu & \text{if } \mu \leq \lambda, \end{cases} \quad (3.2)$$

Similarly the conditions that the same system of functions should be orthogonal and normal in ξ may be written

$$w(l, m) = \int_0^\infty y(l, \xi) y(m, \xi) d\xi = \begin{cases} l & \text{if } l \leq m, \\ m & \text{if } m \leq l. \end{cases} \quad (3.3)$$

A "doubly orthogonal" continuous system of functions, $\{\phi(x, \xi)\}$, is defined to be a system which is orthogonal and normal in both of the variables x and ξ .

4. Integral transforms.

The following properties of a doubly orthogonal continuous system $\{\phi(x, \xi)\}$ are established in this section.

I. If the function $P(x)$ is of summable square in $(0, \infty)$, the equation

$$\Pi(\xi) = \frac{d}{d\xi} \int_0^\infty P(s) \eta(s, \xi) ds$$

defines a function also of summable square in $(0, \infty)$ —the first integral transform of $P(x)$. The second integral transform is

$$\frac{d}{dx} \int_0^\infty P(\sigma) y(x, \sigma) d\sigma.$$

II. $P(x)$ and $\Pi(\xi)$ are connected by Parseval's relation

$$\int_0^\infty P^2(s) ds = \int_0^\infty \Pi^2(\sigma) d\sigma.$$

III. If $Q(x)$ is the second integral transform of $\Pi(\xi)$, *i.e.*

$$Q(x) = \frac{d}{dx} \int_0^\infty \Pi(\xi) y(x, \xi) d\xi,$$

then $Q(x) = P(x)$ almost everywhere in $(0, \infty)$.

The proof of these results depends upon the fundamental approximation theorem that, since $P(x)$ is a function of class $[L^2]$, there exists a function $P_n(x)$, constant in stretches, and such that

$$\int_0^\infty |P(x) - P_n(x)|^2 dx < 1/n^2.$$

It follows from the inequality (2.2) that the sequence $\{P_n(x)\}$ converges strongly to $P(x)$, and that

$$\int_0^\infty P^2(s) ds = \lim_{n \rightarrow \infty} \int_0^\infty P_n^2(s) ds.$$

We now introduce the functions

$$\Pi(\xi, t) = \int_0^t P(s) \phi(s, \xi) ds \quad (4.1)$$

$$\text{and} \quad \Pi_n(\xi, t) = \int_0^t P_n(s) \phi(s, \xi) ds. \quad (4.2)$$

The existence of these integrals for all finite values of t follows from the fact that the functions $P_n(x)$, $P(x)$, and $\phi(x, \xi)$ are all of class $[L^2]$ over any finite interval of x .

If $0, t_1, t_2, \dots, t_m, t_{m+1}, \dots, t$ are the frontier points of the stretches in which $P_n(x)$ is constant, then

$$\Pi_n(\xi, t) = \Sigma P_n(t_m) \cdot \{y(t_{m+1}, \xi) - y(t_m, \xi)\}.$$

Since the system $\{\phi(x, \xi)\}$ is orthogonal in ξ , it follows that

$$\int_0^\infty \Pi_n^2(\sigma, t) d\sigma = \Sigma P_n^2(t_m) \cdot (t_{m+1} - t_m) = \int_0^t P_n^2(s) ds.$$

By a similar argument it follows that

$$\int_0^\infty |\Pi_n(\sigma, t) - \Pi_m(\sigma, t)|^2 d\sigma = \int_0^t |P_n(s) - P_m(s)|^2 ds.$$

Hence there exists a limit function to which the sequence $\{\Pi_m(\xi, t)\}$ converges strongly. It follows from the strong convergence of $P_n(s)$ and from equations (4.1) and (4.2) that this limit function is $\Pi(\xi, t)$.

It may now be deduced that

$$\begin{aligned} \int_0^\infty \Pi^2(\sigma, t) d\sigma &= \lim_{n \rightarrow \infty} \int_0^\infty \Pi_n^2(\sigma, t) d\sigma \\ &= \lim_{n \rightarrow \infty} \int_0^t P_n^2(s) ds = \int_0^t P^2(s) ds. \end{aligned} \quad (4.3)$$

Similarly, since $\{\Pi_n(\xi, t_1) - \Pi_n(\xi, t_2)\}$ converges strongly to

$$\{\Pi(\xi, t_1) - \Pi(\xi, t_2)\},$$

it follows that, if $t_1 < t_2$,

$$\int_0^\infty |\Pi(\sigma, t_1) - \Pi(\sigma, t_2)|^2 d\sigma = \int_{t_1}^{t_2} P^2(s) ds.$$

Hence there exists a limit function $\Pi(\xi)$ to which the sequence $\{\Pi(\xi, t)\}$ converges strongly.

To obtain an explicit expression for $\Pi(\xi)$ we note that

$$\begin{aligned} \int_0^\xi \Pi(\sigma) d\sigma &= \lim_{t \rightarrow \infty} \int_0^\xi \Pi(\sigma, t) d\sigma \\ &= \lim_{t \rightarrow \infty} \int_0^\xi d\sigma \int_0^t P(s) \phi(s, \sigma) ds \\ &= \lim_{t \rightarrow \infty} \int_0^t P(s) ds \int_0^\xi \phi(s, \sigma) d\sigma = \int_0^\infty P(s) \eta(s, \xi) ds. \end{aligned}$$

But, since

$$\left[\int_0^\infty |P(s) \eta(s, \xi)| ds \right]^2 \leq \int_0^\infty P^2(s) ds \int_0^\infty \eta^2(s, \xi) ds \leq \xi \int_0^\infty P^2(s) ds,$$

it follows that the function $\int_0^\xi \Pi(\sigma) d\sigma$ has bounded total variation in any finite interval of values of ξ . Hence this function possesses a finite derivative except at a set of points of measure zero. Hence we have almost everywhere

$$\Pi(\xi) = \frac{d}{d\xi} \int_0^\infty P(s) \eta(s, \xi) ds. \quad (4.4)$$

This completes the proof of the existence of the first integral transform of $P(x)$ (Theorem I). The second integral transform of a function $\Gamma(\xi)$ is similarly defined (*cf.* p. 247) as

$$G(x) = \frac{d}{dx} \int_0^\infty \Gamma(\sigma) y(x, \sigma) d\sigma. \quad (4.5)$$

These two transforms are distinct unless the basic function $\phi(x, \xi)$ is symmetrical in x and ξ .

It follows at once from the strong convergence of $\{\Pi(\xi, t)\}$ to $\Pi(\xi)$ that

$$\begin{aligned} \int_0^\infty \Pi^2(\sigma) d\sigma &= \lim_{t \rightarrow \infty} \int_0^\infty \Pi^2(\sigma, t) d\sigma = \lim_{t \rightarrow \infty} \int_0^t P^2(s) ds \\ &\quad [\text{from equation (4.3)}] \\ &= \int_0^\infty P^2(s) ds. \end{aligned} \quad (4.6)$$

Similarly it may be shown that, if $G(x)$ is the second integral transform of $\Gamma(\xi)$, then

$$\int_0^\infty G^2(s) ds = \int_0^\infty \Gamma^2(\sigma) d\sigma. \quad (4.7)$$

Equations (4.6) and (4.7) constitute the two parts of Parseval's theorem for an asymmetric basic function $\phi(x, \xi)$. (Theorem II.)

The proof of Theorem III depends upon the reciprocal relation

$$\int_0^\infty \Gamma(\sigma) \Pi(\sigma) d\sigma = \int_0^\infty P(s) G(s) ds. \quad (4.8)$$

This may be established by the method used by Titchmarsh (*loc. cit.*) for Fourier transforms. We have

$$\begin{aligned} \int_0^\tau \Gamma(\sigma) \Pi(\sigma, t) d\sigma &= \int_0^\tau \Gamma(\sigma) d\sigma \int_0^t P(s) \phi(s, \sigma) ds \\ &= \int_0^t P(s) ds \int_0^\tau \Gamma(\sigma) \phi(s, \sigma) d\sigma \\ &= \int_0^\tau P(s) G(s, \tau) ds, \end{aligned}$$

where

$$G(s, \tau) = \int_0^\tau \Gamma(\sigma) \phi(s, \sigma) d\sigma.$$

But the sequences $\{\Pi(\sigma, t)\}$, $\{G(s, \tau)\}$ converge strongly to $\Pi(\sigma)$ and

$G(s)$. Hence

$$\lim_{t \rightarrow \infty} \int_0^t \Gamma(\sigma) \{ \Pi(\sigma) - \Pi(\sigma, t) \} d\sigma = 0,$$

and

$$\lim_{\tau \rightarrow \infty} \int_0^\tau P(s) \{ G(s) - G(s, \tau) \} ds = 0,$$

and the reciprocal relation (4.8) follows at once.

Now write $\Gamma(\xi) = \Pi(\xi)$ in this relation. Then

$$\begin{aligned} \int_0^\infty \{ P(s) - G(s) \}^2 ds &= \int_0^\infty P^2(s) ds - 2 \int_0^\infty P(s) G(s) ds + \int_0^\infty G^2(s) ds \\ &= \int_0^\infty P^2(s) ds - 2 \int_0^\infty \Pi^2(\sigma) d\sigma + \int_0^\infty \Pi^2(\sigma) d\sigma \\ &\quad \text{[from (4.7) and (4.8)]} \\ &= 0. \quad \text{[from (4.6)]} \end{aligned}$$

Hence $P(x) = G(x)$ almost everywhere in $(0, \infty)$, i.e.

$$P(x) = \frac{d}{dx} \int_0^\infty \Pi(\xi) y(x, \xi) d\xi$$

almost everywhere (Theorem III).

5. Heaviside's unit function.

The first integral transform of Heaviside's unit function, $H(c-x)$, defined by the equations

$$H(c-x) = \begin{cases} 1 & \text{if } x < c, \\ \frac{1}{2} & \text{if } x = c, \\ 0 & \text{if } x > c, \end{cases}$$

is evidently

$$\int_0^c \phi(s, \xi) ds = y(c, \xi).$$

If $H_\tau(c, x)$ is defined by the integral $\int_0^\tau y(c, \sigma) \phi(x, \sigma) d\sigma$, it follows from the results of § 4 that the sequence $\{H_\tau(c, x)\}$ converges strongly to $H(c-x)$.

The last section discussed the integral transforms of functions of summable square on the assumption that the system $\{\phi(x, \xi)\}$ was doubly orthogonal: this section discusses the integral transforms of summable functions on the additional assumption that the sequence $\{H_\tau(c, x)\}$ has *bounded convergence* to $H(c-x)$ for all positive values of c .

Let $P(x)$ be any function which is summable together with the products $P(x) \phi(x, \xi)$ over $(0, \infty)$. Then the first integral transform of $P(x)$ is

$$\Pi(\xi) = \int_0^\infty P(s) \phi(s, \xi) ds.$$

Since $P(x)$ is summable in $(0, \infty)$ there exists a stretch-wise constant function $P_n(x)$ such that

$$\int_0^\infty |P(x) - P_n(x)| dx < \frac{1}{nA},$$

where A is the upper bound of $|H_\tau(c, x)|$. Therefore

$$\begin{aligned} \int_0^\infty H(c-x) P(x) dx - \int_0^\infty H_\tau(c, x) P(x) dx \\ = \int_0^\infty H(c-x) [P(x) - P_n(x)] dx - \int_0^\infty H_\tau(c, x) [P(x) - P_n(x)] dx \\ + \int_0^\infty P_n(x) [H(c-x) - H_\tau(c, x)] dx. \end{aligned}$$

Now the first and second integrals on the right-hand side are both less, in absolute value, than

$$A \int_0^\infty |P(x) - P_n(x)| dx < \frac{1}{n};$$

and the third integral converges to zero as $\tau \rightarrow \infty$, since the sequence $H_\tau(c, x)$ has strong convergence to $H(c-x)$. Hence

$$\int_0^\infty H(c-x) P(x) dx = \lim_{\tau \rightarrow \infty} \int_0^\infty H_\tau(c, x) P(x) dx.$$

It follows that

$$\begin{aligned} \int_0^c P(x) dx &= \lim_{\tau \rightarrow \infty} \int_0^\infty P(x) dx \int_0^\tau y(c, \sigma) \phi(x, \sigma) d\sigma \\ &= \lim_{\tau \rightarrow \infty} \int_0^\tau y(c, \sigma) d\sigma \int_0^\infty P(x) \phi(x, \sigma) dx \\ &= \int_0^\infty \Pi(\sigma) y(c, \sigma) d\sigma. \end{aligned}$$

Hence, although the integral $\int_0^\tau \Pi(\sigma) \phi(x, \sigma) d\sigma$ may not converge as $\lambda \rightarrow \infty$, the expression obtained by integrating with respect to x under the sign of integration always converges to the integral of $P(x)$. This is the analogue of Fatou's theorem for Fourier series.

6. *The integral representation of functions.*

Sufficient conditions for the convergence of the integral

$$\int_0^\tau \Pi(\sigma) \phi(x, \sigma) d\sigma$$

to the limit

$$\left\{ \frac{1}{2}f(x-0) + \frac{1}{2}f(x+0) \right\}$$

are easily obtained by methods similar to those employed in Part I of this paper. The discussion centres round the null-integral

$$Z_\tau(x, s) = \int_0^\tau \phi(x, \sigma) \phi(s, \sigma) d\sigma.$$

It follows from the convergence of the sequence $H_\tau(c, x)$ that, if ϵ and η are two positive numbers,

$$\lim_{\tau \rightarrow \infty} \int_0^{x-\epsilon} Z_\tau(x, s) ds = \lim_{\tau \rightarrow \infty} \int_0^\tau y(x-\epsilon, \sigma) \phi(x, \sigma) d\sigma = 0.$$

Similarly

$$\lim_{\tau \rightarrow \infty} \int_{x+\eta}^\infty Z_\tau(x, s) ds = 0$$

and

$$\int_{x-\epsilon}^x Z_\tau(x, s) ds \rightarrow \frac{1}{2}, \quad \int_x^{x+\eta} Z_\tau(x, s) ds \rightarrow \frac{1}{2}, \quad \text{as } \tau \rightarrow \infty.$$

We make the final assumption that there exists a summable function $\psi(s)$ such that if $|s-x| > \delta > 0$, then $|Z_\tau(x, s)| < \psi(s)$ for all values of τ . The following results may now be established for any function $P(x)$ which is summable together with the products $|P(x)|\psi(x)$ and $P(x)\phi(x, \xi)$ over $(0, \infty)$.

I. The convergence of the integral $\int_0^\infty \Pi(\sigma) \phi(X, \sigma) d\sigma$ depends only upon the behaviour of $P(x)$ in the immediate vicinity of the point $x = X$.

II. If $P(x)$ has bounded variation in any interval $(X-\delta, X+\delta)$, then

$$\int_0^\infty \Pi(\sigma) \phi(X, \sigma) d\sigma = \frac{1}{2}f(X-0) + \frac{1}{2}f(X+0).$$

There remain for consideration the corresponding problems connected with "mixed" systems of orthogonal functions which contain an enumerable sequence $\{\phi_n(x)\}$ satisfying the conditions of orthogonality of Part I and a continuous system $\{\phi(x, \xi)\}$ satisfying the conditions of Part II. Such mixed systems may be treated by a suitable combination of the methods of Parts I and II.

• ON QUANTITATIVE SUBSTITUTIONAL ANALYSIS

By ALFRED YOUNG.

(Fourth Paper.)

[Received 30 August, 1929.—Read 14 November, 1929.]

The third paper of this series established the theorem that any permutation Q can be expressed in terms of certain substitutional units, in the form

$$Q = \sum a_{r,s}^Q P_r \sigma_{rs} N_s M_s$$

(we write here $N_s M_s$ for what was called there N_s^r). Here we show that

$$P_r \sigma_{rs} N_s M_s = \frac{f}{n!} \sum a_{r,s}^Q Q^{-1},$$

the point being that the numerical coefficient $a_{r,s}^Q$ is the same in both cases.

This gives a simple rule by which the group matrix may be obtained, Theorem I, § 2.

Section 3 has to do with a more detailed examination of the expression M_r .

Prof. Schur, "Über die Darstellung der symmetrischen Gruppe durch lineare homogene Substitutionen", *Berliner Sitz.* (1908), 664–678, has obtained a representation of the symmetric group by linear substitutions which is not merely equivalent to this one, but identical with it; the method is different, and so also is the form of the statement of the result. His paper has suggested Theorem II, § 4, which expresses the connection between the units used here and the *characteristic* of the irreducible linear group. The possibility suggested by Prof. Schur that the coefficients in this set of linear groups may be confined to 0, 1, –1 is shown

in § 5 not to be the case. Further, § 6 establishes one of his fundamental results by these methods.

The last part of this paper proves two theorems by means of which substitutional expressions in $n-1$ letters can be transformed into expressions in n letters. Theorem III does this for a single unit, and Theorem IV for the matrix form.

The three former papers in this series which appeared in these *Proceedings* (1), 33 (1901), 97; (1), 34 (1902), 361; (2), 28 (1927), 255, are quoted as Q.S.A. I, II, III. And a paper with the same title in the *Journal*, 3 (1927), 14, is quoted as Q.S.A.J.

We are here concerned with the group matrix, *i.e.* if the matrices of a linear substitution group be $A, B, C, \dots$, then the group matrix is the sum $\Sigma Ax_A, \dots$; that is, it is a matrix each of whose elements is a linear function of h variables x_A , there being one variable for each operation of the group.

Thus consider the symmetric group of degree 3, $\{abc\}$; there is a representation by means of matrices of two rows and columns, the group matrix can be written

$$\begin{bmatrix} y_1 + x_3 - y_2 - x_2, & y_2 + x_1 - y_3 - x_3 \\ y_3 + x_1 - y_2 - x_2, & y_1 + x_2 - y_3 - x_3 \end{bmatrix}.$$

This means that the several matrices of the group are the matrix coefficients of the different variables. Thus the coefficients of $y_1, y_2, y_3, x_1, x_2, x_3$ are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix},$$

corresponding to 1, (abc) , (acb) , (bc) , (ca) , (ab) .

See Dickson, *Modern algebraic theories* (1926), 251-270.

1. It is convenient to change slightly the notation used in the previous papers. The prepared standard forms formerly written $P_r N_r'^*$ are written here $P_r N_r M_r$, bringing into evidence that they are obtained by introducing a fresh factor M . Further, the numerical factor $f/n!$ will, in general, be supposed absorbed into the symbol $P_r N_r M_r$. This enables us to write

$$(P_r N_r M_r)^2 = P_r N_r M_r.$$

Moreover we use the symbol T with the definition

$$T = \sum_{r=1}^f P_r N_r M_r.$$

* Q.S.A. III, § 4, p. 263.

Then

$$T^2 = T,$$

and we have

$$1 = \sum T_{a_1, a_2, \dots, a_h}.$$

2. It has been shown* that any permutation Q can be expressed in the form

$$\sum a_{rs}^{(Q)} P_r \sigma_{rs} N_s M_s,$$

where $a_{rs}^{(Q)}$ is numerical. The matrices $[a_{rs}^{(Q)}]$ then form a group isomorphic with the symmetric group; there being one such group for each $T_{a_1, \dots, a_h}$. The degree of the matrices is f . The group matrix is $[\sum a_{rs}^{(Q)} x_Q]$.

Consider the operation of $\sum Q x_Q$ on an arbitrary function X . A set of $n!$ equations may be obtained by operating on the equation $\sum Q x_Q X = 0$ with each of the permutations of the symmetric group; they are linear equations satisfied by the $n!$ unknown quantities QX . The condition that there is a non-zero solution is that the numbers x_Q are such that

$$|x_{R^{-1}Q}| = 0. \quad (\text{I})$$

Now, writing Qx_Q in terms of the units $P_r \sigma_{rs} N_s M_s$, we find the condition expressed by k conditions

$$|\sum a_{rs}^{(Q)} x_Q| = 0. \quad (\text{II})$$

Thus the determinants II are the factors of I. Now Frobenius has shown that the determinant I has k irreducible factors, viz. the group determinants of the k irreducible linear substitution groups. Thus the expressions II are the group determinants of the irreducible groups. One of the determinants II cannot be a product of two of the group determinants, for the sum of the squares of the degrees is $n!$ in each case.

Thus $[\sum_Q a_{rs}^{(Q)} x_Q]$ is the group matrix of an irreducible representation of the symmetric group as a linear substitution group.

Now let b_{rs}^Q be the coefficient of the permutation Q in the expansion of $P_r \sigma_{rs} N_s M_s$, so that

$$P_r \sigma_{rs} N_s M_s = \sum_Q b_{rs}^Q Q. \quad (\text{III})$$

Then, since

$$Q = \sum_{\kappa, \lambda} a_{\kappa\lambda}^Q P_\kappa \sigma_{\kappa\lambda} N_\lambda M_\lambda,$$

we find that

$$P_r \sigma_{rs} N_s M_s = \sum_{\kappa, \lambda} \sum_Q b_{rs}^Q a_{\kappa\lambda}^Q P_\kappa \sigma_{\kappa\lambda} N_\lambda M_\lambda.$$

Hence

$$\sum_Q b_{rs}^Q a_{\kappa\lambda}^Q = 0, \quad (\text{IV})$$

* Q.S.A. III, § 7, pp. 267-269.

unless $\kappa = r$ and $\lambda = s$, and $a_{\kappa\lambda}^Q$ belongs to the same T as b_{rs}^Q ; also

$$\sum_Q b_{rs}^Q a_{rs}^Q = 1. \quad (V)$$

Since $[\sum_Q a_{rs}^Q x_Q]$ is an irreducible group matrix*

$$\sum_Q a_{\alpha\beta}^Q a_{\gamma\delta}^{Q^{-1}} = \frac{n!}{f} e_{\alpha\delta} e_{\gamma\beta}, \quad (VI)$$

where $[e_{rs}]$ is the unit matrix.

Keeping r, s fixed there are $n!$ equations IV and V which may be regarded as linear equations to determine the $n!$ unknowns b_{rs}^Q . These are linearly independent since the $n!$ quantities

$$Q = \sum a_{\kappa\lambda}^Q P_\kappa \sigma_{\kappa\lambda} N_\lambda M_\lambda$$

are independent. Hence the solution is unique. Consequently

$$b_{rs}^Q = \frac{f}{n!} a_{sr}^{Q^{-1}}.$$

Moreover, we see from III that $(n!/f) b_{rs}^Q$ is a rational integer. Thus we obtain

THEOREM I. *The group matrix of the irreducible linear substitution group of degree f corresponding to T is obtained from the matrix*

$$[m_{rs}] \equiv \left[\frac{n!}{f} P_s \sigma_{sr} N_r M_r \right]$$

by replacing every permutation Q^{-1} by the variable x_Q , the coefficients being all rational integers.

This gives a simple means of writing down a group of rational integral substitutions isomorphic with the symmetric group.

COROLLARY.
$$\sum_{r=1}^f \frac{n!}{f} P_r N_r M_r = \sum \chi_\sigma \sigma = T \frac{n!}{f},$$

where χ_σ is the characteristic of σ for this irreducible representation.

This follows at once from the above when we remember that the characteristic of a substitution is the sum of the elements in the leading diagonal.

* Schur, "Neue Begründung", *Berliner Sitzungsberichte* (1905), § 2, Theorem IV.

3. The actual form* of M_r in the expression $P_r N_r M_r$ requires a rather more explicit examination. The symbol σ_{rs} will be used here to denote that permutation which changes the tableau F_s to F_r .

Let $N_r P_s$ be not zero and let s be other than r ; then we know that $s > r$. Now N_r must contain a permutation ν_r , for which $\nu_r P_s \nu_r^{-1} = P_r^\dagger$. Similarly P_s contains a permutation ω_s for which $\omega_s N_r \omega_s^{-1} = N_s$.

Hence

$$\sigma_{rs} = \omega_r \nu_r \quad \text{and} \quad \sigma_{sr} = \nu_s \omega_s.$$

Now $\nu_r^{-1} \omega_r \nu_r$ is a permutation of

$$\nu_r^{-1} P_r \nu_r = \nu_r^{-1} \omega_r^{-1} P_r \omega_r \nu_r = \sigma_{sr} P_r \sigma_{rs} = P_s.$$

Hence

$$\sigma_{rs} = \nu_r \omega'_s.$$

This permutation ω'_s is the inverse of ω_s above, since

$$\sigma_{rs} = \omega_s^{-1} \nu_s^{-1} = \omega_s^{-1} \nu_s^{-1} \omega_s \cdot \omega_s^{-1} = \nu_r \omega_s^{-1}.$$

Thus, slightly changing the notation, we have, in this case,

$$\sigma_{rs} = \nu_r \omega_s, \quad (I)$$

where $\nu_r \omega_s \nu_r^{-1}$ belongs to P_r and $\omega_s^{-1} \nu_r \omega_s$ to N_s . Hence when $N_r P_s$ is not zero it is equal to $\pm N_r \sigma_{rs} P_s$ according as ν_r is even or odd. And, so far as P_s alone is concerned,

$$M_r = 1 - \omega_s.$$

It may happen that $N_s P_t \neq 0$ where $t > s$.

$$\begin{aligned} \text{Then (Q.S.A. III)} \quad N_r M_r &= N_r - \omega_s N_s (1 - \omega_t) \\ &= N_r (1 - \omega_s + \omega_s \omega_t). \end{aligned}$$

$$\text{In fact} \quad N_r \sigma_{rt} = N_r \nu_r \omega_s \nu_s \omega_t = N_r \nu'_r \nu_r \omega_s \omega_t = \pm N_r \omega_s \omega_t.$$

Thus, when the values of s for which

$$N_r P_s \neq 0$$

are $r_1, r_2, \dots$; the values of s for which

$$N_{r_\lambda} P_s \neq 0$$

* Q.S.A. III, § 4; Q.S.A.J., p. 15.

† Q.S.A. II, 363, § 13.

are $r_{\lambda 1}$, $r_{\lambda 2}$, etc., and so on; then

$$M_r = 1 - \Sigma \varpi_{r_\lambda} + \Sigma \varpi_{r_\lambda} \varpi_{r_{\lambda\mu}} - \Sigma \varpi_{r_\lambda} \varpi_{r_{\lambda\mu}} \varpi_{r_{\lambda\mu\nu}} + \dots$$

and here

$$r < r_\lambda < r_{\lambda\mu} < r_{\lambda\mu\nu} < \dots$$

It will generally be found convenient to state the law of sequence of tableaux a little differently from the statement in Q.S.A. III, § 4. We say that F_r precedes F_s when each of the letters a_{n2} , a_{n-1} , ..., $a_{n-\lambda+1}$ lies in the same row in both tableaux, but $a_{n-\lambda}$ lies in a lower row in F_r than in F_s . We may call this the sequence from the last letter. It is sometimes useful to use instead the sequence from the first letter, in which F_r precedes F_s when each of the letters a_1 , a_2 , ..., $a_{\lambda-1}$ lies in the same row in both tableaux, but a_λ lies in a higher row in F_r than in F_s . Everything that has been said, for instance, in Q.S.A. III, § 4, applies to either of these sequences.

To find the values of r_λ when the sequence is from the last letter, we take the tableau F_r , and make a permutation in the columns only, such that the last letter which is moved goes to a higher row; the new tableau F'_r is not standard, but, if it can be made standard by a permutation ϖ in its rows only, the resulting tableau F_{r_λ} gives one of the values of r_λ ; and the permutation needed for M_r in respect of it is $\varpi_{r_\lambda} = \varpi^{-1}$.

It is useful to notice that

$$M_r = 1 - \Sigma \pm \sigma_{rr_\lambda} + \Sigma \pm \sigma_{rr_{\lambda\mu}} \dots, \quad (\text{II})$$

the signs being determined in each case by the number of column transpositions.

We shall also use the expression

$$M_r = 1 - \Sigma \varpi_{r_\lambda} M_{r_\lambda},$$

where, for each r_λ , $N_r P_{r_\lambda}$ is not zero, and where

$$\varpi_{r_\lambda}^{-1} N_r \varpi_{r_\lambda} = N_{r_\lambda}.$$

Also $P_r N_r M_r = P_r N_r - \Sigma \pm P_r \sigma_{rr_\lambda} N_{r_\lambda} + \Sigma \pm P_r \sigma_{rr_{\lambda\mu}} N_{r_{\lambda\mu}} \dots$

Denote the matrix $[P_r \sigma_{rs} N_s M_s]$ by I , and the matrix $[P_r \sigma_{rs} N_s]$ by J . Let S be that matrix $[a_{rs}]$, in which a_{rr} is zero and a_{rs} is zero unless $N_s P_r$ is not zero, and then a_{rs} is ∓ 1 according as

$$P_r N_r P_s N_s = \pm P_r \sigma_{rs} N_s.$$

Then

$$I = J(E + S + S^2 + \dots).$$

And if a substitutional expression be $\Sigma X_{rs} P_r \sigma_{rs} N_s M_s = \Sigma Y_{rs} P_r \sigma_{rs} N_s$,

then $[X_{rs}] = [Y_{rs}](E + S + S^2 + \dots)$,

or $[Y_{rs}] = [X_{rs}](E - S)$.

4. We shall use the notation*

$$\Phi_{\alpha_1, \alpha_2, \dots, \alpha_h} = \Sigma \frac{\chi_{\beta_1 \beta_2 \dots \beta_n}}{\beta_1! \beta_2! \dots \beta_n!} \left(\frac{s_1}{1}\right)^{\beta_1} \left(\frac{s_2}{2}\right)^{\beta_2} \dots \left(\frac{s_n}{n}\right)^{\beta_n}$$

to define the characteristic of the irreducible component of the symmetric group of degree n , corresponding to $T_{\alpha_1, \alpha_2, \dots, \alpha_h}$. Then we have

THEOREM II. *When every cycle of degree r ($r = 1, \dots, n$) is replaced by the symbol s_r :*

(i) $T_{\alpha_1 \dots \alpha_h}$ becomes $f_{\alpha_1 \dots \alpha_h} \Phi_{\alpha_1 \dots \alpha_h}$;

(ii) $P_r N_r M_r$ becomes $\Phi_{\alpha_1 \dots \alpha_h}$;

(iii) $P_s \sigma_{sr} N_r M_r$ becomes zero;

(iv) $S.P_r N_r M_r$ becomes $\kappa \left(\frac{n!}{f}\right) \Phi_{\alpha_1 \dots \alpha_h}$, where κ is the coefficient of the identical permutation in $SP_r N_r M_r$, S being any substitutional expression, involving the n letters.

The first statement follows at once from the corollary to Theorem I.

We may write $T_{\alpha_1 \dots \alpha_h} = \left(\frac{f}{n!}\right) \bar{\Sigma} PN$,

where the $\bar{\Sigma}$ extends to all the $n!$ permutations of the n letters in PN . All the terms in the sum are similar, hence PN becomes $\Phi_{\alpha_1 \dots \alpha_h}$.

The tableaux F_r, F_s being both standard, and $r < s$, there is at least one pair of letters bc in the same row in F_r and in the same column in F_s for

$$N_s P_r = 0.$$

Hence the result of interchanging b and c in $P_r \sigma_{rs} N_s$ is merely to change its sign. Thus

$$\bar{\Sigma} P_r \sigma_{rs} N_s = 0.$$

Now we have seen, (II) § 3, that

$$M_r = 1 + \Sigma \pm \sigma_{rs} \quad (s > r);$$

* Schur, *Berliner Sitzungsberichte* (1908), 665.

hence

$$P_r N_r M_r = P_r N_r + \Sigma \pm P_r \sigma_{rs} N_s,$$

and

$$\bar{\Sigma} P_r N_r M_r = \bar{\Sigma} P_r N_r;$$

which proves the second statement.

When $r = f$, s cannot be greater than r and $M_r = 1$ always; and hence

$$\bar{\Sigma} P_s \sigma_{sr} N_r M_r = 0.$$

Let us suppose that this equation has been proved for all values of r greater than a given value. Then, for this value of r ,

$$P_s \sigma_{sr} N_r M_r = P_s \sigma_{sr} N_r (1 - \Sigma \varpi_{r\lambda} M_{r\lambda}) = P_s \sigma_{sr} N_r - \Sigma \pm P_s \sigma_{sr\lambda} N_{r\lambda} M_{r\lambda},$$

and hence

$$\bar{\Sigma} P_s \sigma_{sr} N_r M_r = 0,$$

unless $s = r_\lambda$; i.e. it is such that $N_r P_s$ is not zero. But in this case

$$\bar{\Sigma} P_{r_\lambda} \sigma_{r_\lambda r} N_r M_r = \bar{\Sigma} [P_{r_\lambda} \sigma_{r_\lambda r} N_r - P_{r_\lambda} \sigma_{r_\lambda r} N_r \varpi_{r_\lambda}],$$

since such terms as $\varpi_{r_\lambda} \varpi_{r_\lambda \mu} M_{r_\lambda \mu}$, which are introduced by M_{r_λ} , lead only to

$$\bar{\Sigma} \pm P_{r_\lambda} \sigma_{r_\lambda r_\lambda \mu} N_{r_\lambda \mu} M_{r_\lambda \mu},$$

which is zero by hypothesis. The second term on the right-hand side is derived from the first by the permutation ϖ_{r_λ} , which is contained in P_{r_λ} ; hence this sum is zero, and the third statement is proved.

We know that
$$S.P_r N_r M_r = \Sigma A_{sr} P_s \sigma_{sr} N_r M_r,$$

and hence

$$\bar{\Sigma} S.P_r N_r M_r = \Sigma A_{sr} \bar{\Sigma} P_s \sigma_{sr} N_r M_r.$$

The fourth statement then is the result of comparing the coefficient of s_1^n on the two sides, after replacing permutations by symbols s_r .

5. The linear group of § 2, Theorem I, is identical with that obtained by Schur*. He starts with a set of functions which are unchanged by all the operations of

$$P = \prod_{r=1}^{\lambda} \{a_{r1} \dots a_{r\alpha_r}\} = \prod_{r=1}^{\lambda} P_{\lambda_r},$$

and then considers the whole modul of such functions obtainable from these by the operations of the symmetric group. Amongst these he distinguishes a subdivision A obtained in effect by operating on a member of

* *Loc. cit.*

the first set by a positive symmetric group G , which contains all the letters of P_r and one more from the next lower group; A includes all those functions obtained by permutation of letters from these. He then shows that any member of M can be linearly expressed in terms of a certain set of f functions mod A . This set corresponds exactly to what I have called standard forms. The use of modulus A is in effect the application of the equations to which the forms PN have to submit. And hence the group obtained is the same. On p. 678 Schur remarks that in the case which I call $T_{a11\dots 1}$ the coefficients in the linear group are all either 0, 1, or -1 ; but says that he has not yet determined whether this is always so. The answer to that question is no.

Consider PN ; let us write ϖ for any permutation of P and ν for any permutation of N .

Then, when $\varpi_1\nu_1 = \varpi_2\nu_2$, we must have $\varpi_1 = \varpi_2$ and $\nu_1 = \nu_2$; and hence the coefficient of every permutation in PN when expanded is ± 1 . Similarly this is so in the case of $P_r\tau_{rs}N_s$. It is also true for $P_rN_rM_r$, and for $P_s\tau_{sr}N_rM_r$, in the case in which there is not more than one value of t , other than r , for which

$$N_rP_t \neq 0,$$

and no value of u , other than t , for which

$$N_tP_u \neq 0.$$

We have seen that the coefficients of the linear substitutions are the same as the coefficients of the expressions $P_r\sigma_{rs}N_sM_s$ when different permutations are represented in terms of them. In other words, they are the coefficients when a non-standard form $P\sigma N$ is expressed in terms of standard forms

$$\sum_{r=1}^f A_r P_r \sigma_r N.$$

Using the alphabetical sequence of letters, we find (Q.S.A. 271, Theorem IX)

$$\left\{ \begin{matrix} aefg \\ bcd \end{matrix} \right\} = 2 \left\{ \begin{matrix} abcd \\ efg \end{matrix} \right\} + [\{bcd\} \{efg\}] \left\{ \begin{matrix} abce \\ dfg \end{matrix} \right\}$$

(the operations included in the brackets $[\]$ operate on the letters of the operations which follow them; the tableaux of the P 's only are here expressed, the same N being present in each case).

Thus, in $T_{4,3}$ and naturally in other cases, the coefficients are not confined to ± 1 , as is seen at once by the theorem quoted.

6. Let $P'_1 N'_1$ belong to $T'_{\alpha_1, \dots, \alpha_h}$ with n letters, being derived from the first tableau of the set. Let g be the positive symmetric group of r more letters. And let $T_{\beta_1, \dots, \beta_k}$ refer to the whole $n+r$ letters; we shall take the sequence of letters as beginning with the same sequence as in $P'_1 N'_1$ and ending with the r additional letters. Consider

$$N'_1 T_{\beta_1, \dots, \beta_k} P'_1 g = \Sigma N'_1 P N M P'_1 g.$$

The sequence of letters begins $\alpha_1 \dots \alpha_{a_1}$, and, since these are the letters of the first group of P'_1 , if any one of them appears outside the first row of the tableau for PN , two of them must appear in its first column, and $PNMP' = 0$; hence the product

$$N'_1 P N M P'_1 g = 0$$

unless all the first α_1 letters lie in the first row of F . Let $b_1 b_2 \dots b_{a_2}$ be the next α_2 letters; then N'_1 contains as a factor $\{a_1 b_1\}' \dots \{a_{a_2} b_{a_2}\}'$, and so the product is zero when any of $b_1, \dots, b_{a_2}$ lie in the first row of F . Thus, by continuing the argument, we see that this product is zero, unless F is formed by adding the last r letters $z_1, \dots, z_r$ to the tableau F'_1 at the ends of its different rows and in a final row.

Moreover, owing to the presence of g , the number of letters β_{s+1} in the $(s+1)$ -th row cannot exceed α_s , otherwise two of the z letters will be in the same column of F and the product will again be zero. Hence

$$N'_1 P'_1 g = \Sigma N'_1 T_{\alpha_1 + \gamma_1, \alpha_2 + \gamma_2, \dots, \alpha_h + \gamma_h, r - \Sigma \gamma} P'_1 g,$$

where

$$\gamma_{s+1} \leq \alpha_s - \alpha_{s+1} \quad (s = 1, 2, \dots, h-1).$$

Consider then a particular term, defined by certain values of $\gamma_1, \dots, \gamma_h$. This is equal to

$$\Sigma N'_1 P_s N_s M_s P'_1 g,$$

where each term is derived from a tableau F_s described above. There are $r! / \{\gamma_1! \dots \gamma_h! (r - \Sigma \gamma)!\}$ of these terms.

Consider M_s ; if it contains a term σ_{st} , and any one of the first n letters lies in a lower row in F_t than in F_s , then $N_t P'_1$ is zero. Hence, taking the sequence of tableaux from the first letter, no permutations in which the first n letters are involved need be considered in finding M_s . But, if σ_{st} involve the z letters only, one of these letters must be removed to a higher row, and, since in the column on which it lies in F_s the other letters are unchanged by σ_{st} , $N_s P_t$ is zero; and hence M_s cannot contain permutations involving the last r letters alone. Thus

$$N'_1 P_s N_s M_s P'_1 g = N'_1 P_s N_s P'_1 g.$$

To find the coefficient of the identical permutation, consider first the part free from the last r letters. It is

$$\gamma_1! \gamma_2! \dots \gamma_h! (r - \Sigma \gamma)! \frac{f}{(n+r)!} \frac{n!}{f'} N'_1 P'_1 N'_1 P'_1,$$

where in each $N'_1 P'_1$ the constant $f'/n!$ is supposed to be absorbed,

$$f = f_{a_1 + \gamma_1, \dots, a_h + \gamma_h, r - \Sigma \gamma} \quad \text{and} \quad f' = f_{a_1, a_2, \dots, a_h}.$$

Here too

$$N'_1 P'_1 N'_1 P'_1 = N'_1 P'_1.$$

Thus the coefficient of the identical permutation in $N'_1 P_s N_s M_1 P'_1 g$ is

$$\gamma_1! \dots \gamma_h! (r - \Sigma \gamma)! \frac{f}{(n+r)!};$$

and in $N'_1 T_{a_1 + \gamma_1, \dots, a_h + \gamma_h, r - \Sigma \gamma} P'_1 g$ it is $r! f / (n+r)!$.

Hence, on replacing the permutations by the symbols s_r , $N'_1 P'_1 g$ becomes

$$r! \Sigma \Phi_{a_1 + \gamma_1, \dots, a_h + \gamma_h, r - \Sigma \gamma},$$

where $\gamma_{s+1} \leq a_s - a_{s+1}$ and $r - \Sigma \gamma \leq a_h$. Hence

$$\Phi_{a_1, \dots, a_h} \Phi_r = \Sigma \Phi_{a_1 + \gamma_1, \dots, a_h + \gamma_h, r - \Sigma \gamma},$$

and by repeated use of this we find that

$$\Phi_{a_1} \Phi_{a_2} \dots \Phi_{a_h} = \Phi_{a_1, a_2, \dots, a_h} + \Sigma c_\beta \Phi_{\beta_1, \dots, \beta_k},$$

where the first of the differences

$$\beta_1 - a_1, \quad \beta_2 - a_2, \quad \dots,$$

which does not vanish is positive, and the coefficients c_β are all positive integers*.

7. It is sometimes desirable to pass from what may be called the canonical form of a substitutional expression in a certain $T_{a_1 \dots a_h}$ to the same expression when there is one additional letter added. Before establishing the theorem by which this may be done, it is necessary to prove certain relations. For brevity, we shall understand by f the number $f_{a_1, a_2, \dots, a_h}$; and by $E_r f$ the same expression with a_r changed to $a_r + 1$.

Let

$$\phi(x) = \prod_{r=1}^h (x - a_r + r)$$

and

$$\psi(x) = (x + h + 1) \phi(x).$$

* See Schur, *loc. cit.*, 666.

Then
$$f = n! \frac{\prod_{r \leq s} \{(a_r - r) - (a_s - s)\}}{\prod_r (a_r + h - r)!},$$

and hence
$$E_\lambda f = (n+1)f \frac{\phi(a_\lambda + 1 - \lambda)}{\psi'(a_\lambda - \lambda)}$$

and
$$E_{h+1} f = f_{a_1, \dots, a_h, 1} = (n+1)f \frac{\phi\{1 - (h+1)\}}{\psi'\{- (h+1)\}}.$$

Hence
$$(n+1) \frac{\phi(x+1)}{\psi(x)} f = \sum_{\lambda=1}^{h+1} \frac{E_\lambda f}{x - (a_\lambda - \lambda)}. \quad (I)$$

And, since $\phi(a_r - r) = 0$,

$$\sum_{\lambda=1}^{h+1} \frac{E_\lambda f}{(a_r - r - 1) - (a_\lambda - \lambda)} = 0.$$

It is to be noticed that, in the only case when the denominator can be zero, the numerator is zero; and the term is supposed excluded before giving x a special value.

Let $x_1, x_2, \dots, x_k$ be k different numbers, and let

$$F(x) = (x - x_1)(x - x_2) \dots (x - x_k).$$

Then

$$\sum_{r=1}^{h+1} \frac{E_r f}{F(a_r - r)} = \sum_{r=1}^{h+1} \sum_{s=1}^k \frac{E_r f}{F'(x_s)(a_r - r - x_s)} = - \sum_{s=1}^k \frac{(n+1) \phi(x_s + 1)}{F'(x_s) \psi(x_s)} f.$$

by (I).

Hence, when all the numbers x_s are roots of $\phi(x+1) = 0$,

$$\sum_{r=1}^{h+1} \frac{E_r f}{F(a_r - r)} = 0. \quad (II)$$

8. THEOREM III. A tableau associated with $T_{a_1, a_2, \dots, a_h}$ is $\{a_{rs}\} = F$, the first suffix defining the row, and the second the column; PN is derived from F , then

$$PN = \sum_{\lambda=1}^{h+1} K_\lambda P_\lambda N_\lambda G_\lambda L_\lambda,$$

where $P_\lambda N_\lambda$ is formed from that tableau F_λ of $E_\lambda T_{a_1, a_2, \dots, a_h}$ obtained from F by placing one additional final letter x at the end of the λ -th row (except that in the case when $a_\lambda = a_{\lambda-1}$ this particular term is absent or zero), where

$$K_\lambda = K_{\lambda, h} K_{\lambda, h-1} \dots K_{\lambda, \lambda+1}$$

and

$$K_{\lambda, \mu} = 1 + \sum_{t=1}^{a_\mu} \frac{1}{\alpha_\lambda - \alpha_\mu + \mu - \lambda + 1} (a_{\mu t} x);$$

where

$$G_\lambda = \frac{1}{a_{\lambda-1} - a_\lambda} \{ (a_{1, a_\lambda+1} a_{1, a_\lambda+2} \dots a_{1, a_{\lambda-1}}) (a_{2, a_\lambda+1} \dots a_{2, a_{\lambda-1}}) \dots (a_{\lambda-1, a_\lambda+1} \dots a_{\lambda-1, a_{\lambda-1}}) \} ;$$

and where

$$L_\lambda = L_{\lambda, \lambda-2} L_{\lambda, \lambda-3} \dots L_{\lambda, 1},$$

and

$$L_{\lambda, \mu} = 1 - \frac{1}{a_\mu - a_\lambda + \lambda - \mu - 1} \sum_{t=a_\mu+1}^{a_\mu} \sum_{u=1}^{\mu} (a_{ut} x).$$

To prove this it is necessary to bear in mind the following simple facts :

(i) PN is affected by the coefficient $f/n!$, and $P_\lambda N_\lambda$ is affected by the coefficient

$$\frac{E_\lambda f}{(n+1)!} = \frac{f}{n!} \frac{\phi(a_\lambda + 1 - \lambda)}{\psi'(a_\lambda - \lambda)},$$

with the notation of the last paragraph.

(ii) $\sum_{t=1}^{a_\mu} (a_{\mu t} x)$ is permutable with any permutation of the letters $a_{\mu t}$ alone, and therefore with $\{a_{\mu_1} \dots a_{\mu_{a_\mu}}\}$, and hence with P .

Hence K_λ is permutable with P .

Similarly both G_λ and L_λ are permutable with N .

(iii) When $s \neq u$ and $t \neq v$, and there is a letter a_{sv} , i.e. $v \leq a_s$ (which always happens when $s < v$),

$$P(a_{st} x)(a_{uv} x) N = 0.$$

$$\text{For } P(a_{st} x)(a_{uv} x) N = P(a_{st} a_{uv})(a_{st} x) N = (a_{st} a_{uv}) P'(a_{st} x) N,$$

where P' is obtained from P by the interchange of a_{st} and a_{uv} , and hence contains the factor

$$\{a_{s1} a_{s2} \dots a_{st-1} a_{uv} a_{st+1} \dots a_{sv} \dots\};$$

the result follows at once since N has a factor

$$\{a_{1v} a_{2v} \dots a_{sv} \dots a_{uv} \dots\}'.$$

$$(iv) \quad P(a_{st} x)(a_{sv} x) N = P(a_{st} x) N,$$

$$P(a_{st} x)(a_{ut} x) N = -P(a_{ut} x) N.$$

Now

$$N_\lambda = \left[1 - \sum_{t=1}^{\lambda-1} (a_{t, a_\lambda+1} x) \right] N.$$

$$\text{And } P_\lambda N_\lambda G_\lambda = P_\lambda \frac{1}{a_{\lambda-1} - a_\lambda} \sum_{\mu=a_\lambda+1}^{a_{\lambda-1}} \left[1 - \sum_{t=1}^{\lambda-1} (a_{t\mu} x) \right] N = P_\lambda L_{\lambda, \lambda-1} N.$$

$$\text{Also} \quad P_\lambda = P \left[1 + \sum_{t=1}^{a_\lambda} (a_{\lambda t} x) \right] = P K_{\lambda\lambda}.$$

Hence from (1) and (2) the term

$$\begin{aligned} K_\lambda P_\lambda N_\lambda G_\lambda L_\lambda \\ &= \frac{\phi(a_\lambda + 1 - \lambda)}{\psi'(a_\lambda - \lambda)} P K_\lambda K_{\lambda\lambda} L_{\lambda\lambda-1} L_\lambda N \\ &= \frac{\phi(a_\lambda + 1 - \lambda)}{\psi'(a_\lambda - \lambda)} P \Sigma C(a_{r_1 u_1} x) (a_{r_2 u_2} x) \dots (a_{r_\sigma u_\sigma} x) (a_{s_1 v_1} x) \dots (a_{s_\tau v_\tau} x) N, \end{aligned}$$

where C is a numerical coefficient, and the number of transpositions appearing after it may be anything from 0 to h ; but where

$$h \geq r_1 > r_2 \dots > r_\sigma \geq \lambda > s_t \quad (\text{where } t = 1, \dots, \tau),$$

where, of course, $u_t \leq a_{r_t}$, and also

$$a_\lambda < v_1 < v_2 \dots < v_\tau \leq a_1.$$

Owing to these inequalities no letter can appear twice in these transpositions except x . If $u_1 > u_2$ there is a letter $a_{r_1 u_2}$, and hence, by (iii), the term is zero. If $u_1 = u_2$, we may use (iv) to remove $(a_{r_1 u_1} x)$, changing the sign. Thus we need consider only terms

$$u_1 < u_2 < u_3 \dots < u_\sigma \leq a_\lambda < v_1 < v_2 \dots < v_\tau \leq a_1,$$

Again, when $s_{\tau-1} < s_\tau$ there is always a letter $a_{s_{\tau-1} v_\tau}$, and hence, by (iii), the term is zero. And so we need consider only the terms

$$h \geq r_1 > r_2 \dots > r_\sigma \geq \lambda > s_1 > s_2 \dots s_\tau \geq 1.$$

A product so reduced will be called a reduced product.

For convenience, to simplify statement and proof, the case is stated and proved for $a_1 > a_2 \dots > a_h$; the statement is true in all cases, when it is understood that the λ -th term is missing when $a_{\lambda-1} = a_\lambda$, and the proof needs alteration only in slight detail. A diagram illustrating the λ -th term is given.

$$\begin{array}{cccccccc} \cdot & \cdot & \cdot & \cdot & o & o & o & o & o \\ \cdot & \cdot & \cdot & \cdot & o & o & o & o & \\ \cdot & \cdot & \cdot & \cdot & o & o & & & \\ \times & \times & \times & \times & x & & & & \\ \times & \times & \times & & & & & & \\ \times & \times & & & & & & & \\ \times & & & & & & & & \end{array}$$

Here $\lambda = 4$ and PN belongs to $T_{9, 8, 6, 4, 3, 2, 1}$. The places marked with a cross indicate the letters which appear in $K_\lambda K_{\lambda\lambda}$, and those marked with an o the letters which appear in $L_{\lambda, \lambda-1} L_\lambda$.

Let us consider the coefficient necessarily attached to the transposition $(a_{rs}x)$ in a reduced product belonging to the λ -th term, where $\lambda > r$, and hence $(a_{rs}x)$ appears in $L_{\lambda, \lambda-1} L_\lambda$, and $a_r \geq s > a_\lambda$.

Let s lie between a_{p+1} and a_p , *i.e.*

$$a_{p+1} < s \leq a_p.$$

Then the transposition $(a_{rs}x)$ appears in the factor $L_{\lambda, p}$. We have seen that any term which has a factor $(a_{\sigma\tau}x)$ to the left of this, where $r \leq \sigma$, either is zero or would remove the transposition $(a_{rs}x)$. We may go further and say that a transposition $(a_{\sigma\tau}x)$ does not exist on the left of $(a_{rs}x)$ for which $\sigma \leq \rho$, for then, by (iii), this product is zero, there being a letter $a_{\sigma s}$.

Thus any transposition $(a_{\sigma\tau}x)$ on the left of $(a_{rs}x)$ is such that $\sigma > \rho$.

Similarly a transposition $(a_{r\tau}x)$ on the right of $(a_{rs}x)$, when $r > \sigma$, will still mean a zero term by (iii) unless there is no letter $a_{r\tau}$; *i.e.* unless $\tau > a_r$. But on the right of $(a_{rs}x)$ we have to remember those possible terms $(a_{r\tau}x)$ which are such that

$$P(a_{rs}x)(a_{r\tau}x)N = P(a_{r\tau}x)N.$$

Such terms appear in $L_{\lambda, \rho-1} L_{\lambda, \rho-2} \dots L_{\lambda, r}$, and moreover, from what we have just seen, it is not possible for $(a_{rs}x)$ to appear in a reduced product with any other transposition from these factors. Thus associated with $(a_{rs}x)$ we have the numerical coefficient

$$\frac{-1}{a_p - a_\lambda + \lambda - \rho - 1} \left[1 - \frac{a_{p-1} - a_p}{a_{p-1} - a_\lambda + \lambda - \rho} \right] \dots \left[1 - \frac{a_r - a_{r+1}}{a_r - a_\lambda + \lambda - r - 1} \right]$$

(the factors in this product arising severally from $L_{\lambda, \rho}$, $L_{\lambda, \rho-1}$, ..., $L_{\lambda, r}$)

$$\begin{aligned} &= \frac{1}{(a_\lambda - \lambda) - (a_p - \rho - 1)} \left[\frac{(a_\lambda - \lambda) - (a_p - \rho - 1) - 1}{(a_\lambda - \lambda) - (a_{p-1} - \rho)} \right] \dots \\ &\quad \left[\frac{(a_\lambda - \lambda) - (a_{r+1} - r - 2) - 1}{(a_\lambda - \lambda) - (a_r - r - 1)} \right] \\ &= \left[1 - \frac{1}{(a_\lambda - \lambda) - (a_p - \rho - 1)} \right] \left[1 - \frac{1}{(a_\lambda - \lambda) - (a_{p-1} - \rho)} \right] \dots \\ &\quad \frac{1}{(a_\lambda - \lambda) - (a_r - r - 1)}. \end{aligned}$$

Let us turn to the case where $\lambda \leq r$, and the factor $(a_{rs}x)$ appears in $K_\lambda K_{\lambda\lambda}$. The argument is very similar; this transposition appears in

$K_{\lambda r}$, but there are transpositions $(a_{\sigma s} x)$ which may appear in the factors $K_{\lambda, r+1}, K_{\lambda, r+2}, \dots, K_{\lambda, \rho}$, which have to be taken into account owing to the equation

$$P(a_{\sigma s} x)(a_{rs} x) N = -P(a_{rs} x) N.$$

No other transposition from the $r-\rho$ factors can appear in a reduced product. And thus the coefficient associated with the transposition is of exactly the same form as before. The value of the coefficient depends entirely on the transposition, and not on whether it appears in a K or an L factor. Consider then any reduced product. The different transpositions have coefficients which are each derived from a certain set of the factors $K_{\lambda\mu}, L_{\lambda\mu}$, and these sets cannot overlap, so that the actual coefficient of the whole is expressed as a sum of terms each of which is of the form

$$\frac{E_{\lambda} f}{(n+1) f F(a_{\lambda}-\lambda)},$$

where $F(x)$ is a function all of whose zeros are different and equal to some or all of $a_r - r - 1$, $r = 1, 2, \dots, h$.

The sum of all such terms from $\lambda = 1$ to $\lambda = h+1$ is zero by (II), § 7.

We have only then to calculate the numerical coefficient when x is absent. This is

$$\sum \frac{E_{\lambda} f}{(n+1) f} = 1,$$

a well known result*.

Thus the theorem is proved.

9. That the factors K and L are essentially the same except for the sign of the transpositions may be seen by considering the case where $a_{\lambda-1} = a_{\lambda}$. The λ -th term is absent. The μ -th term ($\mu < \lambda$) has the pair of factors

$$K_{\mu, \lambda} K_{u, \lambda-1} = \left[1 + \frac{1}{a_{\mu} - a_{\lambda} + \lambda - \mu + 1} \sum_{t=1}^{\alpha_{\lambda}} (a_{\lambda, t} x) \right] \left[1 + \frac{1}{a_{\mu} - a_{\lambda-1} + \lambda - \mu} \sum_{t=1}^{\alpha_{\lambda-1}} (a_{\lambda-1, t} x) \right].$$

But, since $a_{\lambda} = a_{\lambda-1}$,

$$\sum_{t=1}^{\alpha_{\lambda}} (a_{\lambda, t} x) \cdot (a_{\lambda-1, u} x) P_{\mu} N_{\mu} = (a_{\lambda-1, u} x) \sum_{t=1}^{\alpha_{\lambda}} (a_{\lambda, t} a_{\lambda-1, u}) P_{\mu} N_{\mu} = -(a_{\lambda-1, u} x) P_{\mu} N_{\mu},$$

* See, for example, Q.S.A., § 3 (2).

and hence

$$K_{\mu, \lambda} K_{\mu, \lambda-1} P_{\mu} N_{\mu} \\ = \left[1 + \frac{1}{\alpha_{\mu} - \alpha_{\lambda} + \lambda - \mu + 1} \left\{ \sum_{t=1}^{\alpha_{\lambda}} (a_{\lambda, t} x) + \sum_{t=1}^{\alpha_{\lambda-1}} (a_{\lambda-1, t} x) \right\} \right] P_{\mu} N_{\mu}$$

in formal analogy to the L factors.

10. Consider a standard tableau F_r of $T_{a_1, a_2, \dots, a_n}$ of n letters. If the last letter a_n be removed a standard tableau $F'_{r'}$ of $n-1$ letters is obtained. Moreover, when $N_r P_s$ is not zero and a_n lies in the same row in F_r and F_s , $N'_{r'} P'_{s'}$ is not zero. And in the same case when $N_r P_s$ is zero, $N'_{r'} P'_{s'}$ is also zero. For otherwise the vanishing would be caused by a letter a_t which is in the same column with a_n in F_r being in the same row with it in F_s . But the change from F_r to F_s can be made by a column permutation followed by a row permutation; the column permutation would have to bring a_t down to the row of a_n , which could only be done by changing the row in which a_n lies; in fact, by moving a_n to a higher row. Thus $P_r N_r M_r$ consists of a sum of terms $\pm P_r \sigma_{rp} N_p$ such that a_n lies in a higher row in F_p than in F_r , together with a sum of terms which becomes $P'_{r'} N'_{r'} M'_{r'}$ when the letter a_n is dropped.

11. Consider now the expression for $P'_{r'} N'_{r'}$ in terms of members of $T_{a_1, \dots, a_n} \equiv T$; i.e. the value of $T.P'_{r'} N'_{r'}$. We shall suppose the letter a_n to be added to the λ -th row, so that only one term $K_{\lambda} P_r N_r G_{\lambda} L_{\lambda}$ in the expression given by Theorem III for $P'_{r'} N'_{r'}$ has to be considered. So far as the rows in the matrix expression for $P'_{r'} N'_{r'}$ are concerned, these are defined by $K_{\lambda} P_r$. First there are terms in the row P_r , the numerical coefficient on this side being unity; in the other terms the row is defined by P_s , where a_n lies in a lower row in F_s than in F_r ; i.e. when we define the sequence of tableaux as from the last letter, these other terms lie in a higher row in the matrix than the first. Moreover, the places in the columns are defined by $N_r G_{\lambda} L_{\lambda}$ entirely independently of $K_{\lambda} P_r$. Thus the effect of K_{λ} is really that of multiplication on the left by a matrix I of f rows and columns like the T matrix, in which all the elements in the leading diagonal are unity and all the elements on the left of and below this diagonal are zero. Moreover, the f'^2 places about the leading diagonal corresponding to that part of the T matrix occupied by terms $P_r \sigma_{rs} N_s M_s$, for which a_n lies at the end of the λ -th row both in F_r and F_s , are occupied by a unit matrix $E_{f'}$.

The terms $N_r G_\lambda L_\lambda$ may be treated in the same way, and hence also the terms

$$N_r M_r G_\lambda L_\lambda = \Sigma \pm \sigma_{ri} N_i G_\lambda L_\lambda,$$

if we remember that we have proved in the last paragraph that $N_r M_r$ differs from that expression obtained from $N'_r M'_r$ by placing a_n at the end of the λ -th row, by terms in which a_n is at the end of a higher row. Thus the effect of the factor $G_\lambda L_\lambda$ is the multiplication on the right of the T matrix expression for $P_r N_r$ by a matrix J . This matrix J has similar properties to I , but we need not dwell on them. For we see now that, if S is a substitutional expression of the first $n-1$ letters, and the matrix equivalent of $S \cdot \Delta_\lambda T$ is $[T_\lambda]$, where $\Delta_\lambda T$ is $T_{a_1, a_2, \dots, a_{\lambda-1}, \dots, a_n}$, then the matrix expression for ST is

$$IMJ,$$

where M is that T matrix which consists of the matrices

$$[T_1], [T_2], \dots, [T_h]$$

down the leading diagonal and zero everywhere else.

Now let $S = 1$, then $[T_\lambda] = E$ and also $IMJ = E$, and therefore

$$J = I^{-1}.$$

THEOREM IV. *The matrix for $S.T$, where S is any substitutional expression involving $n-1$ letters $a_1, a_2, \dots, a_{n-1}$ and $T = T_{a_1, a_2, \dots, a_n}$ involves a last letter a_n , is*

$$IMI^{-1},$$

where M is that T matrix which consists of the matrices $[T_\lambda]$, $\lambda = 1, \dots, h$, down the leading diagonal and zeros elsewhere, and $[T_\lambda]$ is the matrix for $S \cdot \Delta_\lambda T$. The matrix I is of the form $[I_{\lambda\mu}]$, where the $I_{\lambda\mu}$ are partial matrices, $I_{\lambda\lambda}$ occupying the same place in I that $[T_\lambda]$ occupies in M . Then

$$I_{\lambda\lambda} = E, \quad I_{\lambda\mu} = 0 \quad (\mu < \lambda).$$

By applying this theorem successively we find that if S involves only the first m letters, then

$$S.T = IMI^{-1},$$

where M is a matrix which consists of the matrices corresponding to ST' , where T' has to do with the first m letters only, along the leading diagonal, and zeros everywhere else. And I has the same kind of property as in the previous case.

As an example, if $n = 5$ and $T = T_{3,2}$,

$$[T_{3,2}] = I_{3,2} \begin{bmatrix} [T_{3,1}] & 0 \\ 0 & [T_{22}] \end{bmatrix} I_{3,2}^{-1},$$

where
$$I_{3,2} = \begin{bmatrix} E_3 & M \\ 0 & E_2 \end{bmatrix} \quad \text{and} \quad M = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

(Here 0 stands for a zero matrix.)

12. In some cases the form of I can be readily calculated. Thus, when $h = 2$, and we consider $T_{\alpha,\beta}$, it is evident from K that

$$I_{\alpha,\beta} = \begin{bmatrix} E_{\alpha,\beta-1} \frac{1}{\alpha-\beta+2} M_{\alpha,\beta} \\ 0 & E_{\alpha-1,\beta} \end{bmatrix},$$

where $E_{\alpha,\beta}$ is the unit matrix of degree $f_{\alpha,\beta}$, and $M_{\alpha,\beta}$ has $f_{\alpha,\beta-1}$ rows and $f_{\alpha-1,\beta}$ columns. To find the form of $M_{\alpha\beta}$ we notice that

$$\frac{1}{\alpha-\beta+2} M_{\alpha\beta} [T_{\alpha-1,\beta}]$$

corresponds to $[K_{1,2}-1][T_{\alpha-1,\beta}]$. In the sequence of tableaux for $T_{\alpha-1,\beta}$ we take first those, $f_{\alpha-1,\beta-1}$ in number, which have α_{n-1} in the second row, and then those, $f_{\alpha-2,\beta}$ in number, which have α_{n-1} in the first row.

In the first set we consider first those transpositions $(a_t a_n)$ of K_{12} , where $t < n-1$; the effect of these is given by the matrix $M_{\alpha,\beta-1}$, for in all of them α_{n-1} remains in the second row.

In fact we may write

$$[T_{\alpha-1,\beta}] = \begin{bmatrix} [T_{\alpha-1,\beta-1}] & [A] \\ [B] & [T_{\alpha-2,\beta}] \end{bmatrix},$$

where in $[T_{\alpha-1,\beta-1}]$ both P and N tableaux have α_{n-1} in the lower row; while in $[T_{\alpha-2,\beta}]$ in both tableaux α_{n-1} lies in the upper row.

Now let $M_{\alpha\beta} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$, so that

$$M_{\alpha\beta} [T_{\alpha-1,\beta}] = \begin{bmatrix} m_{11} [T_{\alpha-1,\beta-1}] + m_{12} [B] & m_{11} [A] + m_{12} [T_{\alpha-2,\beta}] \\ m_{21} [T_{\alpha-1,\beta-1}] + m_{22} [B] & m_{21} [A] + m_{22} [T_{\alpha-2,\beta}] \end{bmatrix}.$$

The transposition $(\alpha_{n-1} \alpha_n)$ gives rise to the matrix m_{21} ; this transposition invariably changes a standard tableau with α_n above and α_{n-1} below into a standard tableau with α_{n-1} above and α_n below; and $m_{21} = E_{\alpha-1,\beta-1}$.

$m_{11} = M_{\alpha, \beta-1}$; $m_{12} = 0$, for none of the transpositions of K moves an a_{n-1} from the upper to the lower row; and $m_{22} = M_{\alpha-1, \beta}$.

Thus
$$M_{\alpha\beta} = \begin{bmatrix} M_{\alpha, \beta-1} & 0 \\ E_{\alpha-1, \beta-1} & M_{\alpha-1, \beta} \end{bmatrix}.$$

In particular, $I_{\alpha\alpha} = E$,

$$M_{\alpha, \alpha-1} = \begin{bmatrix} M_{\alpha, \alpha-2} \\ E_{\alpha-1, \alpha-1} \end{bmatrix},$$

and

$$M_{\alpha, 1} = [1 \ 1 \ \dots \ 1],$$

a matrix of one row and $\alpha-2$ columns.

In the case of $T_{\alpha, 1, 1, \dots, 1}$, called by Frobenius the case of unit rank,

$$\begin{aligned} K_1 P N &= \left[1 + \frac{1}{\alpha + h - 1} (b_h x) \right] \left[1 + \frac{1}{\alpha + h - 2} (b_{h-1} x) \right] \dots \\ &\quad \left[1 + \frac{1}{\alpha + 1} (b_2 x) \right] \begin{Bmatrix} a_1 a_2 \dots a_\alpha x \\ b_2 \\ \vdots \\ b_h \end{Bmatrix} \\ &= \begin{Bmatrix} a_1 a_2 \dots a_\alpha x \\ b_2 \\ \vdots \\ b_h \end{Bmatrix} + \frac{1}{\alpha + h - 1} \left[\begin{Bmatrix} a_1 a_2 \dots a_\alpha b_h \\ b_2 \\ \vdots \\ b_{h-1} \\ x \end{Bmatrix} - \begin{Bmatrix} a_1 a_2 \dots a_\alpha b_{h-1} \\ b_2 \\ \vdots \\ b_{h-2} \\ b_h \\ x \end{Bmatrix} + \dots \right. \\ &\quad \left. + (-)^h \begin{Bmatrix} a_1 a_2 \dots a_\alpha b_2 \\ b_3 \\ \vdots \\ b_h \\ x \end{Bmatrix} \right]. \end{aligned}$$

Then we may designate the transforming matrix as

$$I_{(\alpha, h)} = \begin{bmatrix} E_r & \frac{1}{\alpha + h - 1} m_{(\alpha, h)} \\ 0 & E_s \end{bmatrix},$$

where

$$r = \binom{\alpha + h - 3}{h - 2}, \quad s = \binom{\alpha + h - 3}{h - 1}.$$

Also

$$m_{(\alpha, h)} = \begin{bmatrix} -m_{(\alpha, h-1)} & 0 \\ E_t & m_{(\alpha-1, h)} \end{bmatrix},$$

where

$$t = \binom{\alpha + h - 4}{h - 2}.$$

ON QUANTITATIVE SUBSTITUTIONAL ANALYSIS

*(Fifth Paper).**By* ALFRED YOUNG.

[Received 30 August, 1929.—Read 14 November, 1929.]

For linear substitution groups of degree two the rotation groups of the regular solids in three dimensions are of fundamental importance. It is natural to believe that the rotation groups of the regular solids in Euclidean space of any number of dimensions must occupy a prominent place in the theory of linear substitution groups of higher degrees. It is well known that in space of n dimensions when $n > 4$ there are only three regular solids, the simplex, the figure corresponding to the cube which we may call the hypercube, and that which corresponds to the octohedron, which we call here the hyper-octohedron. The simplex in n dimensions has $n+1$ vertices, the symmetric permutation group of its vertices is practically the group which includes both the rotations and the reflections of the figure into itself; the alternating group is the pure rotation group. The extended group in the general case, corresponding to Klein's extended tetrahedral group, has been worked out by Schur* as the isomorphic fractional linear group.

The other two regular solids, corresponding to the cube and the octohedron, are reciprocal and hence have the same rotation group. Thus in general there are but two types of rotation group, viz.: the symmetric and the hyper-octohedral. It is thought then worth while to discuss the hyper-octohedral group in the same way as the symmetric. Here it is shown that it lends itself to almost identical treatment by means of this analysis. The irreducible representations as a linear group are obtained both in degree and actual form; like the symmetric group they may all be

* *Journal für Math.*, 139 (1911), 155-250.

represented as linear groups with coefficients which are positive or negative integers. A generating function for the group characters is obtained which is very similar to that of Frobenius for the symmetric group.

In §§ 7-10, a self-conjugate subgroup of index 2 is considered, and its irreducible representations are obtained, again with real and integral coefficients. Finally in § 11 it is shown that the extended tetrahedral group is a subgroup of the hyper-octohedral group in four dimensions.

1. The hyper-octohedral group can be treated in exactly the same way as the symmetric group. We consider such a group of degree n , *i.e.* that related to a hyper-octohedron in Euclidean space of n dimensions. Let the pairs of vertices be $A_1B_1, A_2B_2, \dots, A_nB_n$. Then the Abelian group $\{(A_1B_1), (A_2B_2), \dots, (A_nB_n)\}$ generated by the transpositions (A_rB_r) ($r = 1, 2, \dots, n$) is a self-conjugate subgroup. The factor group is the symmetric group of degree n .

The conjugate sets of operations may be compared to those of the symmetric group $\{a_1a_2\dots a_n\}$. Corresponding to a cycle $(a_1a_2\dots a_r)$ there are two possible permutations, viz.: $(A_1A_2\dots A_r)(B_1B_2\dots B_r)$, and $(A_1A_2\dots A_rB_1\dots B_r)$; in particular, corresponding to a cycle of one letter (a_r) , there are here two possibilities $(A_r)(B_r)$, and (A_rB_r) . Thus corresponding to a permutation of c cycles in the symmetric group, there must here be taken into account the different possibilities for each of these cycles separately.

The notation $\{A_1A_2\dots A_r\}_B$ will be used to denote

$$\{(A_1A_2)(B_1B_2), (A_1A_3)(B_1B_3), \dots, (A_1A_r)(B_1B_r)\},$$

that is the sum of the permutations of the group generated by these permutations, or the sum of the permutations which permute the A 's in any way, and at the same time the B 's in the same way. This will be called the AB positive symmetric group. In the same way $\{A_1A_2\dots A_r\}'_B$ is used to denote the AB negative symmetric group in exact analogy to the ordinary negative symmetric group.

Then $\{A_1A_2\dots A_r\}_{B_1}$ is used to denote

$$\frac{1}{2^r r!} \{A_1A_2\dots A_r\}_B \cdot \{(A_1B_1), (A_2B_2), \dots, (A_rB_r)\},$$

and $\{A_1A_2\dots A_r\}_{B_2}$ to denote

$$\frac{1}{2^r r!} \{A_1A_2\dots A_r\}_B \cdot \{-(A_1B_1), -(A_2B_2), \dots, -(A_rB_r)\}.$$

With the $n - \rho$ letters we form a tableau having β_1 letters in the first row, β_2 in the second, and so on, and β_k in the last, and from this we form the product

$$\frac{f_{\beta_1 \dots \beta_k}}{2^{n-\rho}(\rho!)^{\rho}} P_2 N_2 S'.$$

We take the product of these two expressions (which are permutable), then we sum this product for all permutations of the letters $A_1, A_2, \dots, A_n$ (the letters B being of course permuted simultaneously), and we call this sum

$$\beta_1 \beta_2 \dots \beta_k \bar{T}_{a_1 a_2 \dots a_k}.$$

2. THEOREM I. *The number of conjugate sets in the group is the same as the number of expressions*

$$\beta_1 \beta_2 \dots \beta_k \bar{T}_{a_1 a_2 \dots a_k}.$$

For consider any permutation of the group (as we have seen, it may be derived from a permutation of the symmetric group of n letters) corresponding to any cycle $(a_1 a_2 \dots a_r)$ of the symmetric group; we have either $(A_1 A_2 \dots A_r)(B_1 B_2 \dots B_r)$ or else $(A_1 A_2 \dots A_r B_1 B_2 \dots B_r)$ in our group. Now the permutation of the symmetric group belongs to a conjugate set $t_{\gamma_1 \gamma_2 \dots \gamma_l}$, where $\gamma_1, \gamma_2, \dots, \gamma_l$ are the degrees of its cycles.

Then, in the same way, in our group the permutation belongs to a conjugate set

$$\beta_1 \beta_2 \dots \beta_k t_{a_1 a_2 \dots a_k},$$

where a_r denotes the presence of the product of a pair of cycles $(A_1 A_2 \dots)(B_1 B_2 \dots)$ each of a_r letters, and β_r denotes the presence of a cycle of the form $(A_1 A_2 \dots A_{\beta_r} B_1 B_2 \dots B_{\beta_r})$.

Thus the conjugate sets and the expressions $\bar{T}$ are defined by exactly the same sets of numbers and are therefore equal in number.

THEOREM II. *The product of two different expressions $\bar{T}$ is zero.*

In the first place we observe that, in the expression $P_1 N_1 S$, if σ be any permutation of $P_1 N_1$, then

$$\sigma S = S \sigma,$$

for S contains all the letters contained in $P_1 N_1$, and every permutation of $P_1 N_1$ permutes the B 's in the same way as the A 's. Hence

$$P_1 N_1 S = S P_1 N_1,$$

and similarly

$$P_2 N_2 S' = S' P_2 N_2.$$

Let

$$P_1 N_1 S_1 \cdot P_2 N_2 S'_2$$

belong to $\bar{T}_1$, and $P_3N_3S_3 \cdot P_4N_4S'_4$ to $\bar{T}_2$. Then

$$P_1N_1S_1P_2N_2S'_2 \cdot P_3N_3S_3P_4N_4S'_4 = P_1N_1P_2N_2 \cdot S_1S'_2S_3S'_4 \cdot P_3N_3P_4N_4.$$

Now the groups S are Abelian, and it is at once clear that this is zero unless $S_1 \equiv S_3$ and $S'_2 \equiv S'_4$. But in this case P_1N_1 and P_3N_3 involve exactly the same letters, and also P_2N_2 and P_4N_4 involve the same letters. Hence this product is equal to

$$S_1S'_2P_1N_1P_3N_3 \cdot P_2N_2P_4N_4S_3S'_4,$$

and from the case of the symmetric group we know that this is zero, unless the tableaux for P_1N_1 and P_3N_3 have the same number of letters in each row; and also the tableaux P_2N_2 and P_4N_4 . That is, this product is always zero unless the $\bar{T}$'s are the same. That is, the product of different $\bar{T}$'s is zero.

THEOREM III.

$$1 = \beta_1\beta_2\ldots\beta_k C_{a_1a_2\ldots a_h} \beta_1\beta_2\ldots\beta_k T_{a_1a_2\ldots a_h}$$

where the C 's are numerical.

From the mode of formation it is manifest that

$$\bar{T} = \Sigma \lambda t,$$

where t is the sum of the members of a conjugate set, and the coefficients λ are *real* numbers.

Now the inverse of any permutation belongs to the same conjugate set as itself, and hence the square of such an expression as $(\Sigma \lambda t)$ cannot be zero when the λ 's are all real. Hence the square of $\bar{T}$ is never zero.

There can be no linear relation between the expressions $\bar{T}$. For if

$$\Sigma \mu \bar{T} = 0$$

we may multiply by $\bar{T}_1$ and every term except $\mu_1 \bar{T}_1^2$ vanishes, and hence $\mu_1 = 0$.

The number of expressions $\bar{T}$ is the same as the number of expressions t ; hence the equations

$$\bar{T} = \Sigma \lambda t$$

can be solved in the form

$$t = \Sigma \mu \bar{T}.$$

When we take for t the identical permutation we have the required result.

3. Standard tableaux may now be introduced as in the case of the symmetric group, a standard tableau being one in which the letters follow

the prearranged sequence in every row and in every column. Further we may introduce expressions M so that PNM is a prepared form as in the former case; and we further define PNM as having absorbed the numerical factor $f/n!$

Then, when $r \neq s$, and $P_r N_r M_r$, $P_s N_s M_s$ are two different standard prepared forms involving the same letters, their product is zero, while

$$(P_r N_r M_r)^2 = P_r N_r M_r.$$

Then consider $_{\beta_1 \beta_2 \dots \beta_k} \bar{T}_{\alpha_1 \alpha_2 \dots \alpha_h}$, where

$$\alpha_1 + \alpha_2 + \dots + \alpha_h = \rho,$$

$$\beta_1 + \beta_2 + \dots + \beta_k = n - \rho.$$

We obtain $\binom{n}{\rho} f_{\alpha_1 \alpha_2 \dots \alpha_h} f_{\beta_1 \beta_2 \dots \beta_k}$ standard units

$$P_1 N_1 M_1 S_1 \cdot P_2 N_2 M_2 S'_2,$$

where first a particular set of ρ letters is chosen for the first tableau F'_1 , the remaining letters being assigned to the second tableau F'_2 , which are both arranged in standard form. The combined tableau $F_1 F'_2$ formed by placing them side by side defines the particular term.

Then we take

$$_{\beta_1 \beta_2 \dots \beta_k} T_{\alpha_1 \alpha_2 \dots \alpha_h} = \Sigma P_1 N_1 M_1 S_1 \cdot P_2 N_2 M_2 S'_2,$$

where the Σ extends to forms obtained from standard tableaux only, and the factor $1/2^\rho$ is absorbed in S_1 , and $1/2^{n-\rho}$ in S'_1 .

THEOREM IV. *The product of $P_1 N_1 M_1 S_1 P_2 N_2 M_2 S'_2$ by*

$$P_3 N_3 M_3 S_3 P_4 N_4 M_4 S'_4,$$

i.e. of two different standard forms belonging to the same T , is zero.

This is obvious from the case of the symmetric group when the letters in the tableau F_1 are the same as those of the tableau F_3 . For then

$$P_1 N_1 M_1 P_3 N_3 M_3 = 0$$

or

$$P_2 N_2 M_2 P_4 N_4 M_4 = 0,$$

since F_1 and F_3 cannot be the same and also F'_2 and F'_4 .

If the letters in F_1 and F_3 are different then

$$S_1 S'_4 = 0.$$

Also it is plain that

$$(P_1 N_1 M_1 S_1 P_2 N_2 M_2 S'_2)^2 = P_1 N_1 M_1 S_1 P_2 N_2 M_2 S'_2.$$

Hence

$$T^2 = T.$$

Now we may write

$$T = \Sigma [\Sigma P_1 N_1 M_1 S_1] [\Sigma P_2 N_2 M_2 S'_2]$$

where the first Σ extends to the $\binom{n}{\rho}$ combinations of ρ letters, and the others to the standard groups for the sets of ρ and $n-\rho$ letters.

We know that $\Sigma P_1 N_1 M_1 S_1$ is a linear function of the conjugate sets of the ρ letters, *i.e.* it contains every member of a conjugate set in the same way. The same is true of $\Sigma P_2 N_2 M_2$. Also in T the letters are divided into the two sets in all possible ways, hence

$$T = \Sigma \lambda t,$$

and therefore T and $\bar{T}$ differ by a numerical factor.

Hence we have

$$\text{THEOREM V.} \quad 1 = \Sigma T$$

$$\text{and} \quad T^2 = T.$$

4. The permutations of the hyper-octohedral group may all be written in the form $\sigma\tau$ where σ is a permutation of the A 's multiplied by the same permutation of the B 's, and τ is a product of transpositions $(A_r B_r)$. Remembering the numerical factor absorbed in S_1 , we have

$$S_1^2 = S_1,$$

and hence $P_1 N_1 M_1 S_1 P_2 N_2 M_2 S'_2 = S_1 S'_2 P_1 P_2 N_1 N_2 M_1 M_2 S_1 S'_2$.

Then $\sigma\tau P_1 N_1 M_1 S_1 P_2 N_2 M_2 S'_2 = \pm \sigma P_1 N_1 M_1 P_2 N_2 M_2 S_1 S'_2$.

The operation of σ on the tableau $(F_1 F'_2)$ transforms it into another tableau not necessarily standard. But the process used before* may be applied to each tableau separately. Hence, if we combine the two tableaux in one $(FF')_r$, we obtain

$$\begin{aligned} \sigma\tau \cdot (PP')_r (NMN'M')_r S_r S'_r &= \Sigma \lambda_s (PP')_s \sigma_{sr} (NN'MM')_r S_r S'_r \\ &= \Sigma \lambda_s (SS'PP')_s \sigma_{sr} (NN'MM'SS')_r. \end{aligned}$$

Moreover the number of these *units* is

$$\Sigma \left[\binom{n}{\rho} f_{\alpha_1 \alpha_2 \dots \alpha_h} f_{\beta_1 \beta_2 \dots \beta_k} \right]^2 = \Sigma \binom{n}{\rho}^2 \rho! (n-\rho)! = 2^n \cdot n!$$

* Q.S.A., III, 269.

THEOREM VI. *The $2^n n!$ operations of the hyper-octohedral group can be expressed linearly in terms of the $2^n n!$ units*

$$(SS'PP')_r \sigma_{rs} (NMN'M'SS')_s.$$

The argument of Q.S.A., IV, §2, may be applied with practically no alteration to prove

THEOREM VII. *Corresponding to each $\beta_1 \beta_2 \dots \beta_n T_{\alpha_1 \alpha_2 \dots \alpha_n}$ there is a linear irreducible group isomorphic with the hyper-octohedral group. The group matrix is obtained from the matrix*

$$\left[\frac{2^n n_1! n_2}{f_\alpha f_\beta} (SS'PP')_s \sigma_{sr} (NMN'M'SS')_r \right], \quad (n_1 = \Sigma \alpha, \quad n_2 = \Sigma \beta)$$

by replacing each permutation Q^{-1} by the variable x_Q . Moreover, the coefficients are all real, rational, and integral.

The corollary

$$\Sigma PP'NN'MM'SS' = \frac{f_\alpha f_\beta}{2^n n_1! n_2!} \Sigma \chi_\sigma \sigma$$

follows at once as in the case of the symmetric group.

5. Frobenius* has shown that for the symmetric group the character $\chi_\rho^{(\lambda)}$ is the coefficient of $x_1^{\lambda_1+n-1} x_2^{\lambda_2+n-2} \dots x_n^{\lambda_n}$ in the expansion of

$$\Pi_r (x_1^r + x_2^r + \dots + x_n^r)^{\rho_r} \Delta_r,$$

where Δ_x is the determinant in which the element of the r -th row and s -th column is x_s^{n-r} : the numbers $\lambda_1 \geq \lambda_2 \dots \geq \lambda_n$ define the character (the same as is here defined by $T_{\lambda_1, \lambda_2, \dots}$): and the conjugate set defined by ρ is that which has ρ_r cycles of r letters, $r = 1, 2, \dots$

A corresponding generating function for the characters of the hyper-octohedral group is given as follows:

THEOREM VIII. *The character $\chi_{\rho, \rho'}^{\alpha, \beta}$ belonging to $\beta_1 \beta_2 \dots \beta_n T_{\alpha_1 \alpha_2 \dots \alpha_n}$ of a permutation of the conjugate set (ρ, ρ') , which has ρ_r pairs of cycles of the form $(A_1 \dots A_r)(B_1 \dots B_r)$ and ρ'_s cycles $(A_1 \dots A_s B_1 \dots B_s)$ (where $r, s = 1, 2, \dots$), is the coefficient of $x_1^{\alpha_1+n-1} x_2^{\alpha_2+n-2} \dots x_n^{\alpha_n} y_1^{\beta_1+n-1} y_2^{\beta_2+n-2} \dots y_n^{\beta_n}$ in the expansion of*

$$\Pi_{r,s} (x_1^r + x_2^r + \dots + x_n^r + y_1^r + \dots + y_n^r)^{\rho_r} (x_1^s + x_2^s + \dots + x_n^s - y_1^s - \dots - y_n^s)^{\rho'_s} \Delta_x \Delta_y.$$

When $t > h$, a_t is, of course, zero.

* *Berliner Sitzungsberichte* (1900), 519, etc.

It is to be observed that a pair of cycles (which will be called R_1) $(A_1 \dots A_r)(B_1 \dots B_r)$ is not altered in kind by multiplication by an even number 2κ of permutations $(A_s B_s)$, where s lies between 1 and r ; merely some letters A are exchanged with their corresponding B . Multiplication by an odd number of such permutations changes the kind to a single cycle R_2 , of the form $(A_1 A_2 \dots A_r B_1 \dots B_r)$. Exactly the same remarks apply to the cycle R_2 .

Consider a permutation made up entirely of cycles R_1 . It is asserted that its characters are the coefficients in the expansion

$$\Pi (x_1^r + x_2^r + \dots + x_n^r + y_1^r + \dots + y_n^r)^{p_r} \Delta_x \Delta_y.$$

The corollary to Theorem VII gives

$$\begin{aligned} \Sigma \chi_\sigma \sigma &= \Sigma [\Sigma P N M S] [\Sigma P' N' M' S'] \frac{2^n n_1! n_2!}{f_\alpha f_\beta} \\ &= \Sigma [2^{n_1} \Sigma \chi_{R_1} R_1 . S] [2^{n_2} \Sigma \chi_{R'_1} R'_1 . S'] \end{aligned} \quad (\text{I})$$

When σ is made up entirely of permutations R_1 , it is formed of pairs of cycles with all A 's in one cycle and all B 's in its fellow, multiplied by an even number of permutations $(A_s B_s)$ of the letters of the pair. The first Σ in (I) extends to all possible selections of the letters into the two divisions. The whole of each pair of cycles must appear among the letters of one division, and as the number of permutations $(A_s B_s)$ for each pair is even, these may be selected at will and without introducing a minus sign from S or S' . The letters x are taken to refer to the first division and the letters y to the second. Thus any pair of cycles of $2c_1$ letters introduces a factor

$$(x_1^{c_1} + x_2^{c_1} + \dots + x_n^{c_1} + y_1^{c_1} + y_2^{c_1} + \dots + y_n^{c_1}),$$

and since $\chi_{R_1}^{\alpha}$ is the coefficient of $x_1^{\alpha_1+n-1} \dots x_n^{\alpha_n}$ in

$$\Pi (x_1^c + x_2^c + \dots + x_n^c) \Delta_x$$

it is seen that $\chi_\sigma^{\alpha, \beta}$ is the coefficient of $x_1^{\alpha_1+n-1} \dots x_n^{\alpha_n} y_1^{\beta_1+n-1} \dots y_n^{\beta_n}$ in

$$\Pi (x_1^c + \dots + x_n^c + y_1^c + \dots + y_n^c) \Delta_x \Delta_y.$$

When there is a cycle R_2 of order $2d$ this is derived from a cycle R_1 by multiplication with an odd number of permutations $(A_s B_s)$. This introduces the factor

$$x_1^d + \dots + x_n^d - y_1^d - \dots - y_n^d.$$

6. The degree of the irreducible representation is obtained as the coefficient of

$$x_1^{\alpha_1+n-1} \dots x_n^{\alpha_n} y_1^{\beta_1+n-1} \dots y_n^{\beta_n}$$

in the expansion of

$$(x_1 + x_2 + \dots + x_n + y_1 + y_2 + \dots + y_n)^n \Delta_x \Delta_y.$$

Hence we obtain

$$\beta_1 \beta_2 \dots \beta_k f_{\alpha_1 \alpha_2 \dots \alpha_h} = \left(\begin{matrix} n \\ \Sigma \alpha \end{matrix} \right) f_{\alpha_1 \alpha_2 \dots \alpha_h} f_{\beta_1 \beta_2 \dots \beta_k}.$$

Similarly, any character may be obtained in terms of the characters of the symmetric group. For let

$$(x+y)^p (x-y)^q = \Sigma C_{p,q}^{r,s} x^r y^s.$$

Then

$$\beta_1 \beta_2 \dots \beta_k \chi_{\sigma_1 \sigma_2 \dots \sigma_n}^{\alpha_1 \alpha_2 \dots \alpha_h} = \Sigma (\Pi C_{\rho_r, \sigma_r'}^{\rho_r, \sigma_r'}) \chi_{\rho_1 \rho_2 \dots \rho_n}^{\alpha_1 \alpha_2 \dots \alpha_h} \chi_{\sigma_1' \sigma_2' \dots \sigma_n'}^{\beta_1 \beta_2 \dots \beta_k},$$

where $\rho_r' + \sigma_r' = \rho_r + \sigma_r$.

Thus, in the case $\Sigma \beta = 0$, we obtain

$$\beta_1 \beta_2 \dots \beta_k \chi_{\sigma_1 \sigma_2 \dots \sigma_n}^{\alpha_1 \alpha_2 \dots \alpha_h} = \chi_{\rho_1 + \sigma_1, \rho_2 + \sigma_2, \dots, \rho_n + \sigma_n}^{\alpha_1 \alpha_2 \dots \alpha_h}.$$

And in the case $\Sigma \alpha = 0$ we have

$$\beta_1 \beta_2 \dots \beta_k \chi_{\sigma_1 \sigma_2 \dots \sigma_n}^0 = (-)^{\Sigma \sigma} \chi_{\rho_1 + \sigma_1, \dots, \rho_n + \sigma_n}^{\beta_1 \beta_2 \dots \beta_k}.$$

In fact, the representations ${}_0 f_{\alpha_1 \alpha_2 \dots \alpha_h}$ are just those of the symmetric group, to which the hyper-octohedral group is multiply isomorphic. The representations $\beta_1 \beta_2 \dots \beta_k f_0$ are these again, except for a change of sign which accompanies every transposition $(A_r B_r)$.

The representations for which $\Sigma \beta = 1$ are nearly as simple.

The coefficient of $x^{\rho_1 + \sigma_1 - 1} y$ in $(x+y)^{\rho_1} (x-y)^{\sigma_1}$ is $\rho_1 - \sigma_1$. And hence

$$\beta_1 \beta_2 \dots \beta_k \chi_{\sigma_1 \sigma_2 \dots \sigma_n}^{\alpha_1 \alpha_2 \dots \alpha_h} = (\rho_1 - \sigma_1) \chi_{\rho_1 + \sigma_1 - 1, \rho_2 + \sigma_2, \dots, \rho_n + \sigma_n}^{\alpha_1 \alpha_2 \dots \alpha_h}.$$

7. The hyper-octohedral group has a subgroup of index 2, exactly analogous to the alternating group; its characters may be obtained from those of the alternating group by the methods used above. There is also in this case another subgroup of index 2, which, unlike that corresponding to the alternating group, has characters all real and rational; and, moreover, the linear irreducible groups isomorphic to it can all be expressed with coefficients which are all integers. If the sign of every transposition $(A_r B_r)$ is changed, half the permutations of the hyper-octohedral group change sign; the other half, whose sign remains unaltered, form a group of index 2, which we shall call the (AB) subgroup. This subgroup we proceed to examine.

When the sign of every $(A_r B_r)$ transposition is changed, ${}_{\beta_1 \dots \beta_k} T_{\alpha_1 \dots \alpha_h}$ becomes ${}_{\alpha_1 \dots \alpha_h} T_{\beta_1 \dots \beta_k}$, and every unit of one T is changed into the corresponding unit of the other T . Such a pair of representations T of the group may be called conjugate with respect to the (AB) subgroup. Now the sum of an expression Q (a linear function of the permutations) and Q' , the expression obtained by changing the sign of every (AB) transposition, is a linear function of the permutations of the (AB) subgroup. Thus, if Q_{rs} be a unit of ${}_{\beta_1 \dots \beta_k} T_{\alpha_1 \dots \alpha_h}$, and Q'_{rs} be the corresponding unit of ${}_{\alpha_1 \dots \alpha_h} T_{\beta_1 \dots \beta_k}$, then $Q_{rs} + Q'_{rs}$ will be a unit of the representation ${}_{\beta_1 \dots \beta_k} T_{\alpha_1 \dots \alpha_h} + {}_{\alpha_1 \dots \alpha_h} T_{\beta_1 \dots \beta_k}$ of the (AB) subgroup. Indeed we see that

$$(Q_{rs} + Q'_{rs})(Q_{uv} + Q'_{uv}) = 0$$

unless $u = s$, and then

$$(Q_{rs} + Q'_{rs})(Q_{st} + Q'_{st}) = Q_{rt} + Q'_{rt}.$$

Thus every pair of conjugate representations with respect to the subgroup gives us a single representation of the subgroup.

The only exception arises in the case of the self-conjugate representations ${}_{\alpha_1 \dots \alpha_h} T_{\alpha_1 \dots \alpha_h}$. These occur only when n is even, and then

$$\Sigma \alpha = \frac{n}{2}.$$

The number of these is the number of conjugate classes of permutations, or the number of irreducible representations of the symmetric group of degree $n/2$. From the analogy of the alternating group we should expect there to be a pair of conjugate representations of the subgroup, corresponding to each one of these self-conjugate cases; and this is what, in fact, we find.

8. Let us first consider the conjugate sets in the subgroup.

Let $\Gamma \equiv [\{A_1 \dots A_n\}_{B_2}]$

be an operation which operates on the letters of the permutations which follow it; it is the sum of all the permutations of the (AB) subgroup minus the sum of all the permutations of the hyper-octohedral group which do not belong to this subgroup.

Let s be a permutation of the (AB) subgroup, then, if all the permutations conjugate to s in the original group are also conjugate to s in the subgroup,

$$\Gamma s = 0.$$

But, if these permutations form a pair of conjugate sets in the subgroup,

$$\Gamma s = S_1 - S_2$$

and is not zero.

The permutation $(A_1 \dots A_r)(B_1 \dots B_r)$ belongs to the subgroup; hence $\Gamma(A_1 \dots A_r B_1 \dots B_r) = \Gamma(A_2 \dots A_r A_1 B_2 \dots B_r B_1) = -\Gamma(A_1 \dots A_r B_1 \dots B_r)$, since Γ contains the operation $-(A_1 B_1)$. Hence, when s contains any cycle of the second kind,

$$\Gamma s = 0.$$

Moreover, this equation is true when s contains a pair of cycles of the first kind of odd degree r , for let

$$s = (A_1 A_2 \dots A_r)(B_1 B_2 \dots B_r),$$

then Γ contains the operation $-(A_1 B_1)(A_2 B_2) \dots (A_r B_r)$. Hence the only conjugate sets which break up into a pair of conjugate sets for the subgroup are those which consist entirely of pairs of cycles of even degree of the first kind.

Now the permutations of the first kind all belong to the (AB) subgroup. And it has been shown that a pair of cycles of the first kind of degree δ , multiplied by an even number of $(A_r B_r)$ transpositions of the letters contained in them, is simply again a pair of cycles of the first kind of degree δ ; while, when they are multiplied by an odd number of such transpositions, the result is a single cycle of the second kind. Hence the permutations of the (AB) subgroup all contain an even number of cycles of the second kind. And, *vice versa*, the condition that a permutation is not to belong to this subgroup is that it should contain an odd number of cycles of the second kind. Now the difference between the number of conjugate sets of permutations having an even number of cycles of the second kind and the number of those having an odd number of such cycles is the coefficient of x^n in the expansion of

$$\begin{aligned} & \prod_{r=1}^n (1 + x^r + x^{2r} + \dots) \cdot \prod_{r=1}^n (1 - x^r + x^{2r} - \dots) \\ &= \prod_{r=1}^n \frac{1}{1 - x^r} \cdot \prod_{r=1}^n \frac{1}{1 + x^r} = \prod_{r=1}^n (1 + x^{2r} + x^{4r} + \dots). \end{aligned}$$

The first product relates to the cycles of the first kind, the second to those of the second kind. Hence this difference is equal to the number of conjugate sets which contain only pairs of cycles of the first kind of even degree. Let u , v be the numbers of conjugate sets having an even and odd number respectively of cycles of the second kind. Let w be the

number of conjugate sets having just pairs of even cycles of the first kind. Then

$$u-v=w,$$

and the number of conjugate sets in the (AB) subgroup is $v+2w$.

Now the coefficient of x^n in the expansion of

$$\prod_{r=1}^n (1+x^{2r}+x^{4r}+\dots)$$

is also the number of representations

$$a_1 \dots a_h T_{a_1 \dots a_h}.$$

The total number of representations we know to be $u+v$; since the number which are self-conjugate with respect to the (AB) subgroup is $w=u-v$, the number of conjugate pairs of representations is v .

9. Consider now a self-conjugate representation $a_1 \dots a_h T_{a_1 \dots a_h}$.

$$\varpi = S_{r_1} S'_{r_2} P_{r_1} P_{r_2} \sigma_{rs} N_{s_1} N_{s_2} M_{s_1} M_{s_2} S_{s_1} S_{s_2}$$

is a unit. When the pair $A_1 B_1$ appears both in $S_{r_1} P_{r_1}$ and in $N_{s_1} M_{s_1} S_{s_1}$, the interchange of A_1 and B_1 in ϖ leaves it unaltered; and then

$$\Gamma \varpi = 0.$$

Thus $\Gamma \varpi$ is zero unless the letters of $S_{r_1} P_{r_1}$ are the same as those of $N_{s_2} M_{s_2} S'_{s_2}$, and similarly the letters of $S'_{r_2} P_{r_2}$ are the same as those of $N_{s_1} M_{s_1} S_{s_1}$.

We arrange the sequence of tableaux so that the tableaux $(F_{r_1} F'_{r_2})$ and $(F_{r_2} F'_{r_1})$ are at the same distance from the beginning and end respectively of the sequence. Then, when $\Gamma \varpi$ is not zero, ϖ represents an element on the secondary diagonal of the matrix (that going from the top right-hand to the bottom left-hand corner). If, therefore, W be any substitutional expression,

$$\Gamma W = \lambda m,$$

where m is the matrix whose elements on the secondary diagonal are all unity and the other elements are zero; and λ is a constant, for Γ includes the sum of all the permutations of the (AB) subgroup. Then

$$m^2 = E = T,$$

and hence the pair of new classes is $T+m$ and $T-m$. And m and T are both linear functions of the members of the (AB) subgroup only. In order to find out the appropriate units, let us write

$$[P_{r_1} N_{r_1} M_{r_1} S_{r_1} \cdot P_{r_2} N_{r_2} M_{r_2} S'_{r_2} + P_{r_2} N_{r_2} M_{r_2} S_{r_2} \cdot P_{r_1} N_{r_1} M_{r_1} S'_{r_1}] = p_r.$$

p_r only contains members of the (AB) subgroup. Then, writing

$$p_r(T+m) = p_{r,1}, \quad p_r(T-m) = p_{r,2},$$

and noticing that both T and m are permutable with p_r , we have two complete sets of units, in number together equal to the number of units in T , and in properties identical with those of the units in the other representations.

Let σ be any permutation of the (AB) subgroup; then

$$\sigma(\Gamma W) = (\Gamma W)\sigma.$$

But, when σ is a permutation of the hyper-octohedral group not contained in the (AB) subgroup,

$$\sigma(\Gamma W) = -(\Gamma W)\sigma.$$

Hence, if U' be what any substitutional expression U becomes when the sign of every $(A_r B_r)$ transposition is changed,

$$U(\Gamma W) = (\Gamma W)U'.$$

10. To find the value of m , we have

$$\begin{aligned} m &= \frac{f}{2^n n!} \Gamma S_{r_1} S'_{r_2} P_{r_1} N_{r_1} M_{r_1} P_{r_2} N_{r_2} M_{r_2} \sigma_{r_1 r_2} P_{r_2} N_{r_2} M_{r_2} P_{r_1} N_{r_1} M_{r_1} S_{r_2} S'_{r_1} \\ &= \frac{f}{2^n n!} \Gamma S_{r_1} S'_{r_2} P_{r_1} N_{r_1} M_{r_1} P_{r_2} N_{r_2} M_{r_2} \sigma_{r_1 r_2}. \end{aligned}$$

Let $A_t B_t$ ($t = 1, \dots, \frac{1}{2}n = l$) be the letters of $P_{r_1} N_{r_1} M_{r_1} S_{r_1}$ and $C_t D_t$ ($t = 1, \dots, l$) the letters of $P_{r_2} N_{r_2} M_{r_2} S'_{r_2}$, arranged so that A_t and C_t occupy the same positions in their respective tableaux; and

$$\sigma = (A_1 C_1)(A_2 C_2) \dots (A_l C_l)(B_1 D_1) \dots (B_l D_l).$$

Using the brackets $[]$ again to denote that the enclosed operator operates on the letters of the permutations which follow, we see that

$$\begin{aligned} & [\{ C_1 C_2 \dots C_l \}_D] P_{r_1} N_{r_1} M_{r_1} P_{r_2} N_{r_2} M_{r_2} \sigma_{r_1 r_2} \\ &= [\{ C_1 C_2 \dots C_l \}_D] P_{r_1} N_{r_1} M_{r_1} \sigma_{r_1 r_2} P_{r_1} N_{r_1} M_{r_1} \\ &= [\{ C_1 C_2 \dots C_l \}_D] P_{r_1} N_{r_1} M_{r_1} P_{r_1} N_{r_1} M_{r_1} \sigma_{r_1 r_2} \\ &= [\{ C_1 C_2 \dots C_l \}_D] P_{r_1} N_{r_1} M_{r_1} \sigma_{r_1 r_2}. \end{aligned}$$

For $P_{r_1} N_{r_1} M_{r_1}$ contains permutations of the letters A alone, and the effect of permuting the C 's and D 's in $\sigma_{r_1 r_2}$ is the same as of permuting the A 's and B 's in $\sigma_{r_1 r_2}$. Moreover, Γ contains the group $\{ C_1 \dots C_l \}_D$ as a factor.

Consider, then, a cycle pair and the corresponding part of σ .

$$(A_1 A_2 \dots A_\rho)(B_1 B_2 \dots B_\rho)(A_1 C_1) \dots (A_\rho C_\rho)(B_1 D_1) \dots (B_\rho D_\rho) \\ = (A_1 C_1 A_2 C_2 \dots A_\rho C_\rho)(B_1 D_1 B_2 D_2 \dots B_\rho D_\rho) = \tau,$$

and $P_{r_1} N_{r_1} M_{r_1} \sigma_{r_1 r_2}$ is made up of permutations having only even pairs of cycles of the first kind.

Now the multiplication of this by an even number of transpositions $(A_t B_t)$ or $(C_t D_t)$ ($1 \leq t \leq \rho$) produces a cycle pair of the same kind, and multiplication by an odd number of such transpositions produces a cycle of the second kind. All permutations having such a cycle of the second kind disappear under the operation of Γ .

Now τ is a pair of cycles of the first kind of even degree, and permutations made up of such belong to a conjugate set which breaks up into a pair of sets in the (AB) subgroup. Two such permutations, which differ by the interchange of an odd number of AB pairs, will belong to different conjugate sets of the pair in the subgroup. The multiplication of τ by an even number of transpositions $(A_t B_t)$, or by an even number of transpositions $(C_t D_t)$, produces a pair of cycles obtainable from τ by an even number of AB exchanges. But multiplication by one $(A_t B_t)$ and one $(C_t D_t)$, for instance by $(A_1 B_1)(C_1 D_1)$, produces a pair of cycles obtainable from τ by an odd number of AB exchanges. Hence

$$m = \frac{f}{2^n n!} \Gamma S_{r_1} S'_{r_2} P_{r_1} N_{r_1} M_{r_1} P_{r_2} N_{r_2} M_{r_2} \sigma_{r_1 r_2} \\ = \frac{f}{2^n n!} \Gamma S_{r_1} S'_{r_2} P_{r_1} N_{r_1} M_{r_1} \sigma_{r_1 r_2} \\ = \frac{f}{2^{n+1} n!} \Gamma P_{r_1} N_{r_1} M_{r_1} \sigma_{r_1 r_2}.$$

For the multiplication by $(A_t B_t)(C_u D_u)$ from SS' is accompanied by a change of sign, and the corresponding odd number of (AB) exchanges in Γ is accompanied by a change of sign, so that in the product $S_{r_1} S'_{r_2}$ odd permutations may be ignored, and even permutations with either sign replaced by $+1$. Hence $2m$ may be obtained directly from $T_{a_1 \dots a_h}$, where the suffixes are the same as those of one set of the representation $a_1 \dots a_h T_{a_1 \dots a_h}$ with which we are concerned. We express $T_{a_1 \dots a_h}$ in terms of one set of $\frac{1}{2}n$ letters $A_1, A_2, \dots, A_l$. Then every cycle $(A_1 A_2 \dots, A_r)$ is replaced by the pair of cycles

$$(A_1 C_1 A_2 C_2 \dots A_r C_r)(B_1 D_1 B_2 D_2 \dots B_r D_r)$$

and the result is $2m$.

Thus the group characters of those conjugate sets which are the same for the (AB) subgroup as for the original group are the same as the corresponding characters in the original group in all irreducible representations; and the group characters of those pairs of conjugate sets in the (AB) subgroup which form single sets in the original group are the same for each member of the pair as the corresponding characters in the original group for all irreducible representations which are not self-conjugate, but in a self-conjugate representation $_{a_1 \dots a_h} T_{a_1 \dots a_h}$ the difference between the characters of a pair of sets is the corresponding character in the $T_{a_1 \dots a_h}$ representation of the symmetric group of degree $\frac{1}{2}n$.

11. The extended tetrahedral group appears as a subgroup of the hyperoctohedral group of degree 4 in the following manner. The conjugate sets are:—

$$h_0 = 1, \bar{1}; \quad h_1 = 1, (A_1 B_1)(A_2 B_2)(A_3 B_3)(A_4 B_4);$$

$$h_2 = 6, (A_1 A_2 B_1 B_2)(A_3 A_4 B_3 B_4); \quad h_3 = 4, R^2;$$

$$h_4 = 4, R^4; \quad h_5 = 4, R^5; \quad h_6 = 4, R;$$

where

$$R = (A_1 B_1 A_2 B_2 A_3 B_3)(A_4 B_4).$$

There is, in fact, a set of eight conjugate subgroups of this type; they are also conjugate in the (AB) subgroup.

Compare Frobenius, "Über die Composition der Charaktere einer Gruppe," *Berliner Sitzungsberichte* (1899), 339.

The relations between the characters of the whole group and of the subgroup (using the notation of Frobenius for the latter) are given by the equations:

$${}^0\chi^4 = {}^4\chi^0 = {}^0\chi^{1111} = {}^{1111}\chi^0 = \chi^{(0)},$$

$${}^0\chi^{22} = {}^{22}\chi^0 = \chi^{(1)} + \chi^{(2)},$$

$${}^0\chi^{31} = {}^{31}\chi^0 = {}^0\chi^{211} = {}^{211}\chi^0 = \chi^{(3)},$$

$${}^1\chi^3 = {}^3\chi^1 = {}^1\chi^{111} = {}^{111}\chi^1 = \chi^{(5)} + \chi^{(6)},$$

$${}^2\chi^2 = {}^{11}\chi^{11} = 2\chi^{(3)},$$

$${}^{11}\chi^2 = {}^2\chi^{11} = \chi^{(0)} + \chi^{(1)} + \chi^{(2)} + \chi^{(3)},$$

$${}^1\chi^{21} = {}^{21}\chi^1 = 2\chi^{(4)} + \chi^{(5)} + \chi^{(6)}.$$

The equations apply, of course, only to members of the subgroup.

A PROOF OF A THEOREM ON AVERAGES

By R. E. A. C. PALEY.

[Received and read 12 December, 1929.]

1. Let $\phi(\theta)$ be a function of period 2π absolutely integrable in $(0, 2\pi)$, and let $f(\theta)$ denote the upper bound of

$$\frac{1}{2x} \int_{\theta-x}^{\theta+x} |\phi(t)| dt.$$

Let

$$(1.1) \quad \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) = A_0(\theta) + \sum_{n=1}^{\infty} A_n(\theta)$$

be the Fourier series of $\phi(\theta)$. The significance of $f(\theta)$ lies in the fact that, if $\delta > 0$ and $\sigma_n^\delta(\theta)$ denotes the δ -th Cesàro mean of the series (1.1), for terms whose rank does not exceed n , then

$$|\sigma_n^\delta(\theta)| \leq B_\delta f(\theta),$$

where B_δ is a positive constant depending only on δ . Similarly, if $r < 1$ and $u(r, \theta)$ denotes the sum of the series

$$A_0(\theta) + \sum_{n=1}^{\infty} A_n(\theta) r^n,$$

then

$$(1.2) \quad u(r, \theta) \leq Bf(\theta),$$

where B is an absolute positive constant. These important facts were first pointed out by Hardy and Littlewood in a recent paper* in which

* Hardy and Littlewood, 1.

they consider how the behaviour of $f(\theta)$ can be deduced from that of $\phi(\theta)$. If $k > 1$, and

$$\frac{1}{2\pi} \int_0^{2\pi} |\phi(\theta)|^k d\theta < \infty,$$

then $\{f(\theta)\}^k$ is also integrable, and

$$(1.3) \quad \frac{1}{2\pi} \int_0^{2\pi} \{f(\theta)\}^k d\theta \leq B_k \cdot \frac{1}{2\pi} \int_0^{2\pi} |\phi(\theta)|^k d\theta,$$

where B_k is a constant depending only on k . In the case $k = 1$ Hardy and Littlewood show that this is no longer true. It is true, however, that, if $\phi(\theta)$ satisfies the condition

$$\frac{1}{2\pi} \int_0^{2\pi} |\phi(\theta)| \log^+ |\phi(\theta)| d\theta < \infty,$$

where $\log^+ |\phi(\theta)|$ denotes $\text{Max}(0, \log |\phi(\theta)|)$, then $f(\theta)$ is integrable in $(0, 2\pi)$ and

$$(1.4) \quad \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta \leq B \cdot \frac{1}{2\pi} \int_0^{2\pi} |\phi(\theta)| \log^+ |\phi(\theta)| d\theta + B,$$

where the numbers B are absolute positive constants. Hardy and Littlewood show in their paper that, if the result (1.3) has been established for any value of $k > 1$, then it is possible to deduce the following result.

THEOREM A. *Let $f(z) = f(r, \theta)$ be a function regular in the unit circle, and let*

$$\frac{1}{2\pi} \int_0^{2\pi} |f(r, \theta)|^\lambda d\theta \leq C^\lambda,$$

where $\lambda > 0$. Let $F(\theta)$ denote

$$\text{Max}_{0 \leq r < 1} |f(r, \theta)|.$$

Then

$$\frac{1}{2\pi} \int_0^{2\pi} \{F(\theta)\}^\lambda d\theta \leq A_\lambda C^\lambda,$$

where A_λ is a constant depending only on λ .

In Hardy and Littlewood's proof of the results (1.3), (1.4), the results are based on a theorem about discrete averages, from which the corresponding results about functions are eventually deduced. In this paper I give a shorter but no less curious proof. I first prove the result (1.3) for $k = 2$, using an argument due to Kolmogoroff and Seliverstoff* and after-

* Kolmogoroff and Seliverstoff, 1.

wards extended by Hardy and Littlewood*. Having proved the result (1.3) in the case $k = 2$, it is possible to deduce the remainder of the results by a method similar to that used by Hardy and Littlewood in proving Theorem A. The proof employs some very deep results in the theory of functions of a complex variable. It is very curious that the only two known proofs of an apparently elementary result should be so complicated.

2. THEOREM I. *Let $\phi(\theta)$ be a function of period 2π and of integrable square in $(0, 2\pi)$. Denote by $f(\theta)$ the upper bound of*

$$\frac{1}{2x} \int_{\theta-x}^{\theta+x} |\phi(t)| dt.$$

Then $f(\theta)$ is also of integrable square in $(0, 2\pi)$, and

$$\frac{1}{2\pi} \int_0^{2\pi} \{f(\theta)\}^2 d\theta \leq B \cdot \frac{1}{2\pi} \int_0^{2\pi} \{\phi(\theta)\}^2 d\theta,$$

where B is an absolute constant.

We observe that it is legitimate to assume that $\phi(\theta)$ is everywhere positive or zero. We prove the theorem first in the case where $\phi(\theta)$ is bounded. If $0 \leq r < 1$, we denote by $P(r, t)$ the function

$$\frac{1-r^2}{1-2r \cos t + r^2},$$

always positive in the range considered. $P(r, t)$ may be expanded in powers of r , and written in the form

$$1 + 2 \sum_{n=1}^{\infty} r^n \cos nt.$$

For fixed $r < 1$ this series is absolutely and uniformly convergent.

LEMMA 1.

$$\frac{1}{2\pi} \int_0^{2\pi} P(r_1, \theta-t) P(r_2, \psi-t) dt = P(r_1 r_2, \theta-\psi).$$

* In the proof of a theorem not yet published. Let $\phi(\theta)$ be a function of integrable square in $(0, 2\pi)$, and let $s(n_0, \theta)$ denote the sum to n_0 terms of the Fourier series of $\phi(\theta)$ at the point θ , where n_0 is an integer which varies with θ . Hardy and Littlewood prove that

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\{s(n_0, \theta)\}^2}{\log(n_0+2)} d\theta \leq B \cdot \frac{1}{2\pi} \int_0^{2\pi} \{\phi(\theta)\}^2 d\theta,$$

where B is an absolute constant.

Since the series

$$1 + 2 \sum_{n=1}^{\infty} r_1^n \cos n(\theta - t), \quad 1 + 2 \sum_{n=1}^{\infty} r_2^n \cos n(\psi - t)$$

are absolutely and uniformly convergent in the range $(0, 2\pi)$, we may multiply them together formally and integrate term by term, obtaining the required result.

LEMMA 2. If $t \leq 1 - r$,

$$P(r, t) \geq \frac{1}{2(1-r)}.$$

$$P(r, t) = \frac{(1-r)(1+r)}{(1-r)^2 + 4r \sin^2 \frac{1}{2}t} \geq \frac{1-r}{(1-r)^2 + 4(\frac{1}{2}t)^2} \geq \frac{1-r}{2(1-r)^2},$$

if $t \leq 1 - r$, and this gives the required result.

We employ the letter B to denote positive absolute constants (not always the same constant in different contexts).

LEMMA 3. $P(r_1 r_2, t) \leq B \{P(r_1, t) + P(r_2, t)\}.$

If $r_1 \geq r_2 > \frac{1}{2}$, then

$$\begin{aligned} P(r_1 r_2, t) &= \frac{1 - r_1^2 r_2^2}{(1 - r_1 r_2)^2 + 4r_1 r_2 \sin^2(\frac{1}{2}t)} \leq \frac{1 - r_2^4}{(1 - r_2)^2 + 2r_2 \sin^2(\frac{1}{2}t)} \\ &\leq \frac{2(1 + r_2^2)(1 - r_2^2)}{(1 - r_2)^2 + 4r_2 \sin^2(\frac{1}{2}t)} \leq 4P(r_2, t). \end{aligned}$$

If $r_2 \leq \frac{1}{2}$, then $P(r_2, t) \geq B$. Also, since $r_1 r_2 < \frac{1}{2}$, $P(r_1 r_2, t) \leq B$, from which the required result follows.

LEMMA 4*. $\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| t \frac{d}{dt} \{P(r, t)\} \right| dt \leq B.$

We observe that, for all t ,

$$t \frac{d}{dt} \{P(r, t)\} \leq 0.$$

Now

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} t \frac{d}{dt} \{P(r, t)\} dt &= \frac{1}{2\pi} [tP(r, t)]_{-\pi}^{\pi} - \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, t) dt \\ &= P(r, \pi) - \frac{1}{2} > -\frac{1}{2}. \end{aligned}$$

* Lemmas 4 and 5 are taken from Hardy and Littlewood's proof.

It follows that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| t \frac{d}{dt} \{P(r, t)\} \right| dt < \frac{1}{2}.$$

LEMMA 5. Let $g(\theta)$ denote the upper bound of

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, \theta - t) \phi(t) dt,$$

for $0 \leq r < 1$. Then $g(\theta) \leq Bf(\theta)$.

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, \theta - t) \phi(t) dt &= \frac{1}{2\pi} \int_{\theta - \pi}^{\theta + \pi} P(r, \theta - t) \phi(t) dt \\ (2.1) \qquad &= \frac{1}{2\pi} \left[P(r, \pi) \int_{\theta}^{\theta + \pi} \phi(t) dt - P(r, -\pi) \int_{\theta}^{\theta - \pi} \phi(t) dt \right. \\ &\quad \left. - \int_{\theta - \pi}^{\theta + \pi} \frac{d}{dt} \{P(r, \theta - t)\} \int_{\theta}^t \phi(x) dx dt \right]. \end{aligned}$$

$$\text{Since} \qquad \int_{\theta}^t \phi(x) dx \leq 2|t - \theta| f(\theta),$$

the expression (2.1) is less than

$$2f(\theta) P(r, \pi) + \frac{1}{2\pi} f(\theta) \int_{\theta - \pi}^{\theta + \pi} \left| \frac{d}{dt} \{P(r, \theta - t)\} (t - \theta) \right| dt \leq Bf(\theta).$$

LEMMA 6. Under the hypotheses of Lemma 5, $f(\theta) \leq Bg(\theta)$.

We can find a number $x = x_{\theta} \leq \pi$, such that

$$\frac{1}{2x} \int_{\theta - x}^{\theta + x} \phi(t) dt \geq Bf(\theta).$$

If $x \leq \frac{1}{2}$, we write $r_{\theta} = 1 - x$. Then

$$(2.2) \qquad \frac{1}{2\pi} \int_0^{2\pi} P(r_{\theta}, \theta - t) \phi(t) dt \geq \frac{1}{2\pi} \int_{\theta - x}^{\theta + x} P(r_{\theta}, \theta - t) \phi(t) dt.$$

By Lemma 2, the expression (2.2) is greater than

$$\frac{1}{4\pi x} \int_{\theta - x}^{\theta + x} \phi(t) dt \geq \frac{1}{4\pi} f(\theta).$$

If $x > \frac{1}{2}$, we write $r_{\theta} = \frac{1}{2}$. Then

$$\frac{1}{2\pi} \int_0^{2\pi} P(\tfrac{1}{2}, \theta - t) \phi(t) dt \geq B \int_0^{2\pi} \phi(t) dt \geq \frac{B}{2x} \int_{\theta - x}^{\theta + x} \phi(t) dt = Bf(\theta).$$

It follows that we can associate with each θ a number $r_\theta < 1$, such that

$$\frac{1}{2\pi} \int_0^{2\pi} P(r_\theta, \theta - t) \phi(t) dt \geq Bf(\theta).$$

It follows at once that $g(\theta) \geq Bf(\theta)$.

3. We now proceed to the proof of the theorem. The ideas underlying the proof are roughly as follows. In virtue of Lemmas 5 and 6 it is legitimate to interchange $f(\theta)$ and $g(\theta)$. Thus, where convenient, we may interchange $f(\theta)$ with the Poisson integral

$$\frac{1}{2\pi} \int_0^{2\pi} P(r_\theta, \theta - t) \phi(t) dt.$$

Now, if we apply the ideas of Kolmogoroff and Seliverstoff's theorem to the integral of $\{f(\theta)\}^2$, we may show that

$$\left[\frac{1}{2\pi} \int_0^{2\pi} \{f(\theta)\}^2 d\theta \right]^2 \leq B \cdot \frac{1}{2\pi} \int_0^{2\pi} \{\phi(\theta)\}^2 d\theta \frac{1}{2\pi} \int_0^{2\pi} f(\theta) F(\theta) d\theta,$$

where $F(\theta)$ is the function obtained from $f(\theta)$ in the same way as $f(\theta)$ is obtained from $\phi(\theta)$. We observe that, if we can prove that

$$\frac{1}{2\pi} \int_0^{2\pi} f(\theta) F(\theta) d\theta \leq B \cdot \frac{1}{2\pi} \int_0^{2\pi} \{f(\theta)\}^2 d\theta,$$

we can obtain the required result. But this is deducible from the half strength theorem

$$\frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) f(\theta) d\theta \leq B \cdot \frac{1}{2\pi} \int_0^{2\pi} \{\phi(\theta)\}^2 d\theta.$$

Hence we try Kolmogoroff and Seliverstoff's method again with the integral

$$\frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) f(\theta) d\theta,$$

and this time we are able to obtain the required result.

We first consider the integral of $\phi(\theta) f(\theta)$. We see by Lemma 6 that, by suitable choice of r_θ ,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) f(\theta) d\theta &\leq B \cdot \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) d\theta \frac{1}{2\pi} \int_0^{2\pi} P(r_\theta, \theta - t) \phi(t) dt \\ &= B \cdot \frac{1}{2\pi} \int_0^{2\pi} \phi(t) dt \frac{1}{2\pi} \int_0^{2\pi} P(r_\theta, \theta - t) \phi(\theta) d\theta. \end{aligned}$$

The change in the order of integration, here, and in what follows, is

legitimate since the integrand is bounded. Now applying Schwarz's inequality, we get

$$(3.1) \quad \left[\frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) f(\theta) d\theta \right]^2 \\ \leq B \left[\frac{1}{2\pi} \int_0^{2\pi} \{\phi(t)\}^2 dt \right] \left[\frac{1}{2\pi} \int_0^{2\pi} dt \left\{ \frac{1}{2\pi} \int_0^{2\pi} P(r_\theta, \theta-t) \phi(\theta) d\theta \right\}^2 \right].$$

The second factor on the right-hand side of (3.1) may be written

$$\frac{1}{2\pi} \int_0^{2\pi} dt \frac{1}{2\pi} \int_0^{2\pi} P(r_\theta, \theta-t) \phi(\theta) d\theta \frac{1}{2\pi} \int_0^{2\pi} P(r_\psi, \psi-t) \phi(\psi) d\psi \\ = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \phi(\theta) \phi(\psi) d\theta d\psi \left\{ \frac{1}{2\pi} \int_0^{2\pi} P(r_\theta, \theta-t) P(r_\psi, \psi-t) dt \right\}.$$

By Lemma 1 this is

$$(3.2) \quad \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \phi(\theta) \phi(\psi) P(r_\theta r_\psi, \theta-\psi) d\theta d\psi.$$

By Lemma 3 the expression (3.2) does not exceed

$$(3.3) \quad B \cdot \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \phi(\theta) \phi(\psi) \{P(r_\theta, \theta-\psi) + P(r_\psi, \theta-\psi)\} d\theta d\psi.$$

Now

$$\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \phi(\theta) \phi(\psi) P(r_\theta, \theta-\psi) d\theta d\psi \\ = \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) d\theta \frac{1}{2\pi} \int_0^{2\pi} P(r_\theta, \theta-\psi) \phi(\psi) d\psi \leq B \cdot \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) f(\theta) d\theta.$$

Hence the expression (3.3) does not exceed

$$B \cdot \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) f(\theta) d\theta.$$

Returning to the equation (3.1) we see that

$$\left[\frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) f(\theta) d\theta \right]^2 \leq B \cdot \left[\frac{1}{2\pi} \int_0^{2\pi} \{\phi(\theta)\}^2 d\theta \right] \left[\frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) f(\theta) d\theta \right].$$

Since $\phi(\theta)$ is bounded, we observe that $f(\theta)$ is also bounded. Hence

$$\frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) f(\theta) d\theta < \infty,$$

and thus we have

$$(3.4) \quad \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) f(\theta) d\theta \leq B \cdot \frac{1}{2\pi} \int_0^{2\pi} \{\phi(\theta)\}^2 d\theta.$$

Now suppose that $F(\theta)$ is the upper bound of

$$\frac{1}{2x} \int_{\theta-x}^{\theta+x} f(t) dt,$$

so that $F(\theta)$ is obtained from $f(\theta)$ in the same way as $f(\theta)$ is obtained from $\phi(\theta)$. We have then from (3.4)

$$(3.5) \quad \frac{1}{2\pi} \int_0^{2\pi} f(\theta) F(\theta) d\theta \leq B \cdot \frac{1}{2\pi} \int_0^{2\pi} \{f(\theta)\}^2 d\theta.$$

We see by Lemma 6 that, by suitable choice of r_θ ,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \{f(\theta)\}^2 d\theta &\leq B \cdot \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta \frac{1}{2\pi} \int_0^{2\pi} P(r_\theta, \theta-t) \phi(t) dt \\ &= B \cdot \frac{1}{2\pi} \int_0^{2\pi} \phi(t) dt \frac{1}{2\pi} \int_0^{2\pi} P(r_\theta, \theta-t) f(\theta) d\theta. \end{aligned}$$

Applying Schwarz's inequality

$$(3.6) \quad \left[\frac{1}{2\pi} \int_0^{2\pi} \{f(\theta)\}^2 d\theta \right]^2 \leq B \cdot \left[\frac{1}{2\pi} \int_0^{2\pi} \{\phi(t)\}^2 dt \right] \left[\frac{1}{2\pi} \int_0^{2\pi} dt \left\{ \frac{1}{2\pi} \int_0^{2\pi} P(r_\theta, \theta-t) f(\theta) d\theta \right\}^2 \right].$$

The second factor on the right-hand side of (3.6) may be written

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} dt \frac{1}{2\pi} \int_0^{2\pi} P(r_\theta, \theta-t) f(\theta) d\theta \frac{1}{2\pi} \int_0^{2\pi} P(r_\psi, \psi-t) f(\psi) d\psi \\ = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(\theta) f(\psi) d\theta d\psi \left\{ \frac{1}{2\pi} \int_0^{2\pi} P(r_\theta, \theta-t) P(r_\psi, \psi-t) dt \right\} \\ = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(\theta) f(\psi) d\theta d\psi P(r_\theta r_\psi, \theta-\psi) d\theta d\psi \\ (3.7) \quad \leq B \cdot \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(\theta) f(\psi) \{P(r_\theta, \theta-\psi) + P(r_\psi, \theta-\psi)\} d\theta d\psi. \end{aligned}$$

Now

$$\begin{aligned} \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(\theta) f(\psi) P(r_\theta, \theta-\psi) d\theta d\psi \\ = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta \frac{1}{2\pi} \int_0^{2\pi} P(r_\theta, \theta-\psi) f(\psi) d\psi \leq B \cdot \frac{1}{2\pi} \int_0^{2\pi} f(\theta) F(\theta) d\theta. \end{aligned}$$

By (3.5), this does not exceed

$$B \cdot \frac{1}{2\pi} \int_0^{2\pi} \{f(\theta)\}^2 d\theta.$$

Hence we see from (3.7) that

$$\left[\frac{1}{2\pi} \int_0^{2\pi} \{f(\theta)\}^2 d\theta \right]^2 \leq B \cdot \left[\frac{1}{2\pi} \int_0^{2\pi} \{\phi(\theta)\}^2 d\theta \right] \left[\frac{1}{2\pi} \int_0^{2\pi} \{f(\theta)\}^2 d\theta \right],$$

from which it follows that

$$\frac{1}{2\pi} \int_0^{2\pi} \{f(\theta)\}^2 d\theta \leq B \cdot \frac{1}{2\pi} \int_0^{2\pi} \{\phi(\theta)\}^2 d\theta.$$

This proves the theorem for bounded and measurable $\phi(\theta)$. The extension to the case when ϕ is unbounded is not difficult. Let $\phi_n(\theta)$ denote $\text{Min} \{ \phi(\theta), n \}$. We write

$$h(x, \theta) = \frac{1}{2x} \int_{\theta-x}^{\theta+x} \phi(\theta) d\theta,$$

$$h_n(x, \theta) = \frac{1}{2x} \int_{\theta-x}^{\theta+x} \phi_n(\theta) d\theta.$$

Let $f(\theta)$, $f_n(\theta)$ denote the upper bounds of $h(x, \theta)$, $h_n(x, \theta)$ respectively. Then

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \{h_n(x, \theta)\}^2 d\theta &\leq \frac{1}{2\pi} \int_0^{2\pi} \{f_n(x, \theta)\}^2 d\theta \leq B \cdot \frac{1}{2\pi} \int_0^{2\pi} \{\phi_n(x, \theta)\}^2 d\theta \\ &\leq B \cdot \frac{1}{2\pi} \int_0^{2\pi} \{\phi(\theta)\}^2 d\theta. \end{aligned}$$

Since $h_n(x, \theta)$ everywhere increases to $h(x, \theta)$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \{h(x, \theta)\}^2 d\theta \leq B \cdot \frac{1}{2\pi} \int_0^{2\pi} \{\phi(\theta)\}^2 d\theta,$$

from which it follows that

$$\frac{1}{2\pi} \int_0^{2\pi} \{f(\theta)\}^2 d\theta \leq B \cdot \frac{1}{2\pi} \int_0^{2\pi} \{\phi(\theta)\}^2 d\theta,$$

which is the required theorem.

4. THEOREM II. Let $k > 1$, and suppose that $\phi(\theta)$ is a function of period 2π , such that

$$\frac{1}{2\pi} \int_0^{2\pi} |\phi(\theta)|^k d\theta < \infty.$$

Denote by $f(\theta)$ the upper bound of

$$\frac{1}{2r} \int_{\theta-r}^{\theta+r} |\phi(t)| dt.$$

Then
$$\frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^k d\theta \leq B_k \cdot \frac{1}{2\pi} \int_0^{2\pi} |\phi(\theta)|^k d\theta,$$

where B_k is a positive constant depending only on k .

We assume first that $\phi(\theta)$ is bounded. The theorem is trivial if

$$\int_0^{2\pi} |\phi(\theta)|^k d\theta = 0,$$

for in this case $\phi(\theta)$ is almost everywhere zero, and hence $f(\theta)$ is identically zero. We may therefore assume that $\phi(\theta)$ is not identically zero. Let $U(\theta)$ denote the everywhere positive function $|\phi(\theta)|$ and let $F(\theta)$ denote the upper bound of

$$\frac{1}{2x} \int_{\theta-x}^{\theta+x} U(t) dt.$$

We observe that it is sufficient to prove the theorem for the everywhere positive function $U(\theta)$.

Since $\{U(\theta)\}^k$ is integrable in $(0, 2\pi)$ it follows, by a theorem of M. Riesz*, that there exists a conjugate function $V(\theta)$ satisfying

$$\frac{1}{2\pi} \int_0^{2\pi} |V(\theta)|^k d\theta \leq B_k \cdot \frac{1}{2\pi} \int_0^{2\pi} \{U(\theta)\}^k d\theta,$$

and such that, if $u(r, \theta)$, $v(r, \theta)$ denote respectively the functions

$$\frac{1}{2\pi} \int_0^{2\pi} P(r, \theta-t) U(t) dt, \quad \frac{1}{2\pi} \int_0^{2\pi} P(r, \theta-t) V(t) dt,$$

then $g(r, \theta) = u(r, \theta) + iv(r, \theta)$ represents a function regular in the circle $r < 1$. We write $G(\theta) = U(\theta) + iV(\theta)$, and observe that

$$(4.1) \quad \frac{1}{2\pi} \int_0^{2\pi} |G(\theta)|^k d\theta \leq B_k \cdot \frac{1}{2\pi} \int_0^{2\pi} \{U(\theta)\}^k d\theta.$$

Since $U(\theta)$ is non-negative and not identically zero, it follows that $u(r, \theta)$ is everywhere strictly positive, and that the function

$$h_1(r, \theta) + ih_2(r, \theta) = h(r, \theta) = \{g(r, \theta)\}^{\frac{1}{2k}}$$

is also regular in the unit circle. Since $g(r, \theta)$ tends radially to $G(\theta)$ for almost all values of θ , it follows that $h(r, \theta)$ tends radially to $\{G(\theta)\}^{\frac{1}{2k}}$ for

* M. Riesz, 1.

almost all θ . Also, since

$$\frac{1}{2\pi} \int_0^{2\pi} |g(r, \theta)|^k d\theta$$

is bounded, we see that

$$\frac{1}{2\pi} \int_0^{2\pi} |h(r, \theta)|^2 d\theta$$

is bounded. It follows that $h(r, \theta)$ is the Poisson integral of

$$H_1(\theta) + iH_2(\theta) = H(\theta) = \{G(\theta)\}^{1/k}.$$

Let $K_1(\theta)$, $K_2(\theta)$ denote respectively the upper bounds of

$$\frac{1}{2x} \int_{\theta-x}^{\theta+x} |H_1(t)| dt, \quad \frac{1}{2x} \int_{\theta-x}^{\theta+x} |H_2(t)| dt.$$

If now r_θ is any function of θ satisfying $0 \leq r_\theta < 1$, then

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \{u(r_\theta, \theta)\}^k d\theta &\leq \frac{1}{2\pi} \int_0^{2\pi} |g(r_\theta, \theta)|^k d\theta = \frac{1}{2\pi} \int_0^{2\pi} |h(r_\theta, \theta)|^2 d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \{h_1^2(r, \theta) + h_2^2(r, \theta)\} d\theta. \end{aligned}$$

By Lemma 5, this does not exceed

$$B \cdot \frac{1}{2\pi} \int_0^{2\pi} \{K_1^2(\theta) + K_2^2(\theta)\} d\theta \leq B \cdot \frac{1}{2\pi} \int_0^{2\pi} |H(\theta)|^2 d\theta,$$

by Theorem I. But the last expression is

$$B \cdot \frac{1}{2\pi} \int_0^{2\pi} |G(\theta)|^k d\theta,$$

and it follows from (4.1) that

$$\frac{1}{2\pi} \int_0^{2\pi} \{u(r_\theta, \theta)\}^k d\theta \leq B_k \cdot \frac{1}{2\pi} \int_0^{2\pi} \{U(\theta)\}^k d\theta.$$

But, by Lemma 6, the upper bound of $u(r_\theta, \theta)$ is greater than $BF(\theta)$, and from this we obtain the required theorem. The extension to the unbounded case is similar to that of the last theorem.

5. THEOREM III. Suppose that $\phi(\theta)$ is a measurable function of period 2π , such that

$$\frac{1}{2\pi} \int_0^{2\pi} |\phi(\theta)| \log^+ |\phi(\theta)| d\theta < \infty,$$

where, for $x > 0$, $\log^+ x$ denotes $\text{Max}(0, \log x)$. Let $f(\theta)$ be the upper bound of

$$\frac{1}{2\pi} \int_{\theta-x}^{\theta+x} |\phi(t)| dt.$$

Then $\frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta \leq B \cdot \frac{1}{2\pi} \int_0^{2\pi} |\phi(\theta)| \log^+ |\phi(\theta)| d\theta + B$,

where the numbers B denote positive constants.

Let $U(\theta)$ denote the everywhere positive function $|\phi(\theta)|$. As in the last theorem, it is sufficient to prove the theorem for bounded $U(\theta)$.

By a theorem due to Zygmund*, there exists a conjugate function $V(\theta)$ satisfying

$$\frac{1}{2\pi} \int_0^{2\pi} |V(\theta)| d\theta \leq B \cdot \frac{1}{2\pi} \int_0^{2\pi} U(\theta) \log^+ U(\theta) d\theta + B,$$

and such that, if $u(r, \theta)$, $v(r, \theta)$ denote respectively the functions

$$\frac{1}{2\pi} \int_0^{2\pi} P(r, \theta-t) U(t) dt, \quad \frac{1}{2\pi} \int_0^{2\pi} P(r, \theta-t) V(t) dt,$$

then $g(r, \theta) = u(r, \theta) + iv(r, \theta)$ represents a function regular in the circle $r < 1$. We write $G(\theta) = U(\theta) + iV(\theta)$, and observe that

$$\frac{1}{2\pi} \int_0^{2\pi} |G(\theta)| d\theta \leq B \cdot \frac{1}{2\pi} \int_0^{2\pi} U(\theta) \log^+ U(\theta) d\theta + B.$$

As in the last theorem we may show that, if $0 \leq r_\theta < 1$, then

$$\frac{1}{2\pi} \int_0^{2\pi} u(r_\theta, \theta) d\theta \leq B \cdot \frac{1}{2\pi} \int_0^{2\pi} U(\theta) \log^+ U(\theta) d\theta + B,$$

and, since the upper bound of $u(r_\theta, \theta)$ is greater than a positive multiple of the upper bound of

$$\frac{1}{2x} \int_{\theta-x}^{\theta+x} U(t) dt,$$

we obtain the required result.

References.

- Hardy and Littlewood, 1. "A maximal theorem with function-theoretic applications", *Acta Math.*, 54 (1930), 81-116.
 Kolmogoroff and Seliverstoff, 1. "Sur la convergence des séries de Fourier", *Comptes rendus*, 178 (1924), 303.
 M. Riesz, 1. "Sur les fonctions conjuguées", *Math. Zeitschrift*, 27 (1927), 218-244.
 Zygmund, 1. "Sur les fonctions conjuguées", *Fundamenta Math.*, 13 (1929), 284-303.

* Zygmund, 1.

ON SOME PROBLEMS CONNECTED WITH WEIERSTRASS'S
NON-DIFFERENTIABLE FUNCTION*By* R. E. A. C. PALEY.

[Received 16 December, 1929.—Read 16 January, 1930.]

1. I consider some results connected with Weierstrass's non-differentiable function

$$(1.1) \quad F_c(z) = \sum_{n=1}^{\infty} a^{cn} z^{a^n}.$$

In the first part of the paper we consider an isolated problem in the theory of Weierstrass's function (1.1). Let $z = \rho e^{i\theta}$. Hardy* has shown that

$$F_c(z) = O\{(1-\rho)^{-c}\},$$

uniformly in θ . Also, for every fixed θ , Hardy shows that

$$F_c(z) = \Omega\{(1-\rho)^{-c}\},$$

The question presents itself whether the Ω is uniform in θ , *i.e.* whether it is true that, if $c > 0$, and a is integral, there exist circles $\rho = \text{constant}$ on the whole of which

$$(1.11) \quad |F_c(z)| > K(1-\rho)^{-c},$$

for some positive constant K . (1.11) is known to be true uniformly in θ if c is so large that one term of the series (1.1) dominates the others, and in this case a need not be an integer. In Theorem I I show that the answer to the question is in the negative. In fact, if c is sufficiently

* Hardy, 1.

small, then, on every circle $\rho = \text{constant}$, there exists at least one point for which

$$|F_c(z)| < A,$$

where A is a positive constant depending on c only.

We go on to discuss the properties of functions of the type

$$(1.2) \quad \sum_{n=1}^{\infty} n^{-\beta} z^{a_n}.$$

I discuss the behaviour of the function (1.2) on the radii of the unit circle, and, for integral values of a , I consider the value of the means

$$J_\lambda = \frac{1}{2\pi} \int_0^{2\pi} |F(\rho, \theta)|^\lambda d\theta.$$

The interesting result is that, if $\beta > \frac{1}{2}$, $\lambda > 0$, then J_λ is a bounded function of ρ . All these results extend to the discussion of functions of the type

$$\sum_{n=1}^{\infty} n^{-\beta} \exp(ia_n) z^{a_n}.$$

where the numbers (a_n) are an arbitrary set of arguments. The maximum modulus of the series

$$\sum_{n=1}^{\infty} n^{-\beta} \exp(ia_n) z^{a_n},$$

on the circle $\rho = r$, can be shown to be $\Omega\{\log^{1-\beta}(1/1-r)\}$ for any choice whatever of the arguments a_n . Littlewood* has shown that, corresponding to any set of moduli c_n for which $\sum c_n^2 \leq 1$, there exists a set of arguments such that

$$\sum_{n=1}^{\infty} c_n \exp(ia_n) z^n = o\left\{\sqrt{\log \frac{1}{1-r}}\right\}.$$

By considering values of c_n which are zero except when n is of the form a^m , where m is a positive integer, I show that, if $\phi(1-r) \rightarrow \infty$ more slowly than $\sqrt{\log(1/1-r)}$, then, by appropriate choice of the c_n ,

$$\sum_{n=1}^{\infty} c_n \exp(ia_n) z^n = \Omega\{\phi(1-r)\},$$

for all possible choice of the arguments a_n .

* Littlewood, 1.

Hardy and Littlewood* have obtained a functional equation, proving a connection between the series (1.1) and

$$(1.3) \quad \sum_{n=1}^{\infty} n^{c-\frac{1}{2}} \exp(i\beta n \log n) z^n,$$

where $\beta = 2\pi/\log a$, enabling us to infer the behaviour of the function (1.3) from that of $F_c(z)$. In this paper I give a fresh proof of this result, which enables us to prove a corresponding functional equation for the function

$$\sum_{n=1}^{\infty} a^{cn} n^{\lambda} z^{a^n}.$$

From this proof we see that the functional equation is not so much due to some accidental result of contour integration as to the fact that to each term of the series (1.1) corresponds a group of terms of the series (1.3). The sum of the terms of this group differs only trivially from the single term of (1.1).

The series (1.3) is known to settle certain delicate points in the theory of functions. If c is sufficiently small and negative, then it gives a continuous function for which $\sum |a_n|^{2-\delta}$ is divergent. Given a set of positive numbers b_n , I investigate a set of sufficient conditions for the existence of a continuous function $f(z)$ expressible in the form

$$\sum_{n=1}^{\infty} b_n \exp(ia_n) z^n.$$

By this method we obtain a sequence of continuous functions for which successively the series

$$\sum \frac{|a_n|}{\sqrt{n}}, \quad \sum \frac{|a_n|}{\sqrt{(n \log n)}}, \quad \sum \frac{|a_n|}{\sqrt{(n \log n \log_2 n)}}, \quad \dots$$

are divergent.

The method employed to obtain these functional equations is one which has been employed by van der Corput† to obtain functional equations for series of the type

$$\sum_{n=1}^{\infty} g(n) \exp\{if(n)\},$$

where the functions $f(n)$, $g(n)$ satisfy a prescribed set of conditions.

* Hardy and Littlewood, 1.

† van der Corput, 1.

2. THEOREM I. Let a be a positive integer greater than or equal to 3, and let $c > 0$. Write

$$F(\rho, \theta) = F(z) = \sum_{n=0}^{\infty} a^{cn} z^{a^n}.$$

Then, if c be sufficiently small, on every circle $\rho = r < 1$ there exists a point (r, θ_r) such that

$$|F(r, \theta_r)| < A,$$

where A is a positive constant depending on c .

Let

$$(2.1) \quad \Phi(\rho, \theta) = \sum_{-\infty}^{\infty} a^{cn} \rho^{a^n} \exp(i\theta a^n).$$

The function $\Phi(\rho, \theta)$ is clearly not periodic in θ . We consider the behaviour of $\Phi(\rho, \theta)$ for fixed values of ρ . Let ρ_0 be a number satisfying the equation

$$e^{-\sqrt[3]{a}} \leq \rho_0 \leq e^{-1/\sqrt[3]{a}},$$

and let $0 \leq \theta_0 < 2\pi$. I shall show that, if c is sufficiently small, then, corresponding to any number δ , there exists a number $\theta \equiv \theta_0(2\pi)$, such that

$$|\Phi(\rho_0, \theta)| < \delta.$$

We suppose for simplicity that $a = 3$, and we write $x = a^{-c} < 1$. I choose c sufficiently small to make x greater than $\frac{2}{3}$, and

$$(2.11) \quad xe^{-1/\sqrt[3]{a}} + x^2 e^{-1/\sqrt[3]{a^3}} + x^3 e^{-1/\sqrt[3]{a^9}} + \dots > 24.$$

Let $(\theta - \theta_0)/(2\pi) = n > 0$, and let

$$n = a_1 + 3a_2 + 3^2 a_3 + \dots + 3^m a_{m+1},$$

where $a_k = 0, 1$, or 2 ($1 \leq k \leq m+1$). Now

$$\Phi(\rho_0, \theta) = F(\rho_0, \theta_0) + x\rho_0^{a^{-1}} \exp(i\theta a^{-1}) + x^2 \rho_0^{a^{-2}} \exp(i\theta a^{-2})$$

$$+ \dots + x^k \rho_0^{a^{-k}} \exp(i\theta a^{-k}) + \dots,$$

where, since $x > \frac{2}{3}$, we have

$$|F(\rho_0, \theta_0)| < e^{-1/\sqrt[3]{a}} + x^{-1} e^{-\sqrt[3]{a}} + x^{-2} e^{-\sqrt[3]{a^3}} + \dots < 2.$$

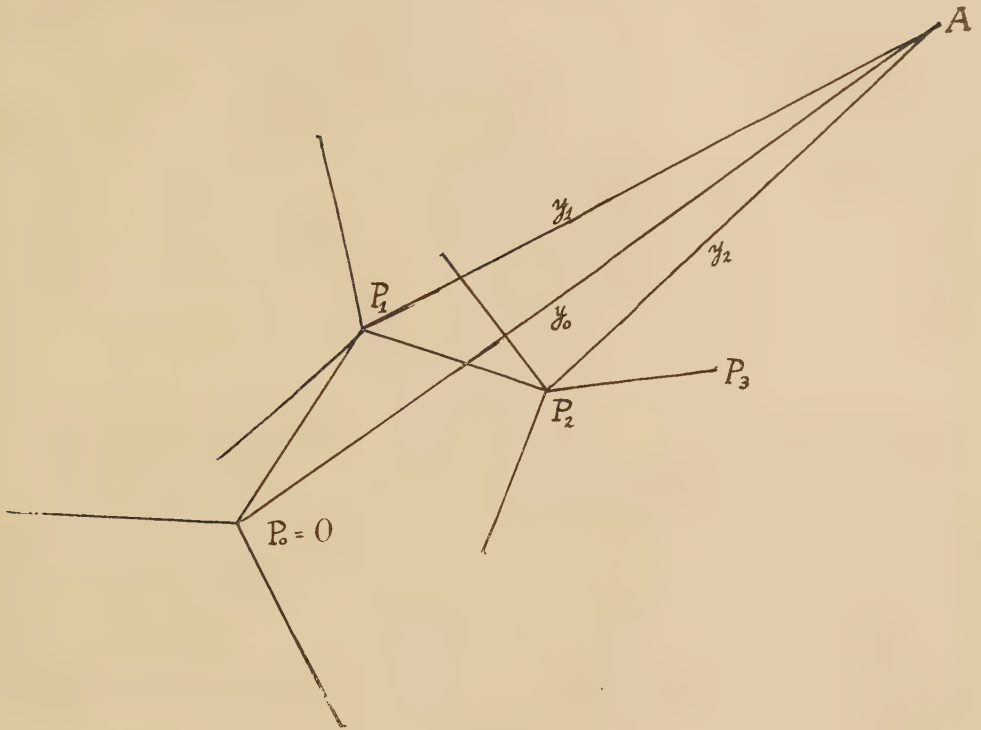


Fig. (i) ($a = 3$).

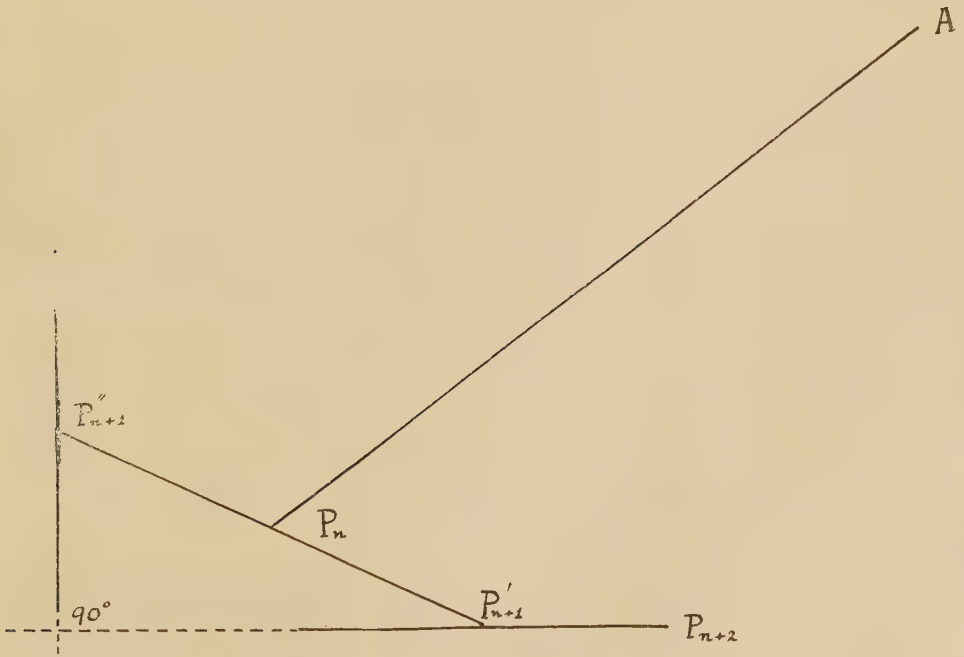


Fig. (ii) ($a = 2$).

We plot on a complex plane the following points :—

A	the point	$-F(\rho_0, \theta_0),$
$P_0 = O$	the origin,	
P_1	the point	$x\rho_0^{a-1} \exp(i\theta a^{-1}),$
P_2	the point	$x\rho_0^{a-1} \exp(i\theta a^{-1}) + x^2 \rho_0^{a-2} \exp(i\theta a^{-2}),$
...	...	...
P_k	the point	$\sum_{\nu=1}^k x^\nu \rho_0^{a-\nu} \exp(i\theta a^{-\nu}),$
...	...	...

Let y_k denote the length AP_k , so that $y_0 = |F(\rho_0, \theta_0)| < 2$. Write $y_{-1} = 0$. I shall give an inductive process for plotting the successive points. Suppose that the points $P_0, P_1, P_2, \dots, P_{k-1}$ have already been obtained by fixing the values of $a_1, a_2, \dots, a_{k-1}$. There are three possible positions of P_k according as $a_k = 0, 1$, or 2 . Let P'_k, P''_k, P'''_k denote these three positions. Then the lines $P_{k-1}P'_k, P_{k-1}P''_k, P_{k-1}P'''_k$ are three lines each of length $x^k \rho_0^{-ak}$, and making with each other angles $2\pi/3$. We choose P_k so that the angle $AP_{k-1}P_k \leq \pi/3$. Thus we define a denumerable set of points P corresponding to a denumerable set of numbers

$$a_1, a_2, \dots, a_k, \dots$$

We show that the set of points P possess A as a limit point. We show in fact that there is an infinite set of numbers satisfying the inequality

$$(2.2) \quad y_{k-1} \leq 2x^k \rho_0^{ak}.$$

For, suppose that this hypothesis is not true; (2.2) is clearly satisfied for $k=0$. Thus the set of numbers for which (2.2) is satisfied is non-null, and, by hypothesis, it is finite. Thus it has a greatest member N . If $N > 0$, we have

$$y_N \leq y_{N-1} + x^N \leq 3x^N,$$

and, if $N=0$, $y_0 < 2 < 3x^0$, so that in either case

$$(2.3) \quad y_N \leq 3x^N,$$

and for all $k \geq N+1$ we have

$$y_{k-1} \geq 2x^k \rho_0^{a-k}.$$

Now since the angle $AP_{k-1}P_k \leq \pi/3$, and, if $k \geq N+1$, we have $y_{k-1} \geq 2x^k \rho_0^{a-k}$ or $AP_{k-1}P_k \geq 2P_{k-1}P_k$, it follows at once from geometrical

considerations that $AP_k < AP_{k-1} - (2 - \sqrt{3}) P_{k-1} P_k$. Thus

$$y_{k-1} - y_k \geq (2 - \sqrt{3}) x^k \rho_0^{-ak} > \frac{1}{4} x^k \rho_0^{-ak}.$$

Hence y_k is steadily decreasing in the range $N \leq k < \infty$, and

$$\sum_{N+1}^{\infty} (y_{k-1} - y_k) > \frac{1}{4} \sum_{N+1}^{\infty} x^k \rho_0^{-ak} > 6x^N,$$

in virtue of (2.11). Also, from (2.3), $6x^N > 2y^N$, and so

$$\sum_{N+1}^{\infty} (y_{k-1} - y_k) > 2y^N.$$

Since the numbers y_k are decreasing in $N \leq k < \infty$, this gives a contradiction. It follows that (2.2) is satisfied for an infinite set of numbers, and, since $x_k \rightarrow 0$, we see that the set of points P possess A as a limit point.

Now we can choose a number k_0 such that, for $k > k_0$,

$$\sum_{\nu=k}^{\infty} x^{\nu} \rho^{\alpha-\nu} < \frac{1}{2}\delta.$$

We now choose a number $k > k_0$ such that $AP_k < \frac{1}{2}\delta$. This number k defines a finite aggregate $(a_1, a_2, \dots, a_{\mu})$, and it is easy to see that, if

$$\theta = \theta_0 + 2\pi(a_1 + 3a_2 + 3^2a_3 + \dots + 3^{\mu-1}a_{\mu}),$$

then

$$(2.4) \quad |\Phi(\rho_0, \theta)| < \delta.$$

By suitable choice of θ , the equation (2.4) is satisfied for any ρ_0 satisfying

$$e^{-\sqrt{a}} \leq \rho_0 \leq e^{-1/\sqrt{a}}.$$

I have proved this only when $a = 3$. The proof is similar when $a > 3$. If $a = 2$, we may obtain a similar result, but the proof is a little harder. We can only ensure that P_{k+2} is nearer A than P_k .

Now from (2.1) it follows at once that, if n is a positive or negative integer, or zero, then

$$(2.5) \quad \Phi(\rho^{a^n}, a^n \theta) = a^{-cn} \Phi(\rho, \theta).$$

Now let $0 < r < 1$. Then we can find a number ρ_0 such that

$$r = \rho_0^a, \quad e^{-\sqrt{a}} \leq \rho_0 \leq e^{-1/\sqrt{a}},$$

and n is a positive or negative integer or zero. Then

$$\Phi(r, a^n \theta) = a^{-cn} \Phi(\rho_0, \theta).$$

If θ is suitably chosen $|\Phi(\rho_0, \theta)| < \delta a^{cn}$. Thus

$$|\Phi(r, a^n \theta)| < \delta.$$

Writing $a^n \theta = \theta_r$, we have

$$|F(r, \theta_r) - \Phi(r, \theta_r)| \leq \sum_{n=1}^{\infty} a^{-cn} \rho^{a^{-n}} \leq \sum_{n=1}^{\infty} 2^{-cn}.$$

Hence

$$|F(r, \theta_r)| \leq \sum_{n=1}^{\infty} 2^{-cn} + \delta < A,$$

where A is a positive constant depending only on c . This proves Theorem I.

3. We now consider the behaviour of the function defined by

$$F_{\beta}(z) = F_{\beta}(\rho, \theta) = \sum_{n=1}^{\infty} n^{-\beta} \rho^{a^n} \exp(ia_n + ia^n \theta),$$

where the numbers a_n are arbitrary, and a is an arbitrary number greater than 1. The interesting case is that for which the numbers a_n are all zero.

THEOREM II.

$$\text{If } \beta < 1, \quad F_{\beta}(\rho, \theta) = F_{\beta}(z) = O\left(\log^{1-\beta} \frac{1}{1-\rho}\right);$$

$$\text{if } \beta = 1, \quad F_{\beta}(\rho, \theta) = F_{\beta}(z) = O\left(\log_2 \frac{1}{1-\rho}\right);$$

$$\text{if } \beta > 1, \quad F_{\beta}(\rho, \theta) - F_{\beta}(1, \theta) = O\left(\log^{1-\beta} \frac{1}{1-\rho}\right);$$

in each case uniformly in θ .

Suppose $\rho^{a^{N+1}} \leq e^{-1} < \rho^{a^N}$. Then

$$F_{\beta}(z) = \sum_{n=1}^N n^{-\beta} \exp(ia_n) z^{a^n} + \sum_{N+1}^{\infty} n^{-\beta} \exp(ia_n) z^{a^n}.$$

Now

$$\begin{aligned} \sum_{N+1}^{\infty} n^{-\beta} \exp(ia_n) z^{a^n} &= O\{(N+1)^{-\beta} + (N+2)^{-\beta} e^{-a} + (N+3)^{-\beta} e^{-a^2} + \dots\} \\ &= O(N^{-\beta}), \end{aligned}$$

uniformly in θ .

$$\begin{aligned}\sum_{n=1}^N n^{-\beta} \exp(i a_n) \{z^{a_n} - \exp(a^n \theta i)\} &= O\left\{\sum_{n=1}^N n^{-\beta}(1-\rho^{a^n})\right\} \\ &= O\left\{N^{-\beta} + \frac{(N-1)^{-\beta}}{a} + \frac{(N-2)^{-\beta}}{a^2} + \dots\right\} = O(N^{-\beta}),\end{aligned}$$

uniformly in θ , but not in a . Hence we see that

$$F_{\beta}(z) = \sum_{n=1}^N n^{-\beta} \exp(i a_n + i a^n \theta) + O(N^{-\beta}).$$

$$\text{If } \beta > 1, \quad \sum_{n=1}^N n^{-\beta} \exp(i a_n + i a^n \theta) = O(N^{1-\beta});$$

$$\text{if } \beta = 1, \quad \sum_{n=1}^N n^{-\beta} \exp(i a_n + i a^n \theta) = O(\log N);$$

$$\begin{aligned}\text{if } \beta < 1, \quad \sum_{n=1}^N n^{-\beta} \exp(i a_n + i a^n \theta) \\ = F_{\beta}(1, \theta) - \sum_{N+1}^{\infty} n^{-\beta} \exp(i a_n + i a^n \theta) = F_{\beta}(1, \theta) + O(N^{1-\beta}).\end{aligned}$$

Since $N = O\{\log 1/(1-\rho)\}$, the results of the theorem follow at once.

THEOREM III. *If a is greater than a constant depending only on β , then*

$$F_{\beta}(z) = \Omega\left\{\log^{-\beta} \frac{1}{1-\rho}\right\},$$

for any value of θ . No better Ω result is possible.

For the direct result we need only observe that, if

$$\rho_1^{a^{N+\frac{1}{2}}} = e^{-1}, \quad \rho_2^{a^{N-\frac{1}{2}}} = e^{-1},$$

$$\text{then} \quad F_{\beta}(\rho_1, \theta) = \sum_{n=1}^N n^{-\beta} \exp(i a_n + i a^n \theta) + O\left(\frac{N^{-\beta}}{\sqrt{a}}\right),$$

$$F_{\beta}(\rho_2, \theta) = \sum_{n=1}^{N-1} n^{-\beta} \exp(i a_n + i a^n \theta) + O\left(\frac{N^{-\beta}}{\sqrt{a}}\right).$$

$$\text{Hence} \quad F_{\beta}(\rho_1, \theta) - F_{\beta}(\rho_2, \theta) = N^{-\beta} \exp(i a_N + i a^N \theta) + O\left(\frac{N^{-\beta}}{\sqrt{a}}\right),$$

$$\text{and} \quad |F_{\beta}(\rho_1, \theta) - F_{\beta}(\rho_2, \theta)| \geq \frac{1}{2} N^{-\beta},$$

for sufficiently large a .

For the negative result, consider the case $a=2$, $a_n=0$ for all n , $\theta = \frac{2}{9}\pi$. Then

$$\sum_1^6 \exp(a^n \theta i) = 0,$$

and $\sum_1^N \exp(a^n \theta i)$ is bounded. It follows that, for all $\beta \leq 0$,

$$\sum_{n=1}^N n^{-\beta} \exp(a^n \theta i) = O(N^{-\beta}),$$

and, if $\beta > 0$,

$$\sum_{n=N+1}^{\infty} n^{-\beta} \exp(a^n \theta i) = O(N^{-\beta}).$$

Hence, for $\beta \leq 0$,

$$F_{\beta}(\rho, \theta) = O\left\{\log^{-\beta} \frac{1}{1-\rho}\right\},$$

and for $\beta > 0$

$$F_{\beta}(\rho, \theta) - F_{\beta}(1, \theta) = O\left\{\log^{-\beta} \frac{1}{1-\rho}\right\}.$$

From this point I shall restrict myself to the case where a is an integer. Let E_n denote the set of θ defined by the inequality

$$|a^n \theta + a^n - 2\pi k| \leq \frac{1}{3}\pi,$$

where k is an integer, and $a \geq 3$. Then, if $\mathcal{E}_n$ denote a connected portion of the set E_n , it is not difficult to see that the set $\mathcal{E}_n E_{n+1} E_{n+2} \dots$ does not vanish. It follows that the set

$$E_1 E_2 E_3 E_4 \dots + E_2 E_3 E_4 \dots + E_3 E_4 \dots$$

is an everywhere dense set of θ . If this set be S , then at every point of S we have, for all $n > N_0$,

$$\cos(a_n + a^n \theta) > \frac{1}{2}.$$

Hence, if θ be a point of S ,

$$|F_{\beta}(\rho, \theta)| > \sum_{n=1}^N n^{-\beta} \cos(a_n + a^n \theta) + O(N^{-\beta}) = \Omega(N^{1-\beta}),$$

if $\beta < 1$. Similarly, if $\beta = 1$,

$$F_{\beta}(\rho, \theta) = \Omega\left(\log \frac{1}{1-\rho}\right),$$

and, if $\beta > 1$,

$$F_{\beta}(\rho, \theta) - F_{\beta}(1, \theta) = \Omega\left(\log^{1-\beta} \frac{1}{1-\rho}\right).$$

This proves that the result of Theorem II is best possible

Littlewood has shown* that corresponding to a set of real numbers a_n , satisfying

$$\sum_{n=1}^{\infty} a_n^2 < \infty,$$

we can define a set of arguments a_n such that

$$\sum_{n=1}^{\infty} a_n \exp(ia_n) z^n = o \sqrt{\left(\log \frac{1}{1-\rho}\right)},$$

uniformly in θ . Now suppose that $a_n = 0$, unless n is of the form 3^μ . Then, writing $b_\mu = a_{3^\mu}$, the argument of the previous paragraph shows that, whatever the choice of the a_n , for a suitable choice of θ , we shall have

$$|\sum b_n \exp(ia_n) z^{3^n}| \geq A \sum_1^N b_n,$$

where $N = \log 1/(1-\rho)$, and A is an absolute positive constant. Now, if $\phi(N)$ be any function of N which decreases to zero as $N \rightarrow \infty$, then, by appropriate choice of the b_n , we can ensure that

$$\sum_{n=1}^N b_n = \Omega\{\sqrt{N} \phi(N)\}, \quad \sum_{n=1}^{\infty} b_n^2 < \infty.$$

Thus we obtain the following result.

THEOREM III. *Let $\epsilon(1-\rho)$ be any function which tends to zero with $1-\rho$. There exists a set of positive numbers b_n satisfying $\sum b_n^2 = 1$, such that, corresponding to any choice of the arguments a_n , we may define a number θ such that*

$$\sum_{n=1}^{\infty} b_n \exp(ia_n) z^{3^n} = \Omega\left\{\sqrt{\left(\log \frac{1}{1-\rho}\right)} \epsilon(1-\rho)\right\}.$$

4. We now consider the behaviour of the means of the function $F_\beta(z)$. As before, we suppose a to be an integer. Let $F(\rho, \theta) = F(z)$ be any function of z regular in $|z| < 1$. Then, if $\lambda > 0$, $\rho < 1$, we write

$$M_\lambda^\lambda(F; \rho_0) = J_\lambda(F; \rho_0) = \frac{1}{2\pi} \int_0^{2\pi} |F(\rho_0, \theta)|^\lambda d\theta.$$

We first need a result due to Hausdorff†.

* Littlewood, 1.

† Hausdorff, 1.

LEMMA 1. (i) $M_\lambda(F; \rho_0)$ is a non-decreasing function of λ for $\lambda > 0$.
(ii) Suppose $0 < \lambda \leq \mu < \nu$, $\lambda < \nu$. Let

$$\vartheta = \frac{\lambda(\nu - \mu)}{\mu(\nu - \lambda)}, \quad 1 - \vartheta = \frac{\nu(\mu - \lambda)}{\mu(\nu - \lambda)}.$$

Then $M_\mu(F; \rho_0) \leq \{M_\lambda(F; \rho_0)\}^\vartheta \{M_\nu(F; \rho_0)\}^{1-\vartheta}$.

Consider the first result. Let $\lambda < \mu$, and write $r = \mu/\lambda > 1$. Then, by a degenerate case of Hölder's inequality, we have

$$\frac{1}{2\pi} \int_0^{2\pi} |F|^\lambda d\theta \leq \left\{ \frac{1}{2\pi} \int_0^{2\pi} |F|^{r\lambda} d\theta \right\}^{1/r},$$

from which it follows that $M_\lambda(F; \rho_0) \leq M_\mu(F; \rho_0)$.

The second result is trivial unless $\lambda < \mu < \nu$; take this to be the case. Then $0 < \vartheta < 1$, and, if

$$r = \frac{\lambda}{\mu\vartheta} = \frac{\nu - \lambda}{\nu - \mu} > 1, \quad r' = \frac{r}{r-1} = \frac{\nu - \lambda}{\mu - \lambda} > 1,$$

then

$$|F|^\mu = |F|^{\lambda/r} |F|^{\nu/r'},$$

and, using Hölder's inequality,

$$M_\mu^\mu = \frac{1}{2\pi} \int_0^{2\pi} |F|^\mu d\theta \leq \left\{ \frac{1}{2\pi} \int_0^{2\pi} |F|^\lambda d\theta \right\}^{1/r} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |F|^\nu d\theta \right\}^{1/r'},$$

from which the result of the lemma follows.

THEOREM IV*. If $\beta > \frac{1}{2}$, then

$$M_\lambda \{F_\beta(z); \rho_0\} = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |F_\beta(\rho, \theta)|^\lambda d\theta \right\}^{1/\lambda}$$

is a bounded function of ρ_0 for all $\lambda > 0$.

In virtue of Lemma 1 it will be sufficient to prove the theorem for the case when λ is an even integer. For if k is an integer, and

$$2k \leq \lambda' \leq 2k+2,$$

then

$$M_{\lambda'} \{F_\beta; \rho_0\} \leq M_{2k}^\vartheta M_{2k+2}^{1-\vartheta},$$

where $0 \leq \vartheta \leq 1$. Hence, if the theorem has been established for $\lambda = 2k$, and $\lambda = 2k+2$, it follows at once for λ' . I shall therefore confine myself to the case where λ is an even integer, and write $\lambda = 2k$.

* This is the extension of a theorem of Hardy and Littlewood. See Hardy and Littlewood, 2.

LEMMA 2. *The number of solutions in positive integers of the equation*

$$(4.1) \quad a^{n_1} + a^{n_2} + \dots + a^{n_k} = \lambda$$

is less than a constant which depends only on k (not on λ).

We use induction, and suppose that the lemma has been established for all numbers less than or equal to $k-1$. Let

$$n_1 \geq n_2 \geq n_3 \geq \dots \geq n_k;$$

$$\text{then} \quad ka^{n_1} \geq \lambda \geq a^{n_1};$$

$$\text{hence} \quad n_1 \log a \leq \log \lambda \leq n_1 \log a + \log k.$$

It follows that n_1 lies between

$$\frac{\log \lambda}{\log a} \quad \text{and} \quad \frac{\log \lambda + \log k}{\log a},$$

and it cannot have more than $1 + \log k / \log a \leq 1 + \log k / \log 2$ values. Since the lemma is true for $k=1$, it follows that it is true generally. The number of solutions of the equation (4.1) is less than a constant $C(k)$.

Now write $\phi(z) = \{F_\beta(z)\}^k = \sum c_n z^n$, where

$$(4.2) \quad c_n = \sum_{a^{n_1} + a^{n_2} + \dots + a^{n_k} = n} (n_1 n_2 \dots n_k)^{-\beta}.$$

Now if N be the number of terms in the sum (4.2), then, by Hölder's inequality,

$$\left\{ \sum_{a^{n_1} + a^{n_2} + \dots + a^{n_k} = n} (n_1 n_2 \dots n_k)^{-\beta} \right\}^2 \leq N \sum_{a^{n_1} + \dots + a^{n_k} = n} (n_1 n_2 \dots n_k)^{-2\beta}.$$

$$\text{It follows that} \quad C_n^2 \leq N \sum_{a^{n_1} + \dots + a^{n_k} = n} (n_1 n_2 \dots n_k)^{-2\beta},$$

$$\sum_{n=1}^{\infty} c_n^2 \leq N \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \dots \sum_{n_k=1}^{\infty} (n_1 n_2 \dots n_k)^{-2\beta} = N \left\{ \sum_{n=1}^{\infty} n^{-2\beta} \right\}^k = O(1).$$

But, by Parseval's theorem,

$$\frac{1}{2\pi} \int_0^{2\pi} |\phi(\rho_0, \theta)|^2 d\theta = 2 \sum c_n^2 \rho_0^{2n} = O(1),$$

from which the result of the theorem follows.

THEOREM V. *If $\lambda > 0$, $\beta < \frac{1}{2}$, then*

$$A(\lambda, \beta) \log^{\frac{1}{2}-\beta} \frac{1}{1-\rho_0} \leq M_\lambda \{F_\beta; \rho_0\} \leq B(\lambda, \beta) \log^{\frac{1}{2}-\beta} \frac{1}{1-\rho_0}.$$

If $\lambda > 0$, $\beta = \frac{1}{2}$, then

$$A(\lambda) \log_2 \frac{1}{1-\rho_0} \leq M_\lambda \{F_\beta; \rho_0\} \leq B(\lambda) \log_2 \frac{1}{1-\rho_0}.$$

The constants A , B depend only on λ and β .

If $\rho_0^{a^N} \leq e^{-1} < \rho_0^{a^{N-1}}$, then

$$F_\beta(z) = \sum_{n=1}^N n^{-\beta} \exp(ia_n + ia^n \theta) + O(N^{-\beta}).$$

The argument of the last theorem shows that, if k is an integer,

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=1}^N n^{-\beta} \exp(ia_n + ia^n \theta) \right|^{2k} d\theta \leq C(k) \left\{ \sum_{n=1}^N n^{-2\beta} \right\}^k;$$

$$J_{2k} \{F_\beta; \rho_0\} = O \left\{ \log^{\frac{1}{2}-\beta} \frac{1}{1-\rho_0} \right\} \quad (\beta < \tfrac{1}{2}),$$

$$= O \left\{ \log_2 \frac{1}{1-\rho_0} \right\} \quad (\beta = \tfrac{1}{2}).$$

This gives the O -result when λ is an even integer. The O -result for general values of λ follows at once when we use the convexity argument of the last theorem. It is easy to see that

$$M_2 \{F_\beta; \rho_0\} \geq A \left\{ \log^{\frac{1}{2}-\beta} \frac{1}{1-\rho_0} \right\} \quad (\beta < \tfrac{1}{2}),$$

$$\geq A \log_2 \frac{1}{1-\rho_0} \quad (\beta = \tfrac{1}{2}),$$

and using the convexity argument again we can prove the whole theorem. For, if $\lambda > 2$, we need only observe that

$$M_\lambda \{F_\beta; \rho_0\} \geq M_2 \{F_\beta; \rho_0\} \geq A \left\{ \log^{\frac{1}{2}-\beta} \frac{1}{1-\rho_0} \right\} \quad (\beta < \tfrac{1}{2}),$$

$$\geq A \log_2 \frac{1}{1-\rho_0} \quad (\beta = \tfrac{1}{2}).$$

If $\lambda < 2$ then, by appropriate choice of $\mathfrak{S}$,

$$[M_\lambda \{F_\beta; \rho_0\}]^3 [M_4 \{F_\beta; \rho_0\}]^{1-3} \geq M_2 \{F_\beta; \rho_0\}.$$

Since $M_4 = O \left\{ \log^{\frac{1}{2}-\beta} \frac{1}{1-\rho_0} \right\}$, $M_2 \geq A \left\{ \log^{\frac{1}{2}-\beta} \frac{1}{1-\rho_0} \right\}$ ($\beta < \frac{1}{2}$),

$$M_4 = O \left(\log_2 \frac{1}{1-\rho_0} \right), \quad M_2 \geq A \log_2 \frac{1}{1-\rho_0} \quad (\beta = \tfrac{1}{2}),$$

we obtain the required result when $\lambda < 2$.

THEOREM VI. If $\beta > \frac{1}{2}$, the series

$$\sum_{n=1}^{\infty} n^{-\beta} \exp (i a_n + i a^n \theta)$$

is convergent for almost all values of θ .

Write
$$F_{\beta}(z) = \sum_{n=1}^{\infty} n^{-\beta} \rho^{a^n} \exp (i a_n + i a^n \theta).$$

Since
$$M_2\{F_{\beta}(z); \rho_0\} = O(1),$$

it follows by a known theorem* that, for almost all values of θ ,

$$\lim_{\rho \rightarrow 1} F_{\beta}(\rho, \theta)$$

exists. Since
$$F_{\beta}(\rho, \theta) = \sum_{n=1}^N n^{-\beta} \exp (i a_n + i a^n \theta) + o(1),$$

where $N \rightarrow \infty$ as $\rho \rightarrow 1$, it follows at once that the series

$$\sum_{n=1}^{\infty} n^{-\beta} \exp (i a_n + i a^n \theta)$$

is convergent for almost all values of θ .

5. We now consider the functional equation for the Weierstrassian function. Consider the series

$$(5.1) \quad \sum_{n=1}^{\infty} n^{c-\frac{1}{2}} \log^{\lambda} n \exp (i \beta n \log n) z^n,$$

where β is any non-zero real number, which for simplicity we suppose to be positive. The argument of the n -th term of the series (5.1) is

$$a_n = a(n; \theta) = \beta n \log n + n \theta.$$

Now
$$a_{n+1} - a_n = \beta \log n + \beta + \theta + O\left(\frac{1}{n}\right),$$

and, provided that $\beta \log n + \beta + \theta$ does not approximate to a multiple of 2π , a group of terms in the neighbourhood of n has no great effect on the value of the sum of the series (5.1). Let n_{ν} (not necessarily an integer) be the solution of the equation

$$\beta \log n + \beta + \theta = 2\pi\nu.$$

* By the Riesz-Fischer theorem, both

$$\sum n^{-\beta} \cos (i a_n + i a^n \theta) \quad \text{and} \quad \sum n^{-\beta} \sin (i a_n + i a^n \theta)$$

are Fourier series, and so they are summable almost everywhere by Poisson's method.

We consider separately the groups of terms of the series (5.1) in the neighbourhood of the numbers n_ν . Consider the sum

$$(5.2)^* \quad \sum'_{n_\nu - n_\nu^{\frac{3}{2}}}^{n_\nu + n_\nu^{\frac{3}{2}}} \{n^{c-\frac{1}{2}} \log^\lambda n \rho^n \exp(i\beta n \log n + in\theta)\}.$$

The sum (5.2) differs from

$$(5.3) \quad n_\nu^{c-\frac{1}{2}} \log^\lambda n_\nu \rho^{n_\nu} \sum_{n_\nu - n_\nu^{\frac{3}{2}}}^{n_\nu + n_\nu^{\frac{3}{2}}} \exp(i\beta n \log n + in\theta)$$

by

$$(5.4) \quad \begin{aligned} O \left\{ \sum_{k=-n_\nu^{\frac{3}{2}}}^{n_\nu^{\frac{3}{2}}} |k| \max_{x=n_\nu - n_\nu^{\frac{3}{2}}}^{n_\nu + n_\nu^{\frac{3}{2}}} \left| \frac{d}{dx} (x^{c-\frac{1}{2}} \log^\lambda x \rho^x) \right| \right\} \\ = O(n_\nu^{\frac{3}{2}}) [n_\nu^{c-\frac{1}{2}} \log^\lambda n_\nu \rho^{\frac{1}{2}n_\nu} \{1 + n_\nu(1-\rho)\}] \\ = O(n_\nu^{c-\frac{1}{2}}) \rho^{\frac{1}{2}n_\nu} \{1 + n_\nu(1-\rho)\}. \end{aligned}$$

Consider the sum

$$(5.5) \quad \sum_{n_\nu - n_\nu^{\frac{3}{2}}}^{n_\nu + n_\nu^{\frac{3}{2}}} \exp(i\beta n \log n + in\theta) = \sum_{n_\nu - n_\nu^{\frac{3}{2}}}^{n_\nu + n_\nu^{\frac{3}{2}}} \exp(i\beta n \log n + in\theta - 2\pi i\nu n).$$

We observe that

$$\left[\frac{d}{dx} \{i\beta x \log x + ix\theta - 2\pi i\nu x\} \right]_{x=n_\nu} = 0;$$

$$\frac{d^2}{dx^2} (i\beta x \log x + ix\theta - 2\pi i\nu x) = O\left(\frac{1}{x}\right).$$

Hence the sum (5.5) differs from

$$(5.6) \quad \int_{n_\nu - n_\nu^{\frac{3}{2}}}^{n_\nu + n_\nu^{\frac{3}{2}}} \exp(i\beta x \log x + ix\theta - 2\pi i\nu x) dx$$

by

$$(5.7) \quad \begin{aligned} O \left\{ \int_{n_\nu - n_\nu^{\frac{3}{2}}}^{n_\nu + n_\nu^{\frac{3}{2}}} \left| \frac{d}{dx} (\beta x \log x + ix\theta - 2\pi i\nu x) \right| dx \right\} \\ = O \left\{ \int_{-n_\nu^{\frac{3}{2}}}^{n_\nu^{\frac{3}{2}}} \frac{|x|}{n_\nu} dx \right\} = O(n_\nu^{\frac{1}{2}}). \end{aligned}$$

* The dash denotes that, if either of the limits of summation is an integer, then only half the corresponding term is to be included in the summation.

Since

$$\begin{aligned} \beta x \log x + x\theta - 2\pi\nu x &= (\beta n_\nu \log n_\nu + n_\nu \theta - 2\pi\nu n_\nu) \\ &\quad + \frac{1}{2}(x - n_\nu)^2 \left[\frac{d^2}{dx^2} (\beta x \log x + x\theta - 2\pi\nu x) \right]_{x=n_\nu} \\ &\quad + O \left\{ \frac{|x - n_\nu|^3}{n_\nu^2} \right\}, \end{aligned}$$

we see that the integral (5.6) is

$$\begin{aligned} \exp \{iF(\nu)\} \int_{-n_\nu^{\frac{3}{2}}}^{n_\nu^{\frac{3}{2}}} \exp \{ix^2 \beta / (2n_\nu)\} dx &+ O \left\{ \frac{n_\nu^{\frac{3}{2}}}{n_\nu^2} \right\} \\ (5.8) \qquad \qquad \qquad &= \sqrt{8} \exp \{iF(\nu)\} \sqrt{\left(\frac{n_\nu}{\beta}\right)} \int_0^{n_\nu^{\frac{1}{2}} \sqrt{\beta}} \exp(i\theta^2) d\theta + O(n_\nu^{\frac{3}{2}}), \end{aligned}$$

where $F(\nu)$ denotes the sum $\beta n_\nu \log n_\nu + n_\nu \theta - 2\pi\nu n_\nu$. Now

$$i \int_{\theta_1}^{\theta_2} (2\theta) e^{i\theta^2} d\theta = \int_{\theta_1}^{\theta_2} d(e^{i\theta^2}) = O(1).$$

$$\text{Hence} \qquad \int_N^\infty e^{i\theta^2} d\theta = i \int_N^\infty \frac{d}{d\theta} \left(\frac{1}{2\theta} \right) \int_N^\theta d(e^{i\phi^2}) d\phi = O(1/N).$$

Hence the expression (5.8) may be written

$$(5.9) \qquad \exp \{iF(\nu)\} \sqrt{\left(\frac{n_\nu}{\beta}\right)} C + O(n_\nu^{\frac{3}{2}}) + O(n_\nu^{\frac{3}{2}}),$$

where C denotes the absolute constant

$$\sqrt{8} \int_0^\infty e^{i\theta^2} d\theta = e^{\frac{1}{2}\pi i} \sqrt{(2\pi)}.$$

Collecting the error terms (5.4), (5.7), (5.9), we see that the sum (5.2) may be written

$$\begin{aligned} (5.91) \quad e^{\frac{1}{2}\pi i} \sqrt{\left(\frac{2\pi}{\beta}\right)} n_\nu^c \log^\lambda n_\nu \rho^{n_\nu} \exp \{iF(\nu)\} \\ + O(n_\nu^{-\delta}) \log^\lambda n_\nu \rho^{\frac{1}{2}n_\nu} \{1 + n_\nu(1 - \rho)\}, \end{aligned}$$

where δ denotes the constant $\frac{1}{10} - c$.

6. We now consider the sum

$$(6.1) \qquad \sum'_{n=n_\nu+n_\nu^{\frac{3}{2}}}^{n_\nu+1-n_\nu^{\frac{3}{2}}} n^{c-\frac{1}{2}} \log^\lambda n \rho^n \exp(i\beta n \log n + in\theta).$$

Let $c(x)$ denote the function $x^{c-\frac{1}{2}} \log^\lambda x \rho^x$. We observe that, for sufficiently large n_0 , $c(x)$ is decreasing for $x > n_0$, and for all values of ρ . We write

$$\gamma_n = a_{n+1} - a_n = \beta \log n + \beta + \theta + O\left(\frac{1}{n}\right).$$

Since $\exp(i\alpha_{n+1}) - \exp(i\alpha_n) = \{\exp(i\gamma_n) - 1\} \exp(i\alpha_n)$,

it follows that, for all N, N' ,

$$\left| \sum_{n=N}^{N'} \{\exp(i\gamma_n) - 1\} \exp(i\alpha_n) \right| \leq 2.$$

Now the sum (6.1) may be written

$$\sum_{n=n_\nu+n_\nu^{\frac{3}{2}}}^{n_{\nu+1}-n_\nu^{\frac{3}{2}}} \left\{ \frac{c(n)}{\exp(i\gamma_n) - 1} \right\} \{\exp(i\gamma_n) - 1\} \exp(i\alpha_n).$$

Using Abel's transformation, we see that this is

$$(6.2) \quad O \left[\sum_{n=n_\nu+n_\nu^{\frac{3}{2}}}^{n_{\nu+1}-n_\nu^{\frac{3}{2}}} \left| \Delta \left(\frac{c(n)}{\exp(i\gamma_n) - 1} \right) \right| \right] + O \left[\frac{c(n_{\nu+1} - n_\nu^{\frac{3}{2}})}{\exp(i\gamma_{[n_{\nu+1} - n_\nu^{\frac{3}{2}}]} - 1)} \right],$$

where Δ denotes the operation defined by

$$\Delta f(n) = f(n+1) - f(n),$$

and $[n_{\nu+1} - n_\nu^{\frac{3}{2}}]$ denotes the integral part of $n_{\nu+1} - n_\nu^{\frac{3}{2}}$. Now, in the range considered,

$$\left| \Delta \left(\frac{c(n)}{\exp(i\gamma_n) - 1} \right) \right| \leq \frac{|\Delta c(n)|}{|\exp(i\gamma_n) - 1|} + \frac{2c(n) |\Delta \exp(i\gamma_n)|}{|\exp(i\gamma_n) - 1|^2}.$$

Hence we see that the expression (6.2) is

$$(6.3) \quad O \left[\text{Max} \left(\frac{1}{|\exp(i\gamma_n) - 1|} \right) \{ \text{Max } c(n) + \Sigma |\Delta c(n)| \} \right. \\ \left. + \text{Max } c(n) \Sigma \left\{ \frac{1/n}{|\exp(i\gamma_n) - 1|^2} \right\} \right].$$

Since $n_{\nu+1} = O(n_\nu)$, we see that

$$\text{Max } c(n) = O \{ n_\nu^{c-\frac{1}{2}} \log^\lambda n_\nu \rho^{n_\nu} \},$$

$$\text{Max} \left\{ \frac{1}{|\exp(i\gamma_n) - 1|} \right\} = O(n_\nu^{\frac{1}{2}}).$$

Also, since $c(n)$ is decreasing for $n > n_0$,

$$\sum_{n_\nu + n_\nu^{\frac{3}{2}}}^{n_{\nu+1} - n_{\nu+1}^{\frac{1}{2}}} |\Delta \{c(n)\}| = O\{c(n_\nu)\}.$$

Now, in the range $\{n_\nu, \sqrt{(n_\nu n_{\nu+1})}\}$ we have

$$|\exp(i\gamma_n) - 1| > A \frac{(n - n_\nu)}{n_\nu},$$

where A is some positive number. Hence

$$\left(\frac{1}{n_\nu}\right) \sum_{n=n_\nu + n_\nu^{\frac{3}{2}}}^{\sqrt{(n_\nu n_{\nu+1})}} \left\{ \frac{1}{|\exp(i\gamma_n) - 1|^2} \right\} = O\left(\frac{1}{n_\nu}\right) \sum_{k=n_\nu^{\frac{3}{2}}}^{\infty} \frac{(n_\nu)^2}{k^2} = O(n_\nu^{\frac{3}{2}}).$$

Similarly

$$\left(\frac{1}{n_\nu}\right) \sum_{n=\sqrt{(n_\nu n_{\nu+1})}}^{n_{\nu+1} - n_{\nu+1}^{\frac{1}{2}}} \left\{ \frac{1}{|\exp(i\gamma_n) - 1|^2} \right\} = O(n_\nu^{\frac{3}{2}}).$$

It follows that the expression (6.3) is

$$(6.4) \quad O(n_\nu^{-\delta} \log^\lambda n_\nu \rho^{n_\nu}).$$

Combining this with the result (5.91) and summing from $\nu = 1$ to μ , we find that

$$(6.5) \quad \sum_{n=1}^{n_\mu + n_\mu^{\frac{3}{2}}} n^{c-\frac{1}{2}} \log^\lambda n \rho^n \exp(i\beta n \log n + in\theta) \\ = e^{i\pi i} \sqrt{\left(\frac{2\pi}{\beta}\right)} \sum_{\nu=1}^{\mu} (n_\nu^{-c}) \log^\lambda n_\nu \rho^{n_\nu} \exp\{iF(\nu)\} \\ + O\left[\sum_{\nu=1}^{\infty} n_\nu^{-\delta} \log^\lambda n_\nu \rho^{n_\nu} \{1 + n_\nu(1-\rho)\}\right].$$

If $c < 1/10$, the series which majorizes the O -term is absolutely convergent, uniformly in (ρ, θ) , and so it is $O(1)$. Now, by the methods already employed, we can prove that, if

$$n_\mu + n_\mu^{\frac{3}{2}} \leq M < n_{\mu+1} + n_{\mu+1}^{\frac{3}{2}},$$

then

$$\sum_{n=n_\mu + n_\mu^{\frac{3}{2}}}^M n^{c-\frac{1}{2}} \log^\lambda n \rho^n \exp(i\beta n \log n + in\theta) = O\{n_\mu^c \log^\lambda n_\mu \rho^{n_\mu} + 1\}.$$

Hence, from (6.5), we obtain the approximate functional equation for

the series (5.1),

$$(6.6) \quad \sum_{n=1}^M n^{c-\frac{1}{2}} \log^\lambda n \rho^n \exp(i\beta n \log n + in\theta) \\ = e^{\frac{1}{2}\pi i} \sqrt{\left(\frac{2\pi}{\beta}\right)} \sum_{n_\nu \leq M} [n_\nu^c \log^\lambda n_\nu \rho^{n_\nu} \exp\{iF(\nu)\}] \\ + O\{n_\mu^c \log^\lambda n_\mu \rho^{n_\mu} + 1\},$$

where $n_\mu \leq M < n_{\mu+1}$.

If we combine the results (5.92) and (6.4), and sum from $\nu = 1$ to ∞ , we obtain the functional equation for the series (5.1),

$$(6.7) \quad \sum_{n=1}^{\infty} n^{c-\frac{1}{2}} \log^\lambda n \rho^n \exp(i\beta n \log n + in\theta) \\ = e^{\frac{1}{2}\pi i} \sqrt{\left(\frac{2\pi}{\beta}\right)} \sum_{\nu=1}^{\infty} n_\nu^c \log^\lambda n_\nu \rho^{n_\nu} \exp iF(\nu) + R,$$

where R is majorized by the expression

$$A \sum_{\nu=1}^{\infty} n_\nu^{-\delta} \log^\lambda n_\nu \rho^{n_\nu} \{1 + n_\nu(1-\rho)\},$$

for some sufficiently large value of A .

7. In the equations (6.6), (6.7), let $a = \exp(2\pi\beta^{-1})$. We observe that the equations (6.6), (6.7) still hold good if for $\log^\lambda n$, $\log^\lambda n_\nu$ we substitute respectively $(\log n + 1 + \theta\beta^{-1})^\lambda$ and $(\log n_\nu + 1 + \theta\beta^{-1})^\lambda$. We do this, and investigate the general term of the (new) series on the right-hand side. We have

$$\beta \log n_\nu + \beta + \theta = 2\pi\nu,$$

from which follows $n_\nu = a^\nu \exp(-1 - \theta\beta^{-1})$;

$$\log n_\nu + 1 + \theta\beta^{-1} = \nu \log a = 2\pi\nu\beta^{-1}.$$

Also, if

$$\rho' = \rho^{\exp(-1 - \theta\beta^{-1})},$$

then

$$\rho^{n_\nu} = (\rho')^{a^\nu}.$$

We have $F(\nu) = \beta n_\nu \log n_\nu + \theta n_\nu - 2\pi\nu n_\nu$, and, substituting for n_ν , $\log n_\nu$, we get

$$F(\nu) = -a^\nu \beta \exp(-1 - \theta\beta^{-1}) = a^\nu \theta',$$

where $\theta' = -\beta \exp(-1 - \theta\beta^{-1})$. We see that the expression (6.7) may be written

$$e^{\frac{1}{2}\pi i} \sqrt{\left(\frac{2\pi}{\beta}\right)} (2\pi\beta^{-1})^\lambda \{\exp(-1 - \theta\beta^{-1})\}^c a^{c\nu} \nu^\lambda (z')^{a^\nu},$$

where $z' = \rho' \exp(i\theta')$. We thus obtain the following theorem.

THEOREM VII. If $c < \frac{1}{10}$, and

$$a = \exp(2\pi\beta^{-1}); \quad z = \rho \exp(i\theta); \quad z' = \rho' \exp(i\theta');$$

$$\rho' = \rho^{\exp(-1-\theta\beta^{-1})}; \quad \theta' = -\beta \exp(-1-\theta\beta^{-1});$$

then

$$(7.2) \quad \sum_{n=1}^{\infty} n^{c-\frac{1}{2}}(\log n + 1 + \theta\beta^{-1})^{\lambda} \exp(i\beta n \log n) z^n \\ = e^{\frac{1}{2}\pi i} (2\pi\beta^{-1})^{\lambda+\frac{1}{2}} \{\exp(-1-\theta\beta^{-1})\}^c \sum_{\nu=1}^{\infty} a^{\nu} \nu^{\lambda} (z')^{\nu} + R,$$

where R is a function continuous in the circle $\rho \leq 1$. Also

$$(7.3) \quad \sum_{n=1}^N n^{c-\frac{1}{2}}(\log n + 1 + \theta\beta^{-1})^{\lambda} \exp(i\beta n \log n) z^n \\ = e^{\frac{1}{2}\pi i} (2\pi\beta^{-1})^{\lambda+\frac{1}{2}} \{\exp(-1-\theta\beta^{-1})\}^c \sum_{\nu=1}^{\beta \log N / (2\pi)} a^{\nu} \nu^{\lambda} (z')^{\nu} \\ + O\{a^{\mu} \mu^{\lambda} (z')^{\mu}\} + O(1),$$

where μ denotes the integer next below $\beta \log N / (2\pi)$.

We observe that, if $c < 0$, or if $c = 0$, $\lambda < -1$, then the series on the left-hand side of (7.2) is uniformly convergent in the circle $\rho \leq 1$, and so it represents a function of z continuous inside the circle.

Arguing in the same way, we may produce equations connecting the sums of the series

$$\sum_{n=1}^{\infty} n^{c-\frac{1}{2}} \log^{\lambda} n \log_2^{\lambda'} n \exp(i\beta n \log n) z^n,$$

and

$$\sum_{\nu=1}^{\infty} a^{\nu} \nu^{\lambda} \log^{\lambda} \nu (z')^{\nu},$$

and so on indefinitely.

8. Theorem VII proves that, if $c < 0$, the series

$$(8.1) \quad \sum_{n=1}^{\infty} n^{c-\frac{1}{2}} \exp\{i\beta n \log n\} z^n$$

represents a function continuous in the unit circle. Writing the series (8.1) in the form $\sum a_n z^n$ we observe that, if c is sufficiently small, the series $\sum |a_n|^{2-\epsilon}$ is divergent. I shall show how to construct a sequence of functions, even more delicate than the function (8.1), for which successively the series

$$\sum \frac{|a_n|}{\sqrt{n}}, \quad \sum \frac{|a_n|}{\sqrt{(n \log n)}}, \quad \sum \frac{|a_n|}{\sqrt{(n \log n \log_2 n)}}, \quad \dots,$$

are divergent. We extend the ideas of Theorem VII to prove the following theorem.

THEOREM VIII*. Let $d(x)$ be a (slowly) increasing function of x . Define

$$\Delta(x) = \int_0^x d(y) dy;$$

$$d\{n(\nu; \theta)\} + \theta = 2\pi\nu;$$

$$F(\nu; \theta) = \Delta\{n(\nu; \theta)\} + \theta n(\nu; \theta) - 2\pi\nu n(\nu; \theta).$$

Let $b(x)$ be a continuous function of x . Suppose that the following conditions are satisfied:—

(1) $d(x)$ is twice differentiable, and

$$d'(x) = O\left(\frac{1}{x}\right); \quad d''(x) = O\left(\frac{1}{x^2}\right);$$

$$\frac{1}{d'(x)} = O\{x \log^H x\},$$

where H is an absolute constant.

(2) $b(x)$ is once differentiable, and

$$b(x) = O(x^{c-\frac{1}{2}}), \quad b'(x) = O(x^{c-\frac{3}{2}}),$$

where c is some number less than $\frac{1}{10}$.

(3) For every positive number ϵ ,

$$\log n(\nu+1; \theta) = O[\{n(\nu; \theta)\}^\epsilon],$$

uniformly in ν, θ .

Write $z = \rho e^{i\theta}$. Then

$$(8.2) \quad \sum_{n=1}^{\infty} b(n) \exp\{i\Delta(n)\} z^n \\ = e^{i\pi i} \sqrt{(2\pi)} \sum_{\nu=1}^{\infty} \frac{b\{n(\nu; \theta)\}}{\sqrt{[d'\{n(\nu; \theta)\}]}} \exp\{iF(\nu)\} + R,$$

where R is a function continuous in the circle $\rho \leq 1$.

We observe that, since $d'(x) < A/x$ for some constant A ,

$$d\{n(\nu+1; \theta)\} - d\{n(\nu; \theta)\} = 2\pi > \frac{2\pi}{A} n(\nu; \theta) \max_{x=n(\nu)}^{n(\nu+1)} d'(x).$$

* The arguments used in the proof of this theorem are similar to those employed by Hille in obtaining O -results for the function $\sum n^{c-\frac{1}{2}} \exp(isn \log n) z^n$. See Hille, 1.

Hence

$$n(\nu+1; \theta) - n(\nu; \theta) > \frac{2\pi}{A} n(\nu; \theta),$$

$$n(\nu+1; \theta) > \left(1 + \frac{2\pi}{A}\right) n(\nu; \theta) = a n(\nu; \theta),$$

where $a > 1$. It follows that, if $\epsilon > 0$, the series

$$\sum_{\nu=1}^{\infty} \{n(\nu; \theta)\}^{-\epsilon} \rho^{n(\nu; \theta)}$$

is absolutely and uniformly convergent in the unit circle.

As in Theorem VII, we consider separately groups of terms

$$(8.3) \quad \sum'_{n=n_{\nu}-n_{\nu}^{\frac{3}{2}}}^{n_{\nu}+n_{\nu}^{\frac{3}{2}}} b(n) \exp \{i\Delta(n)\} z^n,$$

$$(8.4) \quad \sum'_{n=n_{\nu}+n_{\nu}^{\frac{3}{2}}}^{n_{\nu}-n_{\nu+1}^{\frac{3}{2}}} b(n) \exp \{i\Delta(n)\} z^n,$$

where n_{ν} denotes $n(\nu; \theta)$. The sum (8.3) differs from

$$b(n_{\nu}) \rho^{n_{\nu}} \sum_{n=n_{\nu}-n_{\nu}^{\frac{3}{2}}}^{n_{\nu}+n_{\nu}^{\frac{3}{2}}} \exp \{i\Delta(n) + in\theta\}$$

by

$$\begin{aligned} O \left\{ \sum_{k=-n_{\nu}^{\frac{3}{2}}}^{n_{\nu}^{\frac{3}{2}}} |k| \max_{x=n_{\nu}-n_{\nu}^{\frac{3}{2}}}^{n_{\nu}+n_{\nu}^{\frac{3}{2}}} \left| \frac{d}{dx} \{b(x) \rho^x\} \right| \right\} &= O(n_{\nu}^{\frac{3}{2}}) [n_{\nu}^{-\frac{3}{2}} \rho^{\frac{3}{2}n_{\nu}} \{1 + n_{\nu}(1-\rho)\}] \\ &= O(n_{\nu}^{-\frac{3}{2}}) \rho^{\frac{3}{2}n_{\nu}} \{1 + n_{\nu}(1-\rho)\}. \end{aligned}$$

As in the last theorem, the sum

$$\begin{aligned} \sum_{n=n_{\nu}-n_{\nu}^{\frac{3}{2}}}^{n_{\nu}+n_{\nu}^{\frac{3}{2}}} \exp \{i\Delta(n) + in\theta\} &= \int_{n_{\nu}-n_{\nu}^{\frac{3}{2}}}^{n_{\nu}+n_{\nu}^{\frac{3}{2}}} \exp \{i\Delta(x) + ix\theta - 2\pi\nu x\} dx + O(n_{\nu}^{\frac{1}{2}}) \\ &= \exp \{iF(\nu)\} \int_{-n_{\nu}^{\frac{3}{2}}}^{n_{\nu}^{\frac{3}{2}}} \exp \{\tfrac{1}{2}ix^2 d'(\nu)\} dx + O(n_{\nu}^{\frac{1}{2}}) \\ &= \sqrt{\left(\frac{2\pi}{d'(\nu)}\right)} e^{\frac{1}{2}\pi i} \exp \{iF(\nu)\} + O(n_{\nu}^{\frac{1}{2}}). \end{aligned}$$

It follows that the sum (8.3) is

$$(8.5) \quad e^{i\pi i} \sqrt{(2\pi)} \frac{b\{n(\nu; \theta)\}}{\sqrt{[d'_\theta\{n(\nu; \theta)\}]}} \exp\{iF(\nu)\} + R_\nu,$$

where ΣR_ν is majorized by a series uniformly convergent in the unit circle.

9. In dealing with the term (8.4) the analysis is somewhat more difficult, since it is no longer true that $n_{\nu+1} = O(n_\nu)$. We write

$$a_n \equiv \Delta(n) + n\theta,$$

$$\gamma_n = a_{n+1} - a_n,$$

$$c(x) = b(x) \rho^x.$$

Since
$$\sum_{n=N}^{N'} \{\exp(i\gamma_n) - 1\} \exp(ia_n) = O(1),$$

it follows that the series (8.4) is

$$(9.1) \quad O \left[\sum_{n=n_\nu+n_\nu^{\frac{3}{2}}}^{n_{\nu+1}-n_{\nu+1}^{\frac{3}{2}}} \left| \Delta \left\{ \frac{c(n)}{\exp(i\gamma_n) - 1} \right\} \right| + \frac{c(n_{\nu+1} - n_{\nu+1}^{\frac{3}{2}})}{\exp(i\gamma_{n_{\nu+1} - n_{\nu+1}^{\frac{3}{2}}}) - 1} \right].$$

If $\delta = \frac{1}{10} - c$, the last term of (9.1) is

$$(9.2) \quad O\{n_{\nu+1}^{c-\frac{1}{2}} \rho^{n_\nu} n_{\nu+1}^{\frac{3}{2}}\} = O(n_{\nu+1}^{-\delta} \rho^{n_\nu}) = O(n_\nu^{-\delta} \rho^{n_\nu}).$$

We split the first term of (9.1) into three sums

$$(9.3) \quad O \left\{ \sum_{n_\nu+n_\nu^{\frac{3}{2}}}^{N_\nu} + \sum_{N_\nu}^{N_\nu} + \sum_{\frac{1}{2}(n_{\nu+1})}^{n_{\nu+1}-n_{\nu+1}^{\frac{3}{2}}} \right\},$$

where N_ν is the solution of the equation

$$d(n) + \theta = (2\nu + 1)\pi.$$

The first term of (9.3) is

$$(9.4) \quad O \left[\sum_{k=n_\nu^{\frac{3}{2}}}^{n_{\nu+1}} \left\{ \frac{(n_\nu + k) \log^H(n_\nu + k)}{k} |\Delta c(n_\nu + k)| + \frac{(n_\nu + k) \log^{2H}(n_\nu + k)}{k^2} c(n_\nu + k) \right\} \right].$$

Since $|\Delta c(n_\nu + k)| = O\{n_\nu^{c-\frac{3}{2}} + n_\nu^{c-\frac{1}{2}}(1-\rho)\} \rho^{n_\nu}$, we see that

$$\begin{aligned} \sum_{k=n_\nu^{\frac{3}{2}}}^{n_{\nu+1}} \frac{n_\nu}{k} \log^H(n_\nu + k) |\Delta c(n_\nu + k)| &= O\{n_\nu^{c-\frac{3}{2}} \log^{H+1}(n_{\nu+1})\} \{1 + n_\nu(1-\rho)\} \rho^{n_\nu} \\ &= O(n_\nu^{-\delta} \rho^{n_\nu}) \{1 + n_\nu(1-\rho)\}. \\ \sum_{k=n_\nu^{\frac{3}{2}}}^{n_{\nu+1}} \{\log^H(n_\nu + k)\} |\Delta c(n_\nu + k)| \\ &= O\left\{ \sum_{n=n_\nu+1}^{n_{\nu+1}} n^{c-\frac{3}{2}} \log^H(n) \{1 + n(1-\rho)\} \rho^n \right\} = O(n_\nu^{-\delta} \rho^{n_\nu}). \end{aligned}$$

The last term of (9.4) is easily seen to be

$$O\{n_\nu^{c-\frac{1}{2}} \log^{2H+1}(n_{\nu+1}) + n_\nu^{c-\frac{1}{2}} \log^{2H}(n_{\nu+1})\} \rho^{n_\nu} = O\{n_\nu^{-\frac{1}{2}} \rho^{n_\nu}\}.$$

It follows that the expression (9.4) is

$$(9.5) \quad O\{n_\nu^{-\frac{1}{2}} \rho^{n_\nu}\} \{1 + n_\nu(1-\rho)\}.$$

The second term of (9.3) is

$$\begin{aligned} (9.6) \quad & O\left[\sum_{k=\frac{1}{2}n_{\nu+1}}^{n_{\nu+1}-n_\nu} \left\{ \frac{n_{\nu+1} \log^H(n_{\nu+1})}{k} |\Delta c(n_{\nu+1} - k)| \right\} \right. \\ & \quad \left. + \frac{(n_{\nu+1})^2 \log^{2H}(n_{\nu+1})}{k^2} \frac{c(n_{\nu+1} - k)}{(n_{\nu+1} - k)} \right] \\ &= O\{\log^{2H}(n_{\nu+1})\} \left[\sum_{n=n_{\nu+1}}^{n_{\nu+1}} n^{c-\frac{3}{2}} \{1 + n(1-\rho)\} \rho^n \right. \\ & \quad \left. + n_\nu^{c-\frac{1}{2}} \rho^{n_\nu} \sum_{k=\frac{1}{2}(n_{\nu+1})}^{n_{\nu+1}-n_\nu} \frac{1}{n_{\nu+1} - k} \right] \\ &= O(n_\nu^{-\frac{1}{2}} \rho^{n_\nu}). \end{aligned}$$

For the last summation of (9.3) we observe that, in the range considered, $n \geq \frac{1}{2}n_{\nu+1}$. We may argue in the same way as in the last theorem and prove that the sum in question is

$$(9.7) \quad O\{n_{\nu+1}^{-\frac{1}{2}} \rho^{\frac{1}{2}n_{\nu+1}}\} \{1 + n_{\nu+1}(1-\rho)\}.$$

If we write R'_ν for the sum of the terms (9.2), (9.5), (9.6), (9.7), we see that $\Sigma R'_\nu$ is majorized by a uniformly convergent series. We have only to combine this with the result (8.5) to obtain the required theorem.

It is clear that, arguing as in Theorem VII, we may produce an approximate functional equation connecting the functions (8.2), analogous to the equation (6.6).

If, in Theorem VIII, we write $d(x) = \beta \log_2 x$, where $\beta > 0$, $b(x) = n^{c-\frac{1}{2}} \log^{c-\frac{1}{2}}(n) \log_2^\lambda n$, it is clear that the conditions of the theorem are satisfied. We thus obtain the following result.

THEOREM IX. If $c < \frac{1}{10}$, and

$$z = \rho \exp(i\theta); \quad \Phi(x) = \int_{ee}^x \beta \log_2 y \, dy,$$

where β is any positive number, then

$$(9.8) \quad \sum_{n=3^3}^{\infty} n^{c-\frac{1}{2}} \log^{c'-\frac{1}{2}}(n) \log_2^{\lambda} n \exp\{i\Phi(n)\} z^n \\ = e^{\frac{1}{2}\pi i} \sqrt{\left(\frac{2\pi}{\beta}\right)} \sum_{\nu=1}^{\infty} n_{\nu}^c \log^{c'} n_{\nu} \log_2^{\lambda} n_{\nu} \rho^{n_{\nu}} \exp\{iF(\nu)\} + R,$$

where n_{ν} and $F(\nu)$ are defined by the equations

$$\beta \log_2 n_{\nu} + \theta = 2\pi\nu,$$

$$F(\nu) = \Phi(n_{\nu}) + n_{\nu}\theta - 2\pi\nu n_{\nu}.$$

R is a function continuous in the circle $\rho \leq 1$.

10. The function on the right-hand side of (9.8) behaves very much like the function

$$\sum_{\nu=1}^{\infty} a^{ca^{\nu}} a^{c'\nu} \nu^{\lambda} (z')^{a^{\nu}},$$

although no exact relation connecting them can be found. If $c < 0$, or if $c = 0$, $c' < 0$, or if $c = c' = 0$, $\lambda < -1$, then the function on the right-hand side of (9.8) is continuous. Hence the function on the left-hand side is also continuous. If $c = 0$, $c' = \frac{3}{4}$, we have a continuous function for which

$$\sum \frac{|a_n|}{\sqrt{n}}$$

is divergent.

We may produce an approximate functional equation, connecting the functions of (9.8), analogous to the equation (6.6). We may also find functional and approximate functional equations for the function

$$\sum_{n=3^{3^3}}^{\infty} n^{c-\frac{1}{2}} \log^{c'-\frac{1}{2}}(n) \log_2^{\lambda} n \log_3^{\lambda'} n \exp\{i\Phi(n)\} z^n,$$

which behaves like

$$\sum_{\nu=1}^{\infty} a^{ca^{\nu}} a^{c'\nu} \nu^{\lambda} \log^{\lambda'} \nu (z')^{a^{\nu}},$$

and so on indefinitely.

If we write

$$\Psi(x) = \int_{ee^e}^{\infty} \beta \log_3 y \, dy,$$

then, by the arguments of Theorem VIII, we may show that

$$(10.1) \quad \sum_{n=3^{3^3}}^{\infty} n^{c-\frac{1}{2}} \log^{c'-\frac{1}{2}}(n) \log_2^{c''-\frac{1}{2}}(n) \log_3^{\lambda} n \exp \{i\Psi(n)\} z^n \\ = e^{\frac{1}{2}\pi i} \sqrt{\left(\frac{2\pi}{\beta}\right)} \sum_{\nu=1}^{\infty} n_{\nu}^c \log^{c'} n_{\nu} \log^{c''} n_{\nu} \log_3^{\lambda} n_{\nu} \rho^{n_{\nu}} \exp \{iF(\nu)\} + R,$$

where

$$\beta \log_3 n_{\nu} + \theta = 2\pi\nu,$$

$$F(\nu) = \Psi(n_{\nu}) + n_{\nu}\theta - 2\pi\nu n_{\nu},$$

and R is a function continuous in the circle $\rho \leq 1$. If $c = c' = 0$, $c'' = \frac{3}{4}$, we observe that the function on the left-hand side of (10.1) is continuous in the unit circle, and that it satisfies the condition that

$$\sum \frac{|a_n|}{\sqrt{(n \log n)}}$$

is divergent.

By similar arguments we may find continuous functions for which successively

$$\sum \frac{|a_n|}{\sqrt{(n \log n \log_2 n)}}, \quad \sum \frac{|a_n|}{\sqrt{(n \log n \log_2 n \log_3 n)}}, \quad \dots$$

are divergent.

Now let ω be any positive number. In Theorem VIII we write

$$b^2(x) = \frac{d'(x)}{\{d(x)\}^{2+2\omega}}.$$

Since

$$\frac{b(x)}{\sqrt{\{d'(x)\}}} = \frac{1}{\{d(x)\}^{1+\omega}},$$

we see that the series on the right-hand side of (8.2) is majorized by

$$A \sum_{\nu=1}^{\infty} \frac{1}{\nu^{1+\omega}},$$

where A is a positive constant. Hence it represents a function of z continuous in the circle $\rho \leq 1$. Hence, provided that the conditions of the theorem are satisfied, the function

$$\sum_{n=1}^{\infty} b_n \exp \{i\Delta(n)\} z^n$$

is continuous in the circle $z \leq 1$. The series $\sum b^2(n)$ is, by appropriate choice of the function $d(x)$, very slowly convergent. The condition (3)

of Theorem VIII is, however, a very real restriction. The function $n(v, \theta)$ cannot, for instance, diverge as quickly as the series

$$2, 2^2, 2^{2^2}, \dots, 2^{2^{2^{\dots}}}, \dots$$

It follows that, if $b^2(x) = d'(x)\{d(x)\}^{-2-2\omega}$, then the series $\Sigma b^2(n)$ cannot be indefinitely slowly convergent.

References.

- van der Corput **1**, "Beweis einer approximativen Functionalgleichung", *Math. Zeitschrift*, 28 (1928), 238-300.
- Hardy, **1**, "Weierstrass' non-differentiable function", *Trans. American Math. Soc.*, 17 (1916), 301-332.
- Hardy and Littlewood, **1**, "Some problems of Diophantine approximation. A remarkable trigonometrical series", *Proc. National Acad. Sci.*, 2 (1916), 583-586.
- **2**, "Some new properties of Fourier constants", *Math. Annalen*, 97 (1927), 159-209 (201)
- Hausdorff, **1**, "Eine Ausdehnung des Parsevalschen Satzes über Fourierreihen", *Math. Zeitschrift*, 16 (1923), 163-169.
- Hille, **1**, "Note on a power series considered by Hardy and Littlewood", *Journal London Math. Soc.*, 4 (1929), 176-182.
- Littlewood, **1**, "On the mean value of power series", *Proc. London Math. Soc.* (2), 25 (1926), 328-337 (334).

THE ENUMERATION OF THE LATIN RECTANGLE OF DEPTH THREE BY MEANS OF A FORMULA OF REDUCTION, WITH OTHER THEOREMS RELATING TO NON-CLASHING SUBSTITUTIONS AND LATIN SQUARES

By S. M. JACOB.

[Received 10 October, 1928.—Read 8 November, 1928.—Revised 23 October, 1929.]

The paper is divided into the following sections.

In *Section 1* the enumeration of the number of permutations of three non-clashing rows is found, each row consisting of a permutation of the same totality of n letters all differing from each other. The general solution of this problem was first given by MacMahon for s rows in the operational form

$$\frac{(n-1)!}{(n-s)!} n_s = \frac{[D_{a+b+c+\dots}^s (abc \dots)]_{a=1}}{n! \{n!/s!\}^n},$$

where $\{(n-1)!/(n-s)!\} n_s$ is the enumeration for s rows of n letters with the first row and first column fixed in position*. The present paper supplies a non-operational formula from which the enumeration for three rows can be carried out in succession for any value of n from a knowledge of the enumeration for lower values of n .

In *Section 2* two theorems are stated regarding the expression

$$\Sigma \frac{1}{p^x q^y r^z \dots x! y! z! \dots},$$

where $p, q, r, \dots, x, y, z, \dots$ are positive integers which satisfy the condition $xp + yq + rz + \dots = n$ and $p \neq q \neq r \dots$.

The expression is shown to be equal to unity; if the further restriction that $p, q, r \dots$ shall not be unity is imposed, the expression is equal to ${}_2K_n/n!$, where ${}_2K_n$ is Whitworth's sub-factorial n . From this the result

$$\lim_{n \rightarrow \infty} \Sigma \frac{1}{p^x q^y r^z \dots x! y! z! \dots} = \frac{1}{e}$$

is obtained.

In *Section 3* an expression for three rows corresponding to that in *Section 2* for two rows is found.

* *Trans. Camb. Phil. Soc.*, 16 (1898), 262–290.

In *Section 4* the formula of *Section 2* is applied to a classification of Latin Squares according to the partition of the number n to which the operator, converting the first row into the second, corresponds.

In *Section 5* the possible number of "complete cycle" Latin Squares, and thus a lower limit for the number of self-conjugate Latin Squares of degree n is found.

Section 1.

For the enumeration of the number of non-clashing permutations of three rows, each row consisting of n different letters, the totality of letters in each row being identical, the following symbols will be needed: small letters are used when the position of the letters printed is treated as fixed and capital letters when they may take up any position in the group. When dots are used in different rows these indicate that the letters represented by the dots are permutations of the same group of letters, the group itself consisting of n different letters. The possible number of non-clashing permutations (first row fixed) of each type is called its enumeration and the suffix of the enumeration symbol indicates the total number of letters in each row.

Enumeration symbol.

Type I		K_n
		
		
Type II	s	U_n
	S	
	X	
Type III	b	L_n
	T	
	X	
Type IV	$s t$. . .	T_n
	TX . . .	
	XS . . .	
Type V	$s t$. . .	V_n
	ST . . .	
	AX . . .	
Type VI	$s t$. . .	W_n
	SA . . .	
	TX . . .	

The above enumeration symbols apply to the number of possible arrangements when the first row of letters is fixed. In addition to these symbols, however, it will sometimes be more convenient to use the type form itself with a bracket to indicate the possible number of arrangements, thus

$$\left\{ \begin{array}{l} s \dots\dots\dots \\ S \dots\dots\dots \\ T \dots\dots\dots \end{array} \right. \equiv U_n,$$

while

$$\left\{ \begin{array}{l} S \dots\dots\dots \\ S \dots\dots\dots \\ T \dots\dots\dots \end{array} \right. \equiv n! \, U_n.$$

Naturally

$$\left\{ \begin{array}{l} S \dots\dots\dots \\ S \dots\dots\dots \\ T \dots\dots\dots \end{array} \right. = \left\{ \begin{array}{l} S \dots\dots\dots \\ T \dots\dots\dots \\ T \dots\dots\dots \end{array} \right.$$

and, dividing both sides of this equality by $n!$, we have

$$\left\{ \begin{array}{l} s \dots\dots\dots \\ S \dots\dots\dots \\ T \dots\dots\dots \end{array} \right. = \left\{ \begin{array}{l} s \dots\dots\dots \\ T \dots\dots\dots \\ T \dots\dots\dots \end{array} \right.$$

Similarly,

$$\left\{ \begin{array}{l} s \, t \dots\dots\dots \\ S \, T \dots\dots\dots \\ A \, X \dots\dots\dots \end{array} \right. = \left\{ \begin{array}{l} s \, t \dots\dots\dots \\ A \, X \dots\dots\dots \\ A \, X \dots\dots\dots \end{array} \right.$$

Let us consider Type I, and suppose that the first letters in the first column of the second and third rows are fixed (the first row consisting as before of a fixed sequence of letters).

It is easy to prove that

$$K_n = (n-1)(n-2) \, T_n. \tag{1.1}$$

The typical schemas for Type II of which the enumeration symbol is U_n are

(1)

$s \dots\dots\dots$
 $. \, S \dots\dots\dots$
 $x \dots\dots\dots$

(2)

$s \dots\dots\dots$
 $. \, S \dots\dots\dots$
 $. \, x \dots\dots\dots$

(3)

$s \dots\dots\dots$
 $. \, S \dots\dots\dots$
 $. \, . \, X \dots\dots\dots$

where in schema (1) and (2) the x is a small letter to show that it is to be fixed in the same column as s in the first row of schema (1) and as S in the second row of schema (2).

Calling the enumeration of the schemas $U_1(n)$, $U_2(n)$, $U_3(n)$ respectively, we have

$$U_1(n) + U_2(n) + U_3(n) = U_n. \tag{1.2}$$

Now let us examine schema (1) and call the first dot in the second row A , though it may equally well be any of $n-1$ letters. The first column then becomes

$$\begin{matrix} S \\ A \\ X \end{matrix}$$

and we are left with $n-1$ columns of the type $\begin{cases} a \dots \\ S \dots \\ A \dots \end{cases}$

Thus
$$U_1(n) = (n-1) \begin{cases} a \dots \\ S \dots \\ A \dots \end{cases} \text{ to } n-1 \text{ columns.}$$

Exactly the same reasoning applies to the enumeration of schema (2) and we have

$$U_1(n) = U_2(n) = (n-1) U_{n-1}. \tag{1.3}$$

Lastly, since in schema (3) X is not to be in the same column as s in the first row or as S in the second row, X might be replaced by S without producing any correspondences in the columns, so that

$$U_3(n) = K_n. \tag{1.4}$$

From equation (1.2) we thus obtain

$$U_n = 2(n-1) U_{n-1} + K_n. \tag{1.5}$$

If in Type V (for n letters in all) A were replaced by S we should at once reduce it to Type II. Hence the numbers in Type V (first row fixed) are the numbers in Type II *plus* the numbers of arrangements in Type V for which S and A are in the same column.

Using, as before, small letters for fixed positions we have the sub-types

$$\begin{array}{ll} (1) & st\dots \qquad (2) \quad st\dots \\ & \dots ST. \qquad \dots ST. \\ & a\dots X. \qquad \dots AX. \end{array}$$

These are further reducible to

$$\begin{array}{ll} (1a) & st\dots \qquad (1b) \quad st\dots \text{ in which } T \text{ in the second row has} \\ & t\dots S. \qquad \dots S.T \text{ any position except the first} \\ & a\dots X \qquad a\dots X. \text{ two dots.} \\ (2a) & st\dots \qquad (2b) \quad st\dots \text{ in which the column } . \text{ has any} \\ & .s\dots T \qquad \dots ST. \qquad S \\ & .a.X. \qquad \dots AX. \qquad A \\ & \qquad \qquad \qquad \text{position except the first two} \\ & \qquad \qquad \qquad \text{columns.} \end{array}$$

The enumeration of these sub-types is

$$\begin{array}{ll} (1 a) & L_{n-1}. \\ (1 b) & (n-2) \left\{ \begin{array}{l} t b . . . \\ ST . . . \\ BX . . . \end{array} \right. = (n-2) W_{n-1}. \\ (2 a) & L_{n-1}. \\ (2 b) & (n-2) \left\{ \begin{array}{l} s t . . . \\ TB . . . \\ BX . . . \end{array} \right. = (n-2) W_{n-1}. \end{array}$$

Hence we get

$$V_n = U_n + 2L_{n-1} + 2(n-2) W_{n-1}. \tag{1.6}$$

It is now necessary to find reduction formulae for L_n and W_n . Let us take first W_n for which the type is

$$\begin{array}{c} t b . . . \\ ST . . . \\ BX . . . \end{array}$$

We observe that as X is replaced by S the above Type VI is reduced to Type IV, of which the enumeration is T_n .

Hence the numbers in Type VI are the numbers in Type IV *plus* the numbers of Type VI for which S and X appear in the same column.

Now when S and X clash they may appear under t or b or any of the remaining $n-2$ letters of the first row.

These correspond to the sub-types

$$\begin{array}{lll} t b . . . & t b . . . & t b . . . \\ s . . . T & . s . . T & . . . ST \\ x . . B. & . x . B. & . . . XB \end{array}$$

These have respectively the same enumeration as

$$\left\{ \begin{array}{l} b . . . \\ T . . . \\ B . . . \end{array} \right. \left\{ \begin{array}{l} t . . . \\ T . . . \\ B . . . \end{array} \right. (n-2) \left\{ \begin{array}{l} t b . . . \\ TC . . . \\ BC . . . \end{array} \right.$$

for $n-1$ letters.

That is to say the enumeration of the three sub-types is

$$U_{n-1}, \quad U_{n-1}, \quad (n-2) T_{n-1}.$$

Hence we have

$$W_n = T_n + 2U_{n-1} + (n-2) T_{n-1}. \tag{1.7}$$

Now let us examine the enumeration of $b \dots$, namely L_n .

$T \dots$

$X \dots$

This is subdivided into

- | | |
|--|--|
| (1) $b \dots$
$t \dots$
$x \dots$ | (2) $b \dots$ with X in any position but the first dot of the third row. |
| (3) $b \dots$ with T in any position but the first dot of the third row. | (4) $b \dots$ with T and X in any position except in the first column. |

The enumeration of (1) is simply K_{n-1} .

For the enumeration of (2) take C , say, of the $n-1$ available letters for the dot in the first column and we are left with $n-1$ columns of the type

$c \dots$

$C \dots$

$X \dots$

which numbers U_{n-1} . Hence the enumeration of (2) is $(n-1)U_{n-1}$.

Similarly, the enumeration of (3) is $(n-1)U_{n-1}$.

For the enumeration of (4) take, say, C for the first dot of the second row, which can be done in $n-1$ ways, and, say, d for the first dot of the third row, which can be done in $n-2$ ways and we are left with $n-1$ columns of the type

$c d \dots$

$DT \dots$

$XC \dots$

This is Type VI and numbers W_{n-1} . Hence the enumeration of (4) is $(n-1)(n-2)W_{n-1}$. From this we have

$$L_n = K_{n-1} + 2(n-1)U_{n-1} + (n-1)(n-2)W_{n-1}. \quad (1.8)$$

Hence, substituting the value of W_n given by (1.7), we get

$$L_n = K_{n-1} + 2(n-1)U_{n-1} + (n-1)(n-2)T_{n-1} + 2U_{n-2} + (n-3)T_{n-3}. \quad (1.9)$$

Thence (5) gives

$$V_n = U_n + 2(n-2)T_{n-1} + 2K_{n-2} + 8(n-2)U_{n-2} + 4(n-2)(n-3)T_{n-2} \\ + 4(n-2)(n-3)U_{n-3} + 2(n-2)(n-3)(n-4)T_{n-3}. \quad (1.10)$$

We have previously resolved W_n for which the type is

$$\begin{array}{l} t\ b\ .\ .\ . \\ ST\ .\ .\ . \\ BX\ .\ .\ . \end{array}$$

by considering what happens when X is replaced by S ; but we might equally well examine what happens to W_n when X is replaced by T .

If we replace X by T we reduce Type VI to Type II. Hence

$$W_n = U_n \text{ plus the forms of Type VI in which } X \text{ clashes with } t \text{ in the first row or with } t \text{ in the second row.}$$

These clashing forms are

$$\begin{array}{cccc} t\ b\ .\ .\ . & t\ b\ .\ .\ . & t\ b\ .\ .\ . & t\ b\ .\ .\ . \\ s\ .\ .\ .T & .\ .\ .ST & .\ t\ .\ .S & .\ .\ .TS \\ x\ .\ .\ .B & x\ .B\ .\ . & .\ x\ .B\ . & .\ .BX\ . \end{array}$$

for which the enumerations are, respectively,

$$U_{n-1}, \quad (n-2)V_{n-1}, \quad L_{n-1}, \quad (n-2)W_{n-1}.$$

Hence we have the alternative value of W_n given by

$$W_n = U_n + U_{n-1} + L_{n-1} + (n-2)\{V_{n-1} + W_{n-1}\}. \quad (1.11)$$

Solving the six equations, (1.1), (1.5), (1.6), (1.7), (1.8), and (1.11), connecting the six unknowns K , U , V , W , L , and T , we have

$$\begin{aligned} U_n = & n(n-1)U_{n-1} + n(n-1)(n-2)U_{n-2} + (n-1)(n-2)^2U_{n-3} \\ & + 2(n-1)(n-2)(n^2-7n+13)U_{n-4} \\ & - 4(n-1)(n-2)(n-3)(n-4)(n-5)U_{n-5}. \end{aligned} \quad (1.12)$$

When the U 's have been found in succession from this equation, the values of the K 's are given by the equation (1.5), namely,

$$K_n = U_n - 2(n-1)U_{n-1}.$$

We cannot obtain an equation connecting the K 's only; but, by substitution for the U 's and reduction, we have

$$\begin{aligned} K_n = & (n-1)(n-2)K_{n-1} + (3n-4)K_{n-2} + (6n^2-25n+22)K_{n-3} \\ & + 2(6n^3-48n^2+115n-75)K_{n-4} \\ & + 4(n-5)(6n^3-49n^2+122n-87)U_{n-5}. \end{aligned} \quad (1.13)$$

From the above equations the early values of U_n and ${}_3K_n$ (we now reinsert the affix 3 to avoid confusion) have been calculated, namely,

n	U_n	${}_3K_n$
1	0	0
2	0	0
3	2	2
4	36	24
5	840	552
6	29,680	21,280
7	1,429,920	1,073,760
8	90,303,024	70,284,144
9	7,236,080,768	5,791,232,384
10	717,222,055,680	586,972,601,856
11	86,142,372,420,960	71,797,931,307,360
12	12,326,619,129,415,360	10,431,486,936,154,240
13	2,071,953,811,694,777,472	1,776,114,952,588,808,832
14	404,198,557,101,241,753,344	350,327,757,997,177,539,072
15	90,570,776,780,874,724,705,280	79,253,217,182,039,955,611,648

The above values of ${}_3K_n$, up to and including ${}_3K_6$, have been verified independently of the formulae, by forming the possible non-clashing sequences.

The values of the ratio $(n!)^2/{}_3K_n$ for the smaller values of n are as follows :

n	$(n!)^2/{}_3K_n$
3	18.0000
4	24.0000
5	26.0870
6	24.3609
7	23.6567
8	23.1304
9	22.7381
10	22.4340
11	22.1922
12	21.9952
13	21.8318
14	21.6941
15	21.5766

I had originally thought it possible that this sequence converged to $e^3 = 20.0855\dots$; one of the referees, however, pointed out that it was

likely that a convergent expansion of the form

$$\frac{(n!)^2}{{}_3K_n} = A_0 + \frac{A_1}{n} + \frac{A_2}{n(n+1)} + \frac{A_3}{n(n+1)(n+2)} + \frac{A_4}{n(n+1)(n+2)(n+3)} + \dots$$

exists; the entries for $n = 10, 11, \dots, 15$ suggest that the terms in $A_3, A_4, \dots$ are negligible to the degree of approximation adopted, while the values of A_2, A_1, A_0 which are in best agreement with the values of these entries may be found in succession to be

$$A_2 = 46.5, \quad A_1 = 18.85, \quad A_0 = 20.123 \text{ or } 20.124.$$

It consequently appears that the value of the limit exceeds 20.12.

Section 2.

In this section we consider the types of substitution group which produce non-clashing rows.

Take a first fixed row of n letters. Then any substitution group (which we shall briefly refer to as an operator) will produce a second row of letters which does not clash with the first and, provided that the operators represent a distinct partition of the number n , or, if they represent the same partition, contain cyclicly distinct combinations of letters, the resultant second rows of letters will all be distinct from each other.

Thus, if we are dealing with six letters $ABCDEF$ the operators $(ABCDEF)$, $(ABD)(CEF)$, $(AB)(CDEF)$, $(AB)(CD)(EF)$, $(AC)(BD)(EF)$ all produce rows non-clashing with the first fixed row and are distinct from each other.

To enumerate, therefore, the number of non-clashing schemas of two rows, it is necessary only to find the number of cyclicly distinct operators for every possible partition of the number n , excluding unity since this corresponds to an operator of degree 1, which converts a letter into itself.

Let us take a partition of the number n for which there are x " p "s, y " q "s, z " r "s, ..., and so forth. Then the number of cyclicly distinct operators is

$$\begin{aligned} & {}_nC_p \times {}_{n-p}C_p \times {}_{n-2p}C_p \times \dots \times {}_{n-(x-1)p}C_p \\ & \times {}_{n-xp}C_q \times {}_{n-xp-q}C_q \times \dots \times {}_{n-xp-(y-1)q}C_q \\ & \times {}_{n-xp-yq}C_r \times {}_{n-xp-yq-r}C_r \times \dots \times {}_{n-xp-yq-(z-1)r}C_r \\ & \times \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ & \times \frac{[(p-1)!]^x [(q-1)!]^y [(r-1)!]^z}{x! y! z! \dots} \dots \\ & = \frac{n!}{p^x q^y r^z \dots x! y! z! \dots}. \end{aligned} \tag{2.1}$$

Thus, summing for every distinct partition of n (subject to the condition that no number of the partition is less than 2), we obtain another expression for the well-known result in the "Problème des rencontres". Writing Whitworth's "sub-factorial n " as ${}_2K_n$ we have

$${}_2K_n = \sum \frac{n!}{p^x q^y r^z \dots x! y! z! \dots},$$

where the summation extends for all integral values of $p, q, r, \dots, x, y, z, \dots$ consistent with $xp + yq + zr + \dots = n$ and $p \neq q \neq r \neq \dots \neq 1$. The number of terms in the summation is, of course, $P(n|*| \geq 2)$ in the notation explained by Chrystal, *Algebra*, 2 (1900), 555-557.

If we do not impose the condition that $p, q, r, \dots$ are not to be equal to unity we enumerate the number of operators which produce either clashing or non-clashing substitutions, namely $n!$. So we have

$$\sum \frac{1}{p^x q^y r^z \dots x! y! z! \dots} = 1, \quad (2.3)$$

where the summation extends to all integral values of $p, q, r, \dots, x, y, z, \dots$ consistent with $xp + yq + zr + \dots$ integral and $p \neq q \neq r \neq \dots$.

Since ${}_2K_n = \left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!} \right] n!$,

it follows from (2.2) that

$$\lim_{n \rightarrow \infty} \sum \frac{1}{p^x q^y r^z \dots x! y! z! \dots} = \frac{1}{e}, \quad (2.4)$$

where $p \neq q \neq r \neq \dots$.

Prof. Filon has given an alternative method of proving the above results which I give with his permission.

Clearly we may take

$$p = n, \quad q = n-1, \dots \text{down to } 1$$

provided that we include 0 among the possible values of x, y, z ; this will not introduce any irrelevant terms in the summation, since in any case $x, y, z, \dots$ cannot all be zero, and the factors $p^x \cdot x!$ reduce to 1 if $x = 0$.

If we now consider the identity

$$\frac{1}{1-t} = e^{\log 1/(1-t)},$$

value of v consistent with $vs \leq n$, and the summation applies to all the distinct partitions of n , excluding the value s , that is $xp+xq+zs+\dots=n$ and $p \neq q \neq r \neq \dots \neq s$.

Similar expressions for the summation could be found if two or more integers were excluded from the partitions.

Section 3.

In Section 2 the types of operator (substitution group) were considered which, operating on a fixed sequence of n letters, produced a non-clashing sequence of the same n letters. We shall now consider the two sets of operators which, operating on a first fixed sequence of n letters, produce two sequences which clash neither with the first fixed sequence nor with each other, thus forming a Latin rectangle of depth 3, of which the enumeration has been determined. The two sets of operators may correspond to the same or to different partitions of the number n . It is clear that while the first set of operators, which converts the first row into the second, may correspond to any partition of the number n (with the exclusion of unity) and to any allotment of the letters to the brackets of the partition, the second set of operators must be restricted by the condition that in no bracket of the second set must a letter be followed, cyclicly, by the letter which followed it, cyclicly, in the first set.

Calling any derangement of letters which satisfies this condition "appropriate", the number of "appropriate" derangements of a single cycle of letters (Proposition I) and of any number of cycles (Proposition II) will be determined.

This result, combined with that of Section 2, gives another expression for the enumeration of the Latin rectangle of depth 3.

Proposition I.

Number of "appropriate" derangements of a single closed cycle.

Calling this number S_n , take any "appropriate" derangement of the closed cycle $ABCD \dots M$ and add to it the next letter N , at any point, except just before A (in a clockwise sense) or after M , we obtain $(n-2)S_n$ "appropriate" derangements of degree $n+1$.

Again an "appropriate" derangement of degree $n+1$ arises if we take a derangement of degree n , which is "appropriate" except for the conjunction of, say, BC . Such a derangement would occur in S_{n-1} ways, for each one of which we could produce a wholly "appropriate" cycle of degree $n+1$, by inserting N between B and C . This reasoning applies

to the conjunctions $AB, BC, CD, \dots, LM$. It will not apply to the conjunction MA . Thus the number of "appropriate" cycles so produced will be $(n-1)S_{n-1}$.

We have further to add the "appropriate" cycles in which the conjunction MA occurs in the cycles of $n+1$ letters, as this conjunction was not permissible in cycles of n letters, but is permissible in cycles of $n+1$ letters. If this conjunction occurs without any other conjunction, then N can be added in any position except between M and A , *i.e.* in $n-1$ positions. If in the cycle of n letters there is the conjunction of M and of two other letters, say, DE , then N can be interposed between D and E in the cycle of degree n to form an "appropriate" cycle of degree $n+1$. Let the number of such cycles in which there is the conjunction of MA and of one other pair of letters in a derangement of n letters be called V_n .

Then we have

$$S_{n+1} = (n-2)S_n + 2(n-1)S_{n-1} + V_n. \quad (3.1)$$

Similarly, it is easy to show that $V_n = (n-1)S_{n-2}$.

Hence we have finally

$$S_{n+1} = (n-2)S_n + 2(n-1)S_{n-1} + (n-1)S_{n-2}. \quad (3.2)$$

The value of the early S 's are

$$S_2 = 0, \quad S_3 = 1, \quad S_4 = 1, \quad S_5 = 8, \quad S_6 = 36, \quad S_7 = 229,$$

$$S_8 = 1,625, \quad S_9 = 13,208, \quad S_{10} = 120,288.$$

COROLLARY. *The number of derangements of the n letters $ABCD \dots M$ which form an incomplete cycle, that is to say that A does not follow M , such that no letter is followed by a letter that previously followed it, where the new sequence is to form a complete cycle, is $S_n + S_{n-1}$.* (3.3)

Proposition II.

"Appropriate" derangements of two or more cycles.

Denote by $\phi(\bar{p}_1, \bar{p}_2, \dots, \bar{p}_t, q_1, q_2, \dots, q_\nu)$ the number of "appropriate" derangements of t closed (cyclic) sequences consisting of $p_1, p_2, \dots, p_t$ letters, respectively, and q open (incomplete) sequences of $q_1, q_2, \dots, q_\nu$ letters.

Then, taking first the simple case of the formation of a single "appropriate" bracket from two "incomplete" sequences of degree q_1, q_2 respec-

tively, let these be represented by the two incomplete cycles $ABCD$ and $FGHIJ$.

Suppose, now, that we turn the letter F into E , G into F , ..., and J into I ; we shall be left with the sequence $ABCDEFGHJI$ of $q_1 + q_2$ letters, for which the number of "appropriate" derangements [v . equation (3.3)], the conjunction IA being permissible, is $S_{q_1+q_2} + S_{q_1+q_2-1}$.

Now the number of "appropriate" derangements of $ABCD FGHJI$ is the number of "appropriate" derangements of the "incomplete" sequence $ABCDEFGHJI$ plus the number of derangements of the "incomplete" sequence $ABCDEFGHJI$, "appropriate" except for the conjunction DE . The conjunction DE , with no other conjunction of letters, will occur in the derangements of $ABCDEFGHJI$ (treating DE as a single letter) $S_{q_1+q_2-1} + S_{q_1+q_2-2}$ times. Hence the number of "appropriate" derangements of $ABCD FGHJI$ is given by

$$\phi(q_1, q_2) = S_{q_1+q_2} + 2S_{q_1+q_2-1} + S_{q_1+q_2-2}. \quad (3.4)$$

The value of $\phi(q_1, q_2, \dots, q_v)$ is now readily found. For if we form a single bracket cyclic operator from all the letters of the "incomplete" sequence of $q_1, q_2, \dots, q_v$ letters, then, proceeding as in the determination of $\phi(q_1, q_2)$, we take "incomplete" sequences, say, ABC , $FGHIJ$, $LMNO$, ... and change F into D , G into E , ..., L into I , and so on, we obtain an incomplete sequence of $q_1 + q_2 + \dots + q_v$ letters, the number of "appropriate" derangements of which is $S_Q + S_{Q-1}$, where

$$Q = q_1 + q_2 + \dots + q_v.$$

Now the number of "appropriate" derangements of ABC , $FGHIJ$, $LMNO$, ... is the number of "appropriate" derangements of

$$ABCDEFGHIJKL \dots$$

plus the number of derangements of this latter sequence "appropriate" except for the presence of one or more of the conjunctions CD , HI , LM , ...

Now there will be one conjunction only, say of CD , in $S_{Q-1} + S_{Q-2}$ cases, and similarly for one conjunction only of HI , LM , ..., producing $(v-1)(S_{Q-1} + S_{Q-2})$ sequences of one conjunction only. Two conjunctions only (for two specified conjugant pairs, such as CD , LM , say) appear in $S_{Q-2} + S_{Q-3}$ cases, and, since there are ${}_{v-1}C_2$ of such possible conjugant pairs, this gives ${}_{v-1}C_2(S_{Q-2} + S_{Q-3})$ "appropriate" sequences. In the same way for r conjugant pairs the number of "appropriate" sequences is

$${}_{v-1}C_r(S_{Q-r} + S_{Q-r-1}).$$

Hence, adding all the "appropriate" sequences, we have

$$\begin{aligned}\phi(q_1, q_2, \dots, q_v) &= \sum_{r=0}^{v-1} {}_{v-1}C_r (S_{Q-r} + S_{Q-r-1}) \\ &= \sum_{r=0}^r {}_vC_r S_{Q-r}.\end{aligned}\tag{3.5}$$

Value of $\phi(\bar{p}_1, q_1, q_2, \dots, q_v)$

Take as exemplars of these sequences the groups $\overline{ABCD}$, $\overline{FGHIJ}$, $\overline{LMN}$, ..., where the first-named group represents the complete cycle of p_1 letters and the remaining groups form incomplete cycles. In the complete cycle $\overline{ABCD}$ change A into dextro- A and D into laevo- D , that is to say that in the new sequences A cannot have B on its right and D cannot have B on its right and D cannot have C on its left. This converts the complete cycle $\overline{ABCD}$ into the incomplete cycle $ABCD$.

Thus the number of possible "appropriate" sequences of the groups $ABCD$, $\overline{FGHIJ}$, $\overline{LMN}$, ... is the number of "appropriate" sequences of the groups $\overline{ABCD}$, $\overline{FGHIJ}$, $\overline{LMN}$, ... plus the number of "appropriate" sequences of $(DA)\overline{BC}$, $\overline{FGHIJ}$, $\overline{LMN}$, ..., where (DA) is treated as a single letter.

Hence

$$\begin{aligned}\phi(\bar{p}_1, q_1, q_2, q_3, \dots, q_v) &= \phi(p_1, q_1, q_2, q_3, \dots, q_v) - \phi(\overline{p_1-1}, q_1, q_2, q_3, \dots, q_v), \\ &\quad - \phi(\overline{p_1-1}, q_1, q_2, \dots, q_v) \\ &\quad = -\phi(p_1-1, q_2, q_3, \dots, q_v) + \phi(\overline{p_1-2}, q_1, q_2, \dots, q_v), \\ &\quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ (-)^{p_1-1} \phi(\bar{1}, q_1, q_2, \dots, q_v) &= (-)^{p_1-1} \phi(1, q_1, q_2, \dots, q_v).\end{aligned}$$

Thus we have

$$\begin{aligned}\phi(\bar{p}_1, q_1, q_2, \dots, q_v) &= \phi(p_1, q_1, q_2, \dots, q_v) - \phi(p_1-1, q_1, q_2, \dots, q_v) \\ &\quad + (-)^m \phi(p_1-m, q_1, q_2, \dots, q_v) \\ &\quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ &\quad + (-)^{p_1-1} \phi(1, q_1, q_2, \dots, q_v).\end{aligned}$$

Substituting the already determined value of the unbarred ϕ 's we get

$$\phi(p_1, q_1, q_2, \dots, q_v) = + \sum_{m_1=0}^{p_1-1} (-)^{m_1} \sum_{r=0}^{v+1} {}_{v+1}C_r \times S_{p_1-m_1+Q-r},$$

where $Q = q_1 + q_2 + q_3 + \dots + q_v$.

Suppose now that we have two complete cycles, exemplified by the groups $\overline{ABCD}$, $\overline{FGHIJ}$, consisting of p_1 and p_2 letters respectively, it is

easy to show by the same method that

$$\begin{aligned}\phi(\bar{p}_1, \bar{p}_2, q_1, q_2, \dots, q_v) &= \sum_{m_2=0}^{p_2-1} (-)^{m_2} \phi(\bar{p}_1, p_2 - m_2, q_1, q_2, \dots, q_v) \\ &= \sum_{m_2=0}^{p_2-1} (-)^{m_2} \sum_{m_1=0}^{p_1-1} (-)^{m_1} \sum_{r=0}^{v+2} {}_{v+2}C_r S_{p_1+p_2-m_1-m_2+Q-r},\end{aligned}$$

where $Q = q_1 + q_2 + \dots + q_v$.

Hence generally

$$\begin{aligned}\phi(\bar{p}_1, \bar{p}_2, \dots, \bar{p}_t, q_1, q_2, \dots, q_v) \\ = \sum_{m_t=0}^{p_t-1} (-)^{m_t} \dots \sum_{m_2=0}^{p_2-1} (-)^{m_2} \sum_{m_1=0}^{p_1-1} (-)^{m_1} \sum_{r=0}^{v+t} {}_{v+t}C_r S_{P+Q-M-r},\end{aligned}\quad (3.6)$$

where

$$P = p_1 + p_2 + \dots + p_t,$$

$$Q = q_1 + q_2 + \dots + q_v,$$

$$M = m_1 + m_2 + \dots + m_t.$$

This, incidentally, is the solution of the general problem of the number of ways that t groups of letters of degrees $p_1, p_2, \dots, p_t$, respectively, forming complete cycles and v groups of letters forming incomplete sequences of degrees $q_1, q_2, \dots, q_v$ respectively, can be combined in a single complete cycle so that no letter is followed by the letter that followed it in the original groups.

If the original groups contain no incomplete sequences then we have the result

$$\phi(\bar{p}_1, \bar{p}_2, \dots, \bar{p}_t) = \sum_{m_t=0}^{p_t-1} (-)^{m_t} \dots \sum_{m_2=0}^{p_2-1} (-)^{m_2} \sum_{m_1=0}^{p_1-1} (-)^{m_1} \sum_{r=0}^t {}_tC_r S_{P-M-r},\quad (3.7)$$

where P and M have the same values as before.

As special cases of the result (3.6) it is easy to show that

$$\phi(\bar{p}, q) = S_{p+q} + S_{p+q-1} + (-)^{p+1} S_q + S_{q-1}\quad (3.8)$$

and

$$\phi(\bar{p}, \bar{q}) = S_{p+q} + (-)^{q+1} S_p + (-)^{p+1} S_q + (-1)^{p+q}.\quad (3.9)$$

Application to the Latin Rectangle.

If we call the operator which converts the first sequence of letters into the second the first operator, and the operator converting the first sequence into the third, the second (immediate) operator, it now only remains to combine (i) the possible number of first operators and (ii) the number of second operators with "appropriate" derangements of their groups to obtain an expression for the enumeration of the Latin Rectangle of depth 3.

Fixing attention on a definite partition of the number n , we can select from the different parts of the first operator brackets sequences of degrees

$$\left. \begin{array}{l} p_{11}, p_{12}, p_{13}, \dots \\ p_{21}, p_{22}, p_{23}, \dots \\ \dots \quad \dots \quad \dots \end{array} \right\} \begin{array}{l} \text{say, from brackets of} \\ \text{degree } p, \end{array}$$

$$\left. \begin{array}{l} q_{11}, q_{12}, q_{13}, \dots \\ q_{21}, q_{22}, q_{23}, \dots \\ \dots \quad \dots \quad \dots \end{array} \right\} \begin{array}{l} \text{from brackets of degree } q, \end{array}$$

and we may call $\lambda(xp, yq, zr, \dots | p_{11}, p_{12}, \dots, q_{11}, q_{12}, \dots)$ the number of different ways that sequences of the degrees $p_{11}, p_{12}, \dots, q_{11}, q_{12}, \dots$ can be chosen from a *definite allotment of letters* to the first operator brackets. Corresponding to each one of these different ways there will be $\phi(p_{11}, p_{12}, p_{13}, \dots) \times \phi(p_{21}, p_{22}, p_{23}, \dots) \times \phi(q_{11}, q_{12}, q_{13}, \dots), \dots$ possible “appropriate” derangements of the letters, and, therefore, there will be $\lambda\phi \dots$ possible second operators for each definite allotment of letters to the brackets of the first operator. This must be summed for all possible values of the $p_{11}, q_{11}, \dots$

Now from Section 2, the number of different ways in which letters can be allotted to the first operator for a definite partition of the number n is

$$\frac{n!}{p^x q^y r^z \dots x! y! z! \dots}.$$

Hence for each distinct partition of the number n , used for forming the first operator, there correspond

$$\frac{n!}{p^x q^y r^z \dots x! y! z! \dots} \text{Sum}_1 [\lambda(xp, yq, zr, \dots | p_{11}, p_{12}, \dots, q_{11}, q_{12}, \dots) \\ \times \phi(p_{11}, p_{12}, \dots) \phi(p_{21}, p_{22}, \dots) \phi(q_{11}, q_{12}, \dots) \dots]$$

ways of forming the second operator, and for all *possible partitions of the number n* there will be

$$\text{Sum}_2 \left\{ \frac{n!}{p^x q^y r^z \dots x! y! z! \dots} \text{Sum}_1 [\lambda(xp, yq, zr, \dots | p_{11}, p_{12}, \dots, q_{11}, q_{12}, \dots) \right. \\ \left. \times \phi(p_{11}, p_{12}, \dots) \phi(p_{21}, p_{22}, \dots) \phi(q_{11}, q_{12}, \dots) \dots] \right\} \\ = {}_3K_n \text{ ways of forming the second operator,} \tag{3.10}$$

where Sum_1 applies to every possible value of

$$p_{11}, p_{12}, \dots, p_{21}, p_{22}, \dots, q_{11}, q_{12}, \dots,$$

consistent with

$$\left. \begin{array}{l} p_{11} + p_{12} + \dots = p_{21} + p_{22} + \dots = \dots, \text{ say, for } x_1 \text{ values} = p_1, \text{ say} \\ q_{11} + q_{12} + \dots = q_{21} + q_{22} + \dots = \dots, \text{ say, for } y_1 \text{ values} = q_1, \text{ say} \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \end{array} \right\}, (3.11)$$

and $x_1 p_1 + y_1 q_1 + z_1 r_1 + \dots = n, \quad p_1 \neq q_1 \neq r_1 \neq \dots \neq 1$

and Sum applies to every value of $p, q, r, \dots$ consistent with

$$xp + yq + zr + \dots = n, \quad p \neq q \neq r \neq \dots \neq 1.$$

As a simple example of this, take $n = 4$; omitting partitions which contain a single 1 only from the summations, we have

$$\begin{aligned} {}_3K_4 = \frac{4!}{4 \times 1!} [1 \times \phi(\bar{4}) + 3\phi(2)\phi(2) + \phi(1, 1)\phi(1, 1)] \\ + \frac{4!}{2^2 \times 2!} [\phi(\bar{2}, \bar{2}) + 2\phi(1, 1)\phi(1, 1)]. \end{aligned}$$

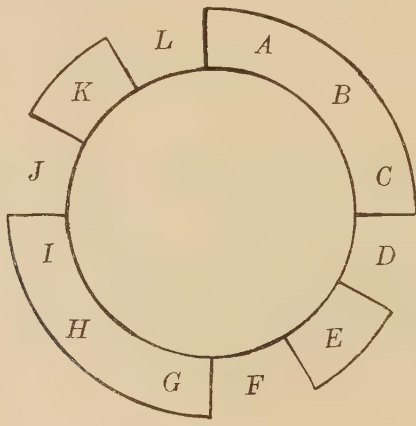
From the results previously given $\phi(\bar{4}) = 1, \phi(2) = 0, \phi(\bar{2}, \bar{2}) = 2, \phi(1, 1) = 1$, so that this gives ${}_3K_4 = 24$, which is true.

Value of the function λ in certain cases.

We need, first, the determination of the number of ways of selecting r non-adjacent letters from a cyclic group of p letters.

To visualise the problem completely, suppose that the p letters are arranged at equal angular intervals round the circumference of a circle and that a template is constructed corresponding to any one of the r partitions of the number $p - r$. (The form of the templates is shown in the diagrams below† for the case of two 4-partitions of the number 8.)

† Template corresponding to the partition 3, 1, 3, 1, of the number 8. As this partition has a twice repeated arrangement it will only produce the desired group of four non-adjacent letters in six positions.



There will be $P(p-r|r|*)$ groups of templates corresponding to the number of cardinal r partitions of the number $p-r$. Let us take a particular cardinal partition, say

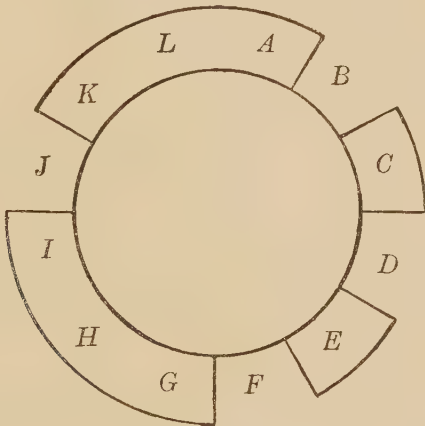
$$\begin{aligned}r_1a_1+r_2a_2+r_3a_3+\ldots &= p-r, \\ r_1+r_2+r_3+\ldots &= r,\end{aligned}$$

where $a_1 \neq a_2 \neq a_3 \neq \ldots$.

With this definite partition we can still make up a number of different templates depending on the cyclic order in which we arrange the teeth of the template. In discussing the number of cyclic arrangements of the teeth clockwise and anti-clockwise, arrangements will be regarded as distinct even when the corresponding templates are identical when inverted.

Let G be the greatest common factor of the numbers $r_1, r_2, r_3, \ldots$ and suppose that $G = S^sT^tU^u\ldots$, where $S, T, U, \ldots$ are primes. Then it is clear that the letters $a_1, a_2, a_3, \ldots$ may be written down so that the letters

Template corresponding to the partition 3, 3, 1, 1, of the number 8. As this partition has no repeated arrangement, it will produce the desired group of four non-adjacent letters in twelve positions.



The full enumeration of cases for the selection of four non-adjacent letters from a 12-cycle is as follows :

Cardinal partition	Repetitive partition	Degree of repetitive partition	Varieties	Different positions of template	Products
5, 1, 1, 1		1	1	12	12
4, 2, 1, 1		1	3	12	36
3, 3, 1, 1		1	1	12	12
	3, 1, 3, 1	2	1	6	6
3, 2, 2, 1		1	3	12	36
2, 2, 2, 2	2, 2, 2, 2	4	1	3	3
Total					105

form recurring groups, the number of such groups being $S^s T^{t'} U^{u'} \dots$ (where $s' \leq s$, $t' \leq t$, ...) in any complete cycle.

Call $r/G = \gamma$, $r_1/G = \gamma_1$, $r_2/G = \gamma_2$, $r_3/G = \gamma_3$, ...

Let λ be the total number of letters in each recurring group, which will, therefore, contain $\lambda r_1/r$, $\lambda r_2/r$, $\lambda r_3/r$, ... of the letters a_1 , a_2 , a_3 , ... respectively. We shall refer to a cyclic arrangement of the letters as of degree λ when there are λ letters in all in each recurring group, so arranged that *there are no recurrences in the group*.

Now suppose that all the letters are arranged round a circle in r/λ groups of degree λ and that an extraneous letter, say E , is introduced into the cycle. It may be inserted in λ cyclicly different positions and hence if we call Z_λ the number of different ways that the $\lambda r_1/r$ letters a_1 , the $\lambda r_2/r$ letters a_2 , and so on, may be arranged without any recurrences *inter se* of lower degree, the introduction of the letter E will be responsible for λZ_λ cyclicly different arrangements. Since the number of ways of arranging the r_1 letters a_1 , the r_2 letters a_2 , the r_3 letters a_3 , ..., and the letter E can be written down at once, we have

$$\text{Sum}(\lambda Z_\lambda) = \frac{r!}{(r_1)! (r_2)! (r_3)! \dots} \quad (3.12)$$

The possible values of λ are $r/S^{s'} T^{t'} U^{u'} \dots$, where $0 \leq s' \leq s$, $0 \leq t' \leq t$, ..., so that λ has $(s+1)(t+1)(u+1) \dots$ values.

In general we have $(s+1)(t+1)(u+1) \dots$ equations of the form

$$\text{Sum}[\gamma S^{s''} T^{t''} U^{u''} \dots Z_{\gamma S^{s'} T^{t'} U^{u'}} \dots] = \frac{(\gamma S^{s'} T^{t'} U^{u'})!}{(\gamma_1 S^{s'} T^{t'} U^{u'})! (\gamma_2 S^{s'} T^{t'} U^{u'})! \dots}, \quad (3.13)$$

where s'' , t'' , u'' , ... have every possible combination of values consistent with $0 \leq s'' \leq s'$, $0 \leq t'' \leq t'$, $0 \leq u'' \leq u'$, ... When these equations are solved the required number of templates corresponding to the given cardinal partition is $\text{Sum}_1 [Z_{\gamma S^{s'} T^{t'} U^{u'}} \dots]$ for all the values of s' , t' , u' , ...

from zero up to and including s , t , u , ... respectively.

Hence the total number of ways of selecting r non-adjacent letters from a cyclic group of p letters is $\text{Sum}_2 ([\text{Sum}_1 Z_{\gamma S^{s'} T^{t'} U^{u'}} \dots])$, where Sum_2 applies to every distinct cardinal r -partition of $p-r$. Thus $\lambda(\bar{p}, 1, 1, 1, \dots, 1)$, for r arguments unity, is equal to

$$\text{Sum}_2 (\text{Sum}_1 [Z_{\gamma S^{s'} T^{t'} U^{u'}} \dots]). \quad (3.14)$$

COROLLARY 1. *If, instead of r single letters, we have to choose r sequences, namely a_1 sequences of degree d_1 , a_2 sequences of degree d_2 , ..., a_r sequences of degree d_r , where $d_1 \neq d_2 \neq \dots \neq d_r$, then, in the particular case in which $a_1, a_2, \dots, a_r$ have no common factor, any*

template which corresponds to a distinct cyclic r -partition of $p-d$ will produce a distinct group of sequences, p being the total number of letters and d being $\sum_{g=1}^r (a_g d_g)$.

Let us denote by $P_{co}(x|y|*)$ the number of cyclicly distinct ordinal partitions (that is, the number of cyclicly distinct "compositions") of x into y parts; then the number of ways of selecting the sequences from among the p cyclicly arranged letters is

$$P_{co}(p-d|r|*) \times p.$$

Hence, in accordance with the previous notation, if $a_1, a_2, \dots, a_r$ are prime to each other and

$$a_1 d_1 + a_2 d_2 + \dots + a_r d_r = d,$$

we have

$$\lambda(\bar{p} | d_1, \dots \text{ for } a_1 \text{ terms, } d_2, \dots \text{ for } a_2 \text{ terms, } \dots) = P_{co}(p-d|r|*) \times p. \quad (3.15)$$

COROLLARY 2. If, instead of r arguments unity, we select r arguments of degree h , then the total number of ways of doing so will be

$$\sum_2 (\sum_1 [Z_{\gamma S \gamma' T \gamma' U \gamma' \dots}]),$$

where the summations now apply to all the distinct cardinal partitions of $p-rh$ into r parts. Hence we may write

$$\lambda(\bar{p} | h, h, \dots, h), \text{ for } r \text{ arguments } h = \sum_2 (\sum_1 [Z_{\gamma S \gamma' T \gamma' U \gamma' \dots}]). \quad (3.16)$$

I have not been able to find the generalized value of

$$\lambda(x\bar{p}, y\bar{q}, z\bar{r}, \dots | p_{11}, p_{12}, \dots, p_{21}, p_{22}, \dots, q_{11}, q_{12}, \dots, r_{11}, r_{12}, \dots),$$

which is, apparently, rather complex.

Section 4.

Classification of Latin Squares.

When a number of individual Latin Squares can be brought to a single form by mere permutation of the complete rows or of the complete columns or by permutation of both complete rows and complete columns, the type form is styled by MacMahon (*Combinatory analysis*, 1, § 208) a "reduced Latin Square". Now every reduced Latin Square can be considered in relation to the types of partitional operator which convert the first row "in succession" into the remaining $n-1$ rows and re-

convert the last row back again into the first. A perfect classification of Latin Squares would take account of each of the " n " successive operators and refer each individual or reduced square to the particular types of operators involved. Thus the Latin Squares of degree 4 belong either to the type (4) or to the type (2) (2)

(4)	(2) (2)
(4)	(2) (2)
(4)	(2) (2).

Again the six types of sets of operators for Latin Squares of degree 5 are

(5):(2) (3):	(5)	:	(5)	:	(5)	:	(2) (3)
(5):(2) (3):(2) (3):	(5)	:	(5)	:	(5)	:	(5)
(5):	(5)	:	(2) (3):(2) (3):	(5)	:	(5)	
(5):	(5)	:	(5)	:	(2) (3):(2) (3):	(5)	
(5):	(5)	:	(5)	:	(5)	:	(2) (3):(2) (3)

The present classification does no more than refer each Latin Square to the type of operator which converts the first row into the second, and this is a step towards the complete classification.

For definiteness the first row and first column of our reduced Latin Square will be represented by $ABCDE\dots$ and the operator converting the first row into the second must necessarily contain the letters AB in association.

All Latin Squares, then, for which the operator converting the first row into the second corresponds to the same partition of the number n , will be classed as of the same *family*. (For this purpose the Latin Squares arising from an operator such as $(AB)(CDE)$ are regarded as of different family from those arising from $(ABC)(CDE)$, since the association AB is fixed by our convention.)

Next, all Latin Squares for which the first operators correspond not only to the same partition of the number n , but to the identical grouping of letters in each bracket of the operator, will be said to be of the same *genus*.

And lastly, all Latin Squares referable to the same "reduced" form will be said to be of the same *species*.

Now the number of reduced (or individual) Latin Squares is the same for each genus of the same family. If this be not regarded as obvious it may be proved as follows.

Let $g(xp, yq, zr, \dots)$ be the number of reduced Latin Squares which arise from an operator (converting the first row into the second) in which

there are x substitutions of degree p , y of degree q , and so on, the association AB occurring, say, in a bracket of degree p .

Suppose now that we take two blank operators and fill in the first with the letters $ABPQR \dots$ in succession and the other with $ABP'Q'R' \dots$. Let the first operator operate on a Latin Square whose first row and first column are $ABPQR \dots$ and the second operator on a Latin Square whose first row and column are $ABP'Q'R' \dots$. Then clearly the number of possible Latin Squares is the same for both these genera, since they differ only in substituting P' for P , Q' for Q and so on, both the squares being in their reduced form and belonging to the same family.

Hence to enumerate the number of all possible Latin Squares it will be necessary only to determine the number of reduced squares for each particular genus and multiply this by the number of possible genera in each family, finally summing for all possible families.

It is now necessary only to determine the number of genera in the family whose operator corresponds to the partition $xp, yq, zr, \dots$ of the number n , where $p \neq q \neq r \neq \dots \neq 1$. We have, in fact, to find the permutations of the letters $CDEFG \dots$ in such an operator as

$$(abCDE \dots)(FGH \dots)(IJK \dots) \dots,$$

the association AB being fixed in a substitution of degree p . The number of species (reduced squares) in a family of this type is thus given by

$$\begin{aligned} & {}_{n-2}C_{p-2} \times {}_{n-p}C_p \times {}_{n-2p}C_p \times \dots \times {}_{n-(x-1)p}C_p \\ & \times {}_{n-xp}C_q \times {}_{n-xp-q}C_q \times \dots \times {}_{n-xp-(y-1)q}C_q \\ & \times {}_{n-xp-yq}C_r \times {}_{n-xp-yq-r}C_r \times \dots \times {}_{n-xp-yq-(z-1)r}C_r \\ & \times \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ & \times \frac{(p-2)! [(p-1)!]^{x-1} [(q-1)!]^{y-1} [(r-1)!]^{z-1} \dots}{x! y! z! \dots} \\ & \times g(\underline{xp}, yq, zr, \dots) \\ & = \frac{(n-2)!}{p^{x-1} q^y r^z \dots x! y! z! \dots} g(\underline{xp}, yq, rz, \dots). \end{aligned}$$

The total number of species is, therefore,

$${}_n k_n = \sum \frac{(n-2)!}{p^{x-1} q^y r^z \dots x! y! z!} g(\underline{xp}, yq, rz, \dots), \quad (4.1)$$

where the summation extends to every integral solution of

$$xp + yq + zr + \dots = n,$$

provided that $p \neq q \neq r \neq \dots \neq 1$, and to the allotment of the associated letters AB to each partition of different degree.

The total number of Latin Squares is, of course,

$${}_nK_n = n! (n-1)! \sum \frac{(n-2)!}{p^{x-1} q^y r^z \dots x! y! z!} g(\underline{xp}, \underline{yq}, \underline{zr}, \dots), \quad (4.2)$$

the summation having the same extension as before.

As examples, the values of ${}_5K_5$ and ${}_6K_6$ can be found when the g 's have been calculated *a priori*. Thus, having found

$$g(5) = 6, \quad g(\underline{3}, 2) = g(2, \underline{3}) = 4, \quad g(6) = 168, \quad g(\underline{4}, 2) = g(2, \underline{4}) = 176, \\ g(\underline{3}, \underline{3}) = 192, \quad g(\underline{2}, \underline{2}, \underline{2}) = 224,$$

we get

$${}_5K_5 = 5! 4! 3! \left\{ g(5) + \frac{g(\underline{3}, \underline{2})}{2} + \frac{g(2, \underline{3})}{3} \right\} \\ = 5! 4! \times 56. \\ {}_6K_6 = 6! 5! 4! \left\{ g(6) + \frac{g(\underline{4}, \underline{2})}{2} + \frac{g(2, \underline{4})}{4} + \frac{g(\underline{3}, \underline{3})}{6} + \frac{g(\underline{2}, \underline{2}, \underline{2})}{24} \right\} \\ = 6! 5! \times 8192.$$

MacMahon's result for the number of reduced Latin Squares of degree 5, *Combinatory analysis*, 1 (1916), § 209, is 52, whereas the above result shows that the number is 56, made up of 25 (not 23) pairs of conjugate and six self-conjugate squares†. The six self-conjugate reduced Latin Squares are given by MacMahon (*loc. cit.*), while one square from each pair of the 25 conjugate pairs is shown in the appendix to this paper.

Section 5.

The number of "complete cycle" Latin Squares.

Let us operate on any group of degree n by a cyclic substitution of degree n of the same letters. The Latin Square produced by $n-1$ such operations in succession will be spoken of as a "complete cycle" Latin Square. Call any such cyclic substitution Q . Then, by operating directly on the first row of letters with the operators $Q, Q^2, \dots, Q^{n-1}$, a Latin Square is obtained, whatever order of the letters in Q is adopted. Not all the $(n-1)!$ possible arrangements of letters of Q , however, lead to different Latin Squares considered in their reduced form.

† The value of $1_2 = 1$ given by MacMahon is clearly a slip for $1_2 = 2!$.

Thus, if we take $Q \equiv (a_1 a_2 a_3 a_4 \dots a_n)$ then

$$Q^f = (a_1 a_{1+f} a_{1+2f} \dots a_{1+rf} \dots)$$

which is a complete cycle operator of degree n if f is prime to n , but breaks down, if f and n have a common factor s , into s smaller cycles.

Hence, to each operator such as Q , there corresponds an operator with a permutation of the letters $a_1 a_2 \dots a_n$ which is identical with Q^f , f being prime to n .

Now the Latin Squares produced by the successive operations

$$Q, Q^2, Q^3, \dots, Q^{n-1} \quad (5.1)$$

and

$$Q^f, Q^{2f}, Q^{3f}, \dots, Q^{(n-1)f} \quad (5.2)$$

are identical in their reduced form. This is clear since, if we divide $f, 2f, 3f, \dots, (n-1)f$ by n , the remainders are $1, 2, 3, \dots, n-1$, though in different order, so that the operations (5.1) and (5.2) lead to the same sequences. Thus the reduced Latin Squares resulting from these two groups of operations are identical.

So, for each operator corresponding to a particular permutation of the letters $a_1 a_2 a_3 \dots a_n$, which we call Q , there is another permutation Q^f , which leads to the same reduced Latin Square. This is true whenever f is prime to n , so that the number of permutations of the letters which lead to differing reduced Latin Squares is $(n-1)!/\Phi(n)$, where $\Phi(n)$ is the number of integers, including unity, which are prime to and less than n . The total number of "complete cycle" Latin Squares is $n!(n-1)!$ times this number, so that we may write

$$C_n = \frac{n! [(n-1)!]^2}{\Phi(n)}, \quad (5.3)$$

where C_n is the number of "complete cycle" Latin Squares of degree n .

We shall now show that each distinct "complete cycle" operator produces, in its reduced form, one self-conjugate Latin Square.

A Latin Square is defined as "self-conjugate"† when the letter in the r -th column and s -th row is the same, for all values of r and s , as the letter in the s -th column and the r -th row, or, as we may write it, when $[r, s] = [s, r]$.

Now if the first row of our Latin Square be $a_1 a_2 \dots a_r \dots a_n$, where the letters are written in their "natural" order, and it is operated on in succession by the operator $Q \equiv (a_1 a_2 \dots a_r \dots a_n)$, then a "complete cycle" Latin Square will be formed for which $[r, s]$ will be the $(r+s-1)$ -th letter

† MacMahon, *Combinatory analysis*, 1 (1916), § 209.

of Q , following the letters on in their cyclic order if $r+s-1 > n$. Similarly $[s, r]$ will be the $(s+r-1)$ -th letter of Q . Hence, $[r, s] \equiv [s, r]$ and the square so formed will be self-conjugate, and will be in its "reduced" form.

Now take any other "complete cycle" operator such as

$$Q' \equiv (a_1 a_3 \dots a_n \dots a_r \dots a_p)$$

in which the letters are no longer in their "natural" order and let it operate on the first row of letters $a_1 a_3 \dots a_n \dots a_r \dots a_p$; then for the Latin Square so formed $[r, s]$ and $[s, r]$ must be identical. In order to bring this Latin Square to its "reduced" form (the first row and first column being at present $a_1 a_3 \dots a_n \dots a_r \dots a_p$) it will be necessary to transpose the complete rows and complete columns, and if the r -th row is transferred to the r' -th row, the r -th column must be transferred to the r' -th column, and, if the s -th column is transferred to the s' -th column, the s -th row must be transferred to the s' -th row. Hence, if $[r, s]$ in the "unreduced" square becomes $[r', s']$ in the "reduced" square, $[s, r]$ will become $[s', r']$ and since $[r, s] \equiv [s, r]$, it follows that $[s', r'] \equiv [r', s']$, so that the new "reduced" Latin Square is also self-conjugate.

Thus each "complete cycle" operator operating "in succession" produces, in its "reduced" form, one self-conjugate Latin Square.

The converse proposition, that every self-conjugate Latin Square arises from a "complete cycle" operator is not, however, true as may be seen, for example, from the four types of Latin Square of degree 4, namely,

I	II	III	IV
$ABCD$	$ABCD$	$ABCD$	$ABCD$
$BCDA$	$BADC$	$BDAC$	$BADC$
$CDAB$	$CDBA$	$CADB$	$CDAB$
$DABC$	$DCAB$	$DCBA$	$DCBA$

of which Types I-III arise from the "complete cycle" operators

$$(ABCD) \equiv (ADCB), \quad (ACBD) \equiv (ADBC), \quad (ABDC) \equiv (ACBD),$$

respectively, whereas Type IV, although it is self-conjugate, cannot be produced from any "complete cycle" operator. Thus the value of C_n given in (5.3) is merely a lower limit for the number of self-conjugate Latin Squares. For squares of degree 5 the number of squares produced by complete cycle operators is equal to the number (6) of self-conjugate Latin Squares, but I cannot prove whether such a result holds for all Latin Squares of prime degree.

ON A NEGLECTED PRINCIPLE IN ELEMENTARY TRIGONOMETRY

By H. LOB and H. W. RICHMOND.

[Received 7 March, 1929.—Read 14 March, 1929.]

1. *Introductory.* In the writing of our paper upon “Malfatti’s problem” (4) it gradually became clear that, after one solution had been obtained, every detail of the work could be adapted by means of a suitable substitution so as to apply immediately to any one of the thirty-two solutions foreseen by Steiner (12). This was done, in the first instance, by a trigonometrical substitution which, it was seen, would bring the equations to be solved into the required form; the first eight solutions (which form Class I in the table of solutions, and in which the centres of the three “Malfatti” circles lie upon the internal bisectors of the angles of the triangle) were thus derived from the original solution. Then the geometrical significance of the process was recognized, and the remaining solutions were obtained in a more scientific and more satisfactory manner.

This method of derivation illustrates a general principle of plane trigonometry which seems to be worth considering both for itself and for its applications to other problems besides that of Malfatti. The principle may be enunciated as follows :—

(a) *An equation among the sides, angles, etc., of a triangle ABC remains true when, for the angles A, B, C, are substituted*

$$\text{either} \quad (i) \quad \alpha_1 \pi + A, \quad \beta_1 \pi + B, \quad \gamma_1 \pi + C;$$

$$\text{or} \quad (ii) \quad \alpha_2 \pi - A, \quad \beta_2 \pi - B, \quad \gamma_2 \pi - C;$$

$\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$ being integers, positive or negative, subject only to a condition which makes the sum of the angles π in either case, viz.

$$\alpha_1 + \beta_1 + \gamma_1 = 0, \quad \alpha_2 + \beta_2 + \gamma_2 = 2.$$

(b) *The equations to which these substitutions can be applied are either trigonometrical relations or equations of loci in areal coordinates.*

It is to be understood that for the sides of the triangle must be substituted numbers a, b, c , *proportional in sign as well as in magnitude* to the sines of these angles. Other magnitudes connected with the triangle are to be expressed in terms of the sides and angles by the common formulae of the text-books, which, it should be noted, remain consistent when the restriction as to the size of the angles is discarded. A formula containing a square root or other ambiguity may on that account require consideration; ambiguous formulae are here for the most part avoided.

2. *Historical.* The facts which led us to the results in our former paper are so elementary that it seemed impossible that the method had been overlooked. The simplest transformation of the second type, when $\alpha_2, \beta_2, \gamma_2$ have the values 0, 1, 1, is the "transformation continue" of Lemoine*; the text-books do not as a rule include it, but it is fairly well known as a means of deriving results concerning the escribed circles of a triangle from those of the inscribed circle. We have, however, failed to discover any place where the principle is enunciated as a general principle of elementary plane trigonometry. This is almost incredible; because, if once the legitimacy of Lemoine's transformation is admitted, so that $\alpha_2, \beta_2, \gamma_2$ are allowed to assume the values 0, 1, 1, or 1, 0, 1, or 1, 1, 0, it is an obvious step to combine these substitutions, and so derive a group of transformations. To establish the principle no more than this is necessary.

In contrast to this, the analogous but far more abstruse theory of the spherical triangle has been investigated with extreme thoroughness by Study† and others‡, among them Klein and Grace Chisholm-Young. A very full account of Study's work will be found under 16; the article in the *Encyk. der Math. Wiss.*, III₁, 835-854, is shorter, but shows more clearly the reason for the investigation and its importance. In view of the elementary character of our work it seems absurd to base the simple theory which we require upon elaborate results for a spherical triangle, though such a course is no doubt possible. In his book, 12, on pages 116 and 180, Study makes a brief mention of the plane as a

* 3. See also *Encyk. der Math. Wiss.*, III₁, 775, Note 4.

† *Encyk. der Elementar-Mathematik* (Weber und Wellstein), II, *Elemente der Geometrie*. Book 2 of this volume (303-438) deals with Trigonometry, 36 pages being allotted to plane and 100 to spherical. Through about a quarter of these 100 pages the reader is led from the Euler'sche, Möbius'sche, Gauss-Study'sche, up to the Study'sche Dreiecksbegriff. The upper half of p. 385 has a special interest in connexion with the present paper.

‡ *Encyk. der Math. Wiss.*, III₁, 835-854. Further references there given in foot-notes include Study, 12, and Klein, 2. We wish to express our thanks to Messrs. E. H. Neville, G. N. Watson, and F. P. White for their help with these and other references.

limiting case of the results which he has obtained; but for the purpose of application to simple properties of plane triangles the form is not inviting.

3. *On continuous variation of the sides and angles of a triangle. Lemoine's and other geometrical transformations.* Lemoine arrived at his theorem that *a relation connecting the sides a, b, c and the angles A, B, C of a triangle ABC remains true when these quantities are replaced by $a, -b, -c, -A, \pi-B, \pi-C$* , by supposing that two vertices B and C of the triangle and the direction of the side BA were kept fixed, while the side CA turned about C so that the vertex A moved along the fixed line BA away from B . The triangle continued as an ordinary triangle, and the usual relations among the sides and angles held good, until the side CA became parallel to BA . The lines then ceased to form a triangle. But if the turning of CA about C was continued, the lines would again form a triangle whose sides and angles a', b', c', A', B', C' would satisfy the usual relations. These quantities, however (with the exception of a' , which has not varied), are not what a, b, c, A, B, C would have become had they gone on changing continuously. (Whether the word continuous applies to the variation of b and c through infinite values need not be argued.) By continuous change, says Lemoine, the values which a, b, c, A, B, C would have assumed are *not* (a', b', c', A', B', C') , but $(a', -b', -c', -A', \pi-B', \pi-C')$. On this reasoning Lemoine based his theorem.

Other modes of continuous variation of the parts of a triangle lead more simply to similar results*: it is enough to consider one. In the initial position of the triangle ABC , let H be the foot of the perpendicular from A to the base BC , and let B' and C' be the points of the base such that BB' and CC' have H for their middle point. Suppose that A remains fixed, while B and C move through H to B' and C' respectively. The final triangle $AB'C'$ has the same absolute dimensions as ABC , but is reversed as by reflection. There has been no moving off to infinity, and it is easy to follow out the continuous change which any side or angle has undergone. Thus b and c have diminished down to the length AH and then have increased up to their original values. B and C have passed through

* A method which is of interest because it leads to six different results may be noticed:— Given the triangle in its initial position ABC ; let A' be the reflection of A in the base BC ; let B and C be kept fixed while the third vertex moves from A to A' , either along the finite line AA' , or along a circular arc through A and A' , or along the infinite segment of the line AA' . The base is divided into three segments by the points B and C and the point at infinity; and the result depends upon the position of the point where the moving vertex crosses the base, in one or other segment or at one or other of the three points.

$\frac{1}{2}\pi$ at H to the final values $\pi-B$ and $\pi-C$. A and a have diminished to zero, have become negative, and continued to diminish to the final values $-A$ and $-a$. The initial and final values, linked by continuous variation, are therefore

$$(a, b, c, A, B, C) \quad \text{and} \quad (-a, b, c, -A, \pi-B, \pi-C).$$

These are not in exact agreement with the values obtained by Lemoine, for the signs of the sides are all reversed. But this is of no consequence, because lengths enter any valid relation homogeneously. Either from the geometrical theory of similitude or from the general theory of dimensions, we know that a trigonometrical formula remains true when ka , kb , kc are written in place of a , b , c , the angles A , B , C being left unchanged. When, as here, negative values of the sides are permissible, k must be allowed negative values, and, in particular, the value -1 . Lemoine's result is thus reached without the introduction of infinity.

It is, moreover, possible to visualize and follow out the variations which other magnitudes related to the triangle undergo. When the points B and C coincide in the course of the variation, the circumcircle of ABC becomes the circle which passes through A and touches the base at the point of coincidence, and the escribed circles opposite to B and C become the two circles which touch the base and also touch the coincident sides AB and AC at A . There is no difficulty here; the circumcircle passes from its initial to its final position without vanishing: the two escribed circles vary continuously, being always of finite size, but the escribed circle opposite to B changes into the escribed circle opposite to C' . Our formulae confirm this, for they show, when the changed sides and angles are substituted in the usual formulae, that R is unaffected, while r_2 and r_3 are interchanged. But when B and C coincide, the inscribed circle and the escribed circle opposite to A vanish, and the matter is slightly more difficult. It appears (and it is confirmed by the formulae) that the incentre moves down to the base, crosses it at a finite angle, and so becomes the escribed centre opposite to A , the sign of the radius having at the same time changed from positive to negative in its passage through zero*. Although geometrical intuition, helped by trigonometrical formulae, will carry us as far as Lemoine's results and further, we shall be able to make fuller use of the method, and feel more security if our proofs are built upon an analytical basis.

* In the variation suggested in foot-note (*), p. 357, where the vertex A moves to infinity along the line at right angles to BC , the inscribed and first escribed circle interchange rôles without the radii vanishing.

4. *The group of substitutions and the principle.* A triangle ABC being given, we may denote by h the length of the perpendicular AH from A to the base BC , and by ϕ and ψ the angles HAB and HAC respectively. We suppose that ϕ and ψ are measured in opposite senses from the fixed line AH , so that ϕ and ψ are positive when the base angles B and C are acute; if one base angle is obtuse, either ϕ or ψ is negative. The triangle ABC is defined by the values of h , ϕ , ψ , and its sides and angles can be expressed simply in terms of h , ϕ , ψ ; in fact,

$$\left. \begin{aligned} a &= h(\tan \phi + \tan \psi), & b &= h \sec \psi, & c &= h \sec \phi \\ A &= \phi + \psi, & B &= \frac{1}{2}\pi - \phi, & C &= \frac{1}{2}\pi - \psi \end{aligned} \right\}. \quad [G]$$

Now, with polar coordinates, it is the custom to allow the vectorial radius and angle to be capable of assuming any value, and ϕ and ψ are vectorial angles: convention (quite unnecessarily, as we maintain) limits a , b , c to positive values, and A , B , C to positive values less than π . To conform to convention we must impose conditions upon ϕ and ψ :

$$\frac{1}{2}\pi > \phi > -\frac{1}{2}\pi, \quad \frac{1}{2}\pi > \psi > -\frac{1}{2}\pi, \quad \phi + \psi > 0.$$

But the area, the perimeter, the radii of the circumscribed, inscribed, and escribed circles of ABC can all be expressed in terms of h , ϕ , ψ ; any trigonometrical relation that holds for the triangle can be transformed into a relation in h , ϕ , ψ , and so verified. Further, unless the relation is one which holds only for triangles of some particular scale, h will enter homogeneously into the transformed relation and may be removed: the relation to be verified will be one depending only upon ϕ and ψ . And, lastly, if the relation in ϕ , ψ takes the form of an algebraic relation among the trigonometrical functions of these angles and their multiples or submultiples, we argue that such a relation, if it holds for all values of ϕ and ψ limited as above, must hold universally. The limitations of ϕ and ψ are unnecessary; as also are those of a , b , c , A , B , C , if only the relations [G] above are satisfied.

The pith of the principle is this:—that a series of values of ϕ and ψ lead to triangles which, for many purposes, can be ranked as identical with the given triangle. Lemoine, by his continuous transformation, has indicated one such triangle; to obtain it and its properties we make ϕ and ψ diminish continuously through zero to the values $-\phi$ and $-\psi$. From these two triangles the other triangles of the series are obtained even more simply; it is only necessary to increase or diminish ϕ and ψ by any multiples whatever of π . The lines AB and AC are made to rotate about A through arbitrary multiples of two right angles, so that the vertices of the new triangles coincide with those of the first two. The

effect of this upon the angles is obvious from equations [G]; they are increased by three multiples of π whose sum is zero. Operating upon the angles A, B, C of the given triangle, we derive the series of angles of type (i) in the first section of this paper: operating upon the angles of Lemoine's triangle, we derive the series of angles of type (ii). As regards the effect on the sides, we see from equations [G] that, in all circumstances,

$$a : b : c = \sin A : \sin B : \sin C ;$$

and it has been observed that the ratios of the sides, and not the actual values, alone concern us. Thanks to this rule, there is no longer any necessity to discuss the sides; given the angles, the ratios of the sides to one another are defined. In fact, h, ϕ, ψ have served their purpose, and are no longer needed; eliminating them from equations [G] we find the relations among a, b, c, A, B, C , which are really fundamental for our ends, viz. :

$$A+B+C = \pi, \quad a : b : c = \sin A : \sin B : \sin C. \quad [H]$$

In conclusion, certain comments may be made. Lemoine's transformation of the angles, quoted in § 3, is one of three such transformations, $\alpha_2, \beta_2, \gamma_2$ taking the values 0, 1, 1 in various orders. If two of these transformations are made in succession, one of the angles is increased and another diminished by π . Hence the statement made in § 2, that the whole group of substitutions could be derived from Lemoine's theorem, is justified. Secondly, if we start from three angles unrestricted as to size but having the sum π , and if we take three numbers $\sigma_1, \sigma_2, \sigma_3$ proportional to the sines of these angles, any two of the three moduli $|\sigma_1|, |\sigma_2|, |\sigma_3|$ are always together greater than the third. An ordinary triangle can therefore be constructed with sides proportional to these moduli, and one of the substitutions of § 1, applied to the angles of this triangle, will derive the three unrestricted angles from which we started.

Thirdly, if a point has polar coordinates (r, θ) referred to A as pole and AH as initial line, the perpendiculars from it to the sides of ABC are

$$h - r \cos \theta, \quad r \sin (\psi - \theta), \quad r \sin (\phi + \theta),$$

and, save for possible changes of sign, these continue to represent the lengths of the perpendiculars throughout the variations of the triangle. Trilinear coordinates referred to the triangle therefore remain trilinear coordinates, except that it may be necessary to change their signs.

The fundamental identity

$$a\alpha + b\beta + c\gamma = 2\Delta$$

suggests a simple rule for the signs, which can be verified otherwise, viz., that the signs of α, β, γ should be changed in conformity with those of a, b, c . What is even better, *areal coordinates*, being proportional to $a\alpha, b\beta, c\gamma$, require no change whatever.

We shall not attempt to expound a theory of extended conceptions of the notion of a triangle, as has been done in spherical trigonometry. To do so might be interesting, but does not seem essential for the simple applications we are about to make. The principle enunciated in § 1 is of real importance in connection with results depending upon fractional parts of the angles of the triangle. Among such results are (1) inscribed and escribed circles, (2) Malfatti's problem, (3) the extended theorem of Feuerbach, (4) Morley's theorem on the trisectors of the angles of a triangle, (5) results concerning epi- and hypo-cycloids which touch the sides of a given triangle. These serve to illustrate the principle.

Applications of the principle.

5. *The simplest uses of the principle.* Applied to the commonest trigonometrical relations among the sides, angles, etc., of a triangle, the principle tells us little: the equations are merely reproduced, possibly with signs changed throughout. The same is true of the areal coordinates of the simplest points, such as the orthocentre, or of the equations (ordinary or tangential) in areal coordinates of the simplest loci connected with the triangle, such as the circumcircle.

Lemoine recognized the fundamental fact that his "transformation continue" enabled him to derive the confusing formulae of the escribed circles from the well known and simpler formulae of the inscribed circle. He regarded this as an end; but it is the simplest of the applications of the principle. It is convenient to describe the two types of substitution (i) and (ii) of § 1 as the *plus* type and the *minus* type. We will consider the simplest examples of each type, viz., those in which

$$(i) \quad (\alpha_1, \beta_1, \gamma_1) = (0, +1, -1); \quad (ii) \quad (\alpha_2, \beta_2, \gamma_2) = (0, 1, 1);$$

the latter being Lemoine's. These changes in the angles carry with them the following consequential changes in other dimensions of the triangle (see 3):

$$\begin{array}{l} (i) \quad \left| \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c} A & B & C & \Delta & a & b & c & R & r & r_1 & r_2 & r_3 \\ \hline A & B+\pi & C-\pi & \Delta & a & -b & -c & R & -r_1 & -r & r_3 & r_2 \\ \hline -A & \pi-B & \pi-C & -\Delta & a & -b & -c & -R & r_1 & r & -r_3 & -r_2 \end{array} \right|. \\ (ii) \end{array}$$

A relation satisfied by the quantities in the first row remains true when the quantities in the second or third row replace them; it is easy to test this in any number of special cases. As a consequence of the law of homogeneity it has been pointed out that the signs of the last eight quantities of a row may be changed *en bloc*. Substitution (i) will be found to be as effective as Lemoine's (ii); and both are applicable to equations in areal coordinates as well as to the trigonometrical formulae for which Lemoine employed them*.

It will be found that the other substitutions of § 1 give nothing more than these simplest ones; the principle leads us from the inscribed to the three escribed circles, but not further. The reason for this is that the results relating to the inscribed circle depend upon the sines, cosines, etc., of the complete angles of the triangle, and upon the tangents of the half-angles, *i.e.* functions of the angles with period 2π . To alter the integers $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$ by an even number has no effect upon such formulae. Therefore, in dealing with the inscribed and escribed circles, it is legitimate and convenient to restrict the values of α, β, γ to 0 and 1 in the substitutions of both types, and to replace the conditions regarding their sums by

$$\alpha_1 + \beta_1 + \gamma_1 \equiv 0 \pmod{2}, \quad \alpha_2 + \beta_2 + \gamma_2 \equiv 0 \pmod{2}.$$

Since, in this set of four circles, the different members are obtained by addition of π to the angles, and since it is possible to distinguish angles which differ by π by defining a positive and negative direction of the bounding line, these four circles are sometimes distinguished by regarding lines and circumferences as "oriented" or "directed". It is not easy to see how this method can be used to distinguish angles which differ by 2π or 4π , as will be necessary in other applications of the principle. The method is attractive because it is so purely geometrical, but its use appears to be limited to this case. When it is a question of distinguishing the members of a family of 16 or 18 or 32, the algebraic method of increasing the angles by 2π or 3π or 4π seems to be the only one possible.

Malfatti's problem has been discussed at length in our former paper.

* J. S. Mackay, *Proc. Edinburgh Math. Soc.*, 12 (1894), 87, gives a long list of results obtained by Lemoine's transformation. An earlier list mentioned by Mackay would have historical interest; but we have failed to trace the reference, *Lady's and Gentleman's Diary for 1871*, 93. E. Lucas, 5, had the idea of variation of the sides and angles of a triangle before Lemoine, but fell into errors. A second paper, 6, is mainly a list of formulae, but makes a mention of spherical trigonometry. E. Lemoine published his theorem in 3 and the other places named in *Encyk. der Math. Wiss.*, III, 775, Note 4.

But we take this opportunity of pointing out that our form of solution of the original problem, equations (vi), p. 292, had been anticipated, *Math. Gazette*, 14 (1929), 143, by L. J. Rogers.

6. *The extended theorem of Feuerbach.* The inscribed and escribed centres, I, I_1, I_2, I_3 , of a triangle ABC give rise to four triangles, $I_1I_2I_3$, II_2I_3 , II_3I_1 , II_1I_2 , and the circumcircle of ABC is the nine-points circle of each triangle: the circle ABC is therefore touched by the circles inscribed or escribed to each triangle. The sixteen circles of this family illustrate the principle in a way that displays many of the same features as Malfatti's problem, but is rather simpler, for it is found that the two types of substitutions of § 1 lead to the same solutions. Hence the number sixteen instead of thirty-two.

Consider the circle inscribed in the triangle $I_1I_2I_3$. If ρ is its radius and α, β, γ are the perpendicular distances of its centre from the sides of ABC , we find that

$$\alpha = \rho (\sin \tfrac{1}{2}B + \sin \tfrac{1}{2}C - \sin \tfrac{1}{2}A)$$

with similar equations for β and γ . Also

$$\rho (\cos \tfrac{1}{2}A + \cos \tfrac{1}{2}B + \cos \tfrac{1}{2}C) = \tfrac{1}{2}(a+b+c).$$

If we denote $\tan \tfrac{1}{4}A, \tan \tfrac{1}{4}B, \tan \tfrac{1}{4}C$ by u, v, w , we can prove that the areal coordinates of the point where this circle touches the circumcircle of ABC are given by

$$x:y:z = a \frac{v+w}{v-w} : b \frac{w+u}{w-u} : c \frac{u+v}{u-v},$$

and that the equation of their common tangent is

$$\left(\frac{v-w}{v+w}\right)^2 x + \left(\frac{w-u}{w+u}\right)^2 y + \left(\frac{u-v}{u+v}\right)^2 z = 0.$$

These are clumsy results; but it is clear that the trigonometrical functions involved (sines and cosines of the halves, and tangents of the quarters, of A, B, C) all have the period 4π . We may therefore adopt the same procedure as in § 5, with the appropriate modification. We restrict the values of $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$ to 0, 1, 2, and 3, and we replace the conditions concerning their sums by

$$\alpha_1 + \beta_1 + \gamma_1 \equiv 0 \pmod{4}; \quad \alpha_2 + \beta_2 + \gamma_2 \equiv 2 \pmod{4}.$$

But when we consider the substitution of the minus type for which $\alpha_2 = \beta_2 = \gamma_2 = 2$, we find that it gives us nothing beyond the solution found above; and in the same way any two substitutions of the plus and

minus types in which

$$\alpha_1 + \alpha_2 \equiv \beta_1 + \beta_2 \equiv \gamma_1 + \gamma_2 \equiv 2 \pmod{4}$$

lead to results which, in spite of superficial differences, are identical. Two such substitutions of different types were paired together in our paper on Malfatti's problem, but did not there lead to the same result. Here the substitutions of the minus type may be disregarded, and all the variants of the theorem deduced from the sixteen substitutions of the plus type, *i.e.* substitutions in which we write

$$(A + \alpha\pi, B + \beta\pi, C + \gamma\pi) \text{ for } (A, B, C) \text{ where } \alpha + \beta + \gamma \equiv 0 \pmod{4}.$$

The values of α, β, γ in the sixteen substitutions of the plus type are exhibited in the attached Table. With each is shown the circle derived by that substitution from the circle inscribed in the triangle $I_1 I_2 I_3$.

α	β	γ	Triangle	Circle obtained
0	0	0	$I_1 I_2 I_3$	Inscribed circle
0	2	2	"	Escribed circle opposite to I_1
2	0	2	"	" " I_2
2	2	0	"	" " I_3
2	1	1	$II_2 I_3$	Inscribed circle
2	3	3	"	Escribed circle opposite to I
0	3	1	"	" " I_2
0	1	3	"	" " I_3
1	2	1	$II_3 I_1$	Inscribed circle
3	2	3	"	Escribed circle opposite to I
1	0	3	"	" " I_3
3	0	1	"	" " I_1
1	1	2	$II_1 I_3$	Inscribed circle
3	3	2	"	Escribed circle opposite to I
3	1	0	"	" " I_1
1	3	0	"	" " I_2

7. Morley's theorem upon the trisectors of the angles of a triangle.

A very remarkable series of properties of the lines which trisect the angles of a triangle has been discovered by F. Morley*. The principle throws

* 11. This is a fuller statement of the original paper, 9; see also 10. There are three papers by Taylor and Marr, 15, 14, 8, to which reference is made later. Several proofs, among them a very simple geometrical one by J. M. Child, 1, appeared in the *Mathematical Gazette* in 1922.

light upon them, although they were obtained originally in quite a different way.

Let the angles A, B, C of a triangle ABC be trisected internally by the lines AY, AZ, BZ, BX, CX, CY , so that X is the point of intersection of those trisectors of B and C which are next to BC ; and similarly for Y and Z .

The triangle XYZ is an equilateral triangle.

With the usual notation, we have

$$CAY = YAZ = ZAB = \frac{1}{3}A; \quad BXC = \frac{1}{3}(A + 2\pi);$$

$$ABZ = ZBX = XBC = \frac{1}{3}B; \quad CYA = \frac{1}{3}(B + 2\pi);$$

$$BCX = XCY = YCA = \frac{1}{3}C; \quad AZB = \frac{1}{3}(C + 2\pi).$$

From this we deduce that

$$AY = b \sin \frac{1}{3}C \operatorname{cosec} \frac{1}{3}(B + 2\pi); \quad AZ = c \sin \frac{1}{3}B \operatorname{cosec} \frac{1}{3}(C + 2\pi);$$

$$AY : AZ = \sin \frac{1}{3}(B + \pi) : \sin \frac{1}{3}(C + \pi);$$

the sum of the angles $\frac{1}{3}(B + \pi)$ and $\frac{1}{3}(C + \pi)$ is the same as the sum of AZY and AYZ , and their sines are also proportional; hence the angles are equal. Thus

$$CYX = XZB = \frac{1}{3}(A + \pi); \quad AZY = YXC = \frac{1}{3}(B + \pi);$$

$$BXZ = ZYA = \frac{1}{3}(C + \pi).$$

Finally we have the result

$$YZ = ZX = XY = 8R \sin \frac{1}{3}A \sin \frac{1}{3}B \sin \frac{1}{3}C,$$

showing that the triangle XYZ is equilateral; also we notice that all the angles in the figure are expressed in terms of $\frac{1}{3}A, \frac{1}{3}B, \frac{1}{3}C$, and $\frac{1}{3}\pi$.

The trilinear coordinates of X, Y, Z are respectively

$$(1 : 2 \cos \frac{1}{3}C : 2 \cos \frac{1}{3}B), \quad (2 \cos \frac{1}{3}C : 1 : 2 \cos \frac{1}{3}A), \quad (2 \cos \frac{1}{3}B : 2 \cos \frac{1}{3}A : 1);$$

and the principle enables us to derive from these a number of other equilateral triangles, by substituting for A, B, C the angles given in the first section of this paper. Any such substitution will give a set of three points which lie at the corners of an equilateral triangle; and the number of derived triangles will be finite, because all the trigonometrical functions which we have used have period 6π . But to determine the exact number of triangles requires a more careful investigation. It was remarked that the signs of trilinear coordinates must be changed in conformity with those of $\sin A, \sin B, \sin C$, while areal coordinates need no

such change. Now the areal coordinates of X are in the ratio

$$1 : 2 \cos \frac{1}{3}C \sin B \operatorname{cosec} A : 2 \cos \frac{1}{3}B \sin C \operatorname{cosec} A,$$

and are not affected when one of the angles A, B, C is increased and another diminished by 3π ; the same is true for Y and Z . In consequence of this, we see that the values of $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$ may be supposed even; for since the sum of the three is even they are either all even or else one even and two odd; and, if the latter, an addition and a subtraction of three turns the odd numbers into even ones. Since the trigonometrical functions have period 6π and the numbers α, β, γ are even, we may restrict their values to 0, 2, 4, provided that at the same time we modify the conditions as to their sums into

$$\alpha_1 + \beta_1 + \gamma_1 \equiv 0 \pmod{6}, \quad \alpha_2 + \beta_2 + \gamma_2 \equiv 2 \pmod{6}.$$

We notice also that when α, β, γ are even, the difficulty about the signs of the sides, or of the trilinear coordinates, is avoided. Trilinear coordinates will therefore be used in preference to areal.

There are nine possible sets of values of $\alpha_1, \beta_1, \gamma_1$, viz.

$$(0, 0, 0), (2, 2, 2), (4, 4, 4), (0, 2, 4), (2, 4, 0), (4, 0, 2), \\ (0, 4, 2), (4, 2, 0), (2, 0, 4),$$

and nine possible sets of values of $\alpha_2, \beta_2, \gamma_2$, viz.

$$(0, 0, 2), (0, 2, 0), (2, 0, 0), (0, 4, 4), (4, 0, 4), (4, 4, 0), \\ (2, 2, 4), (2, 4, 2), (4, 2, 2);$$

therefore XYZ is one out of a set of eighteen equilateral triangles. In any one of these triangles one vertex X' lies either on the trisector BX or on one of the lines through B inclined to BX at an angle $\frac{1}{3}\pi$; X' lies also either on the trisector CX or on one of the lines through C inclined to CX at an angle $\frac{1}{3}\pi$; so that, if we disregard multiples of the period π ,

$$3 \times CBX' = CBA; \quad 3 \times BCX' = BCA.$$

Although the extended figure contains eighteen equilateral triangles, there are only nine possible positions of the vertex X' , viz. the nine intersections of three lines through B with three lines through C ; each point X' is therefore a vertex of two of the eighteen equilateral triangles. This is proved also when we derive the coordinates of X' from those of X by means of the two sets of nine substitutions; with any one of the first nine we can associate one of the second nine in which $\beta_2 = \beta_1$ and $\gamma_2 = \gamma_1$, and these give the same coordinates for X' . Similar results hold for the other vertices.

The equations of the sides of the eighteen equilateral triangles. It is plain that the angle between ZY and BC is $\frac{1}{3}(B-C)$, and that the various substitutions which derive from ZY the corresponding sides of other equilateral triangles either leave this expression unaltered, or change its sign, or add or subtract $\frac{2}{3}\pi$. When we find the equations of the sides of XYZ and apply the substitutions, we find that the sides of the eighteen triangles are nine straight lines and no more. The equation of YZ is found to be

$$a \sin \frac{1}{3}(A+\pi) + \beta \sin \frac{1}{3}(B-\pi) + \gamma \sin \frac{1}{3}(C-\pi) = 0,$$

and we observe that the sum of the three angles in brackets vanishes. When we replace A, B, C by $A+\alpha_1\pi$, etc., or by $\alpha_2\pi-A$, etc., we still have three angles whose sum is zero; moreover, the argument for restricting the values of $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$, first to even values and then to the values 0, 2, 4, is still sound. The principle derives from the equation of YZ the lines

$$a \sin \frac{1}{3}(A+\rho\pi) + \beta \sin \frac{1}{3}(B+\sigma\pi) + \gamma \sin \frac{1}{3}(C+\tau\pi) = 0,$$

where ρ, σ, τ are either odd numbers so chosen that the sum of the three angles in brackets vanishes, or have the values 1, 3, 5, so chosen that the sum of the angles vanishes to modulus 6π . It is clear that the second type of substitution gives the same equations with the sign of the left-hand member changed throughout, which represent the same lines as before; also it is clear that the principle generalizes ZX and XY into the same system of nine lines.

At this point we must refer to the two separate papers **8** and **14**, and one joint paper, **15**, by W. L. Marr and F. G. Taylor. Marr has in reality illustrated the principle which we have here expounded; his method needs to be justified, and sundry details and difficulties elucidated, by help of the principle.

Having obtained equations in trilinear coordinates respecting the equilateral triangle determined by the internal trisectors of the angles, Marr considered the effect of replacing A, B, C in these equations by $A+2r\pi, B+2s\pi, C+2t\pi$: r, s, t being unrestricted integers. Now to limit the increments of A, B, C to *even* multiples of π would be fatal both in Lemoine's work upon the inscribed and escribed circles, and in the applications of the principle which we have previously made; but in this theorem we have seen that the increments may be so restricted, and also that trilinear coordinates may be used, without confusion or loss of generality. Marr next investigates the effect of thus modifying the

equations of the sides of the equilateral triangle, and finds that a new equilateral triangle is derived if $r+s+t$ divided by 3 has remainder 0 or 2, but not if the remainder is 1. This strange result is explained by the principle: the remainder 0 is obtained from a substitution of the plus type, and the remainder 2 from one of the minus type, it being observed that the equations of the sides depend only upon cosines of angles, and are unaffected when the signs of these angles are reversed. The work of Marr and Taylor, and the interesting further results which they obtain, may therefore be associated with the general principle of this paper.

8. *The epicycloids and hypocycloids which touch the sides of a given triangle.* A short note upon the application of the principle must suffice. The (p, ψ) equation of such a curve is

$$p = k \cos(n\psi + \kappa)$$

k, κ, n being constants, of which n depends only upon the species of the curve, *i.e.* the ratio of the radii of the generating circles. The pole from which the perpendicular (p) is let fall upon a tangent is the centre of the "fixed" circle; and, if an epicycloid or hypocycloid of known species touches the sides of a given triangle, the locus of its pole is a family of straight lines, all of which can be derived by the principle from the equation in areal coordinates of one of their number. [If n is an odd integer the locus is a single straight line.] In areal coordinates the equations of the lines are given by

$$\frac{x}{\sin A} \sin n(A + \alpha_0 \pi) + \frac{y}{\sin B} \sin n(B + \beta_0 \pi) + \frac{z}{\sin C} \sin n(C + \gamma_0 \pi) = 0,$$

where $\alpha_0, \beta_0, \gamma_0$ receive integer values such that the sum of the three angles in the brackets is zero, *i.e.*

$$\alpha_0 + \beta_0 + \gamma_0 + 1 = 0.$$

Substitutions of the minus type give no results other than those obtainable by substitutions of the plus type.

When $n = \frac{1}{3}$, the curve is an epicycloid of one cusp, the generating circles being equal, *i.e.* is a cardioid, and the locus is composed of the nine lines which form the sides of the equilateral triangles in § 7.

9. *Conclusion.* In plane trigonometry, as presented not only in text-books but in treatises such as that of Hobson, there is a curious incon-

sistency. Except in one part, the subject from beginning to end is developed on the understanding that an angle is capable of assuming any magnitude. The exception is in the chapters which deal with the geometry of triangles. Here a thoroughly reactionary spirit prevails. Not only are all angles supposed to be less than two right angles, but the triangle itself is regarded as Euclid regarded a figure, as something fixed and immutable. The fundamental idea of trigonometrical periodicity is scarcely used; and the commonplace geometrical notion of the figure undergoing a continuous variation, its points describing loci, its angles increasing or diminishing, and possibly passing beyond the Euclidean limits, is put aside and forgotten. Yet, when a consistent system of trigonometrical formulae has been constructed with no little care and elaboration, it is unreasonable to deny them significance except for the Euclidean triangle with its restricted angles. That is the basis and justification for the principle which we have discussed.

References.

1. J. M. Child, "Proof of Morley's theorem", *Math. Gazette*, 11 (1923), 171.
2. F. Klein, *Über die hypergeometrische Funktion* (Göttingen, 1894).
3. E. Lemoine, "Sur la transformation continue", *Bull. de la Soc. Math. de France*, 19 (1891), 136-141.
4. H. Lob and H. W. Richmond, "On Malfatti's problem", *Proc. London Math. Soc.* (2), 30 (1930), 287-304.
5. E. Lucas, "Sur l'emploi dans la géométrie d'un nouveau principe des signes", *Nouvelle Corr. Math.*, 2 (1876), 384-391.
6. ———— *Ibid.*, 3 (1877), 1-5.
7. J. S. Mackay, "Formulae connected with the radii of the incircle and the excircles of a triangle", *Proc. Edinburgh Math. Soc.*, 12 (1894), 86-105.
8. W. L. Marr, "Morley's trisection theorem: an extension and its relation to the circles of Apollonius", *Proc. Edinburgh Math. Soc.*, 32 (1914), 136-150.
9. F. Morley, "On reflexive geometry", *Trans. Amer. Math. Soc.*, 8 (1907), 14-24.
10. ———— "On the intersections of the trisectors of the angles of a triangle", *Journal of the Math. Assoc. of Japan for Secondary Education*, 6 (1924), 260-262.
11. ———— "Extensions of Clifford's chain theorem", *Amer. J. of Math.*, 51 (1929), 465-472.
12. J. Steiner, "Einige geometrische Betrachtungen", *Journal für Math.*, 1 (1826), 161-184.
13. E. Study, "Sphärische Trigonometrie, orthogonale Substitutionen und elliptische Funktionen", *Leipziger Abh.*, 20 (1893), 83-232. (Also published in book form, Leipzig, 1893.)
14. F. G. Taylor, "The relation of Morley's theorem to the Hessian axis and circumcentre", *Proc. Edinburgh Math. Soc.*, 32 (1914), 132-135.
15. F. G. Taylor and W. L. Marr, "The six trisectors of each of the angles of a triangle", *Proc. Edinburgh Math. Soc.*, 32 (1914), 119-131.
16. H. Weber and J. Wellstein, *Encyclopädie der Elementar-Mathematik*, 2 (1915).
17. G. C. Young (G. Chisholm), *Algebraisch-gruppentheoretische Untersuchungen zur sphärischen Trigonometrie* (Dissertation, Göttingen, 1895).
18. ———— "Sulla varietà razionale normale M_3^4 di S_6 rappresentante della trigonometria sferica", *Atti di Torino*, 34 (1899), 587-596.

ON SUMMABLE TRIGONOMETRIC SERIES

By S. VERBLUNSKY.

[Received 21 September, 1929.—Read 14 November, 1929.]

1. Suppose that we are given a trigonometric series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1)$$

which satisfies the condition (F), namely that

$$\sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2} \quad (2)$$

is the Fourier series of a continuous function. A number of theorems have been proved by Zygmund† of the following general character. If the upper and lower Poisson sums of (1) satisfy certain conditions, then (1) is a Fourier series. The problem now is to find the least restrictive condition which the coefficients of (1) must satisfy in order that, subject to the specified behaviour of the Poisson sums, condition (F) may be satisfied. It is a remarkable fact that in most of the cases considered by Zygmund the condition

$$a_n = o(n), \quad b_n = o(n) \quad (3)$$

is sufficient. The proof of this forms the main object of the present paper. As regards the method employed, attention may be directed to the use of the idea of relatively uniform convergence (§ 3). In § 10 we give an application of one of the results to the proof of a Tauberian theorem.

2. In this section we enunciate as lemmas two known results, which will be required in the sequel.

† Zygmund, *Math. Zeitschrift*, 25 (1926), 274–290.

LEMMA 1 (Rajchman and Zygmund†). Let $\bar{P}(x)$, $\underline{P}(x)$ denote the upper and lower Poisson sums of

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \quad (4)$$

Let
$$-\sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2}$$

be the Fourier series of a function $F(x)$ integrable in the sense of Lebesgue in $(0, 2\pi)$. Then, if $F(x)$ is continuous at $x = x_0$, the intervals

$$[\bar{D}^2 F(x_0), \underline{D}^2 F(x_0)], [\bar{P}(x_0), \underline{P}(x_0)]$$

have at least one point in common; where $\bar{D}^2 F(x)$, $\underline{D}^2 F(x)$ denote respectively the upper and lower generalized second derivatives of $F(x)$.

LEMMA 2 (de la Vallée Poussin‡). If $F(x)$ is continuous in (a, b) , and

$$\bar{D}^2 F(x) \geq f(x) \geq \underline{D}^2 F(x) \quad (a < x < b)$$

where $f(x)$ is finite and integrable (in the sense of Denjoy-Perron) in (a, b) , then

$$F(x) = \int_a^x dy \int_a^y f(t) dt + Ax + B \quad (a < x < b) \quad (5)$$

where A and B are constants. The equation (5) holds if $f(x)$ is infinite at an enumerable set of points, provided that at each such point ξ

$$\lim_{h \rightarrow 0} \frac{F(\xi + h) + F(\xi - h) - 2F(\xi)}{h} = 0. \quad (6)$$

3. A sequence $\{f_n(x)\}$ of finite single-valued functions defined in an interval (a, b) is said to converge relatively uniformly to $f(x)$ if there exists a finite function $\sigma(x)$ such that, given $\epsilon > 0$, there is an n_ϵ satisfying

$$|f(x) - f_n(x)| < \epsilon \sigma(x) \quad (n \geq n_\epsilon)$$

for all x in (a, b) .

The importance of this mode of convergence for our purposes arises from the following theorem due to E. W. Chittenden§.

† Rajchman and Zygmund, *Math. Zeitschrift*, 25 (1926), 261-273 (268).

‡ Cf. Hobson, *Theory of functions of a real variable*, 2 (1926), 671.

§ E. W. Chittenden, *Trans. Amer. Math. Soc.*, 20 (1919), 179-184.

If a sequence $\{f_n(x)\}$ of continuous functions converges relatively uniformly in (a, b) to $f(x)$, then if P be any perfect set contained in (a, b) , the points of discontinuity of $f(x)$ with respect to P are non-dense on P .

We shall say that $f(x)$ possesses the property R if it is the limit of a relatively uniformly convergent sequence of continuous functions.

The idea can, of course, be extended to the case in which we do not have a sequence of functions, but a family of functions which depend on a continuous parameter. For example, we may have a family $\{f_a(x)\}$ of finite single-valued functions defined in (a, b) , which we consider for values of a which tend to a certain value of a_0 . If, given $\epsilon > 0$, there is an $\eta = \eta(\epsilon)$ such that

$$|f(x) - f_a(x)| < \epsilon \sigma(x) \quad (|a - a_0| \leq \eta)$$

for all x in (a, b) , where $\sigma(x)$ is a finite function, then the family $\{f_a(x)\}$ will be said to converge relatively uniformly to $f(x)$; and the theorem of Chittenden holds in this case.

4. LEMMA 3. *If the upper and lower Poisson sums of*

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{7}$$

are everywhere finite, then the sum of the series

$$\sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n} r^n$$

converges relatively uniformly as $r \rightarrow 1$.

Writing
$$\phi_r(x) = \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) r^{n-1},$$

we have, by hypothesis,

$$\lim_{r \rightarrow 1} |\phi_r(x)| < \infty$$

for all x , and consequently

$$\text{Max}_{0 \leq r \leq 1} |\phi_r(x)| < \infty$$

for all x . Now

$$\sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n} r^n = \int_0^r \phi_r(x) dr.$$

Hence

$$\left| \lim_{r \rightarrow 1} \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n} r^n - \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n} r^n \right| \\ = \left| \int_r^1 \phi_r(x) dr \right| \leq (1-r) \max_{0 \leq r \leq 1} |\phi_r(x)|.$$

If we put $\max_{0 \leq r \leq 1} |\phi_r(x)| = \sigma(x)$,

we have the required result.

LEMMA 4. *Let*

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad [a_n = o(n), \quad b_n = o(n)] \quad (8)$$

be a trigonometric series whose upper and lower Poisson sums are everywhere finite. Then

$$- \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2} \quad (9)$$

converges everywhere to a function $F(x)$ which possesses the property R .

By applying Lemma 3 twice we infer that

$$- \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2} r^n \quad (10)$$

converges relatively uniformly as $r \rightarrow 1$. Since the coefficients in (9) are $o(1/n)$, it follows, by Tauber's theorem, that (9) converges everywhere to a function $F(x)$, to which (10) converges as $r \rightarrow 1$, and which therefore possesses the property R . We observe that (9) is the Fourier series of $F(x)$, which is integrable L .

5. THEOREM I. *Let the upper and lower Poisson sums of*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad [a_n = o(n), \quad b_n = o(n)] \quad (11)$$

be finite and integrable D (i.e. in the sense of Denjoy-Perron). Then (11) is a Fourier series.

We may suppose without loss of generality that $a_0 = 0$. Write

$$F(x) = - \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2}, \quad (12)$$

where, by Lemma 4, $F(x)$ is integrable L and possesses the property R ;

i.e. the points of discontinuity of $F(x)$ with respect to any perfect set P are non-dense on P .

In any interval (α, β) in which $F(x)$ is continuous, we write

$$f(x) = \text{Min} \{ \bar{D}^2 F(x), \bar{P}(x) \}.$$

Then, by Lemma 1, we have in (α, β)

$$\bar{D}^2 F(x) \geq f(x) \geq \underline{D}^2 F(x); \quad \bar{P}(x) \geq f(x) \geq \underline{P}(x). \quad (13)$$

Let $f(x)$ be defined in this manner in all intervals such as (α, β) which are contained in a fixed interval Δ arbitrarily assigned. At each point x of $\Delta - \Sigma(\alpha, \beta)$ we put

$$f(x) = \bar{P}(x). \quad (14)$$

It follows that the second relation in (13) is satisfied at all points of Δ , and hence that $f(x)$ is integrable D in Δ .

We now show that $F(x)$ cannot have an isolated discontinuity in Δ . For suppose that ξ is such a point of discontinuity. Then there is an $\epsilon > 0$ such that ξ is the only point of discontinuity of $F(x)$ in $(\xi - \epsilon, \xi + \epsilon)$. By Lemma 2 and the first relation in (13) we have

$$F(x) = \int_{\xi - \epsilon}^x dy \int_{\xi - \epsilon}^y f(t) dt + Ax + B \quad (\xi - \epsilon < x < \xi).$$

We have similarly

$$F(x) = \int_x^{\xi + \epsilon} dy \int_y^{\xi + \epsilon} f(t) dt + A'x + B' \quad (\xi < x < \xi + \epsilon).$$

Hence $F(x)$ is of bounded variation in $(\xi - \epsilon, \xi + \epsilon)$, so that ξ is a point of ordinary discontinuity. But, since the series (12) is everywhere convergent and has coefficients which are $o(1/n)$, it follows, by a classical theorem of Fatou†, that $F(x)$ is a differential coefficient, and so it cannot have an ordinary discontinuity.

We have thus proved that the points of discontinuity of $F(x)$ on Δ form a set E which is dense in itself. Consequently, E is everywhere dense in a perfect set Π ; and in virtue of the fact that $F(x)$ possesses the property R , Π is non-dense in Δ .

We denote the contiguous intervals of Π by (a_n, b_n) . Then for $a_n < x < b_n$, we have

$$F(x) = \int_{a_n}^x dy \int_{a_n}^y f(t) dt + A_n x + B_n, \quad (15)$$

where A_n and B_n are constant in (a_n, b_n) .

† Fatou, *Acta. Math.*, 30 (1906), 335-400 (345, 385).

The end points of Δ being denoted by a, b , consider the function

$$F_1(x) = F(x) - \int_a^x dy \int_a^y f(t) dt,$$

which also possesses the property R . By (15) it is linear in every interval contiguous to Π . Moreover, since it differs from $F(x)$ by a continuous function, its points of discontinuity on Δ form the set E previously mentioned, which is everywhere dense in Π .

By definition of the property R , we can find an interval δ containing points of Π such that $F_1(x)$ is continuous in $\Pi\delta$. Let p be a point of $\Pi\delta$. It is the limit point of end points of intervals contiguous to $\Pi\delta$. We know that $F_1(x)$ is linear within each interval contiguous to $\Pi\delta$; further, $F_1(x)$ is an everywhere existent differential coefficient, so that it is linear in each contiguous interval taken closed. Consequently, since the maximum and minimum of $F_1(x)$ in each contiguous interval are attained at the end points, *i.e.* at points of $\Pi\delta$, and since $F_1(x)$ is continuous at p on $\Pi\delta$, it is continuous at p on Δ . Thus $F_1(x)$ is continuous on δ . This contradicts the fact that the points of discontinuity of $F_1(x)$ on Δ are everywhere dense in Π . Hence Π is a null set and $F_1(x)$ is continuous. So therefore is $F(x)$.

We now have, for every positive x ,

$$F(x) = \int_0^x dy \int_0^y f(t) dt + Ax + B;$$

$$i.e. \quad - \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2} = \int_0^x dy \int_0^y f(t) dt + Ax + B.$$

$$\text{Hence} \quad \sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n} \sim \int_0^x f(t) dt + A.$$

In order that the second number may represent a periodic function of period 2π , we must have

$$\int_0^{2\pi} f(t) dt = 0.$$

Further,

$$\begin{aligned} \frac{a_n}{n} &= \frac{1}{\pi} \int_0^{2\pi} \sin nx \left(\int_0^x f(t) dt \right) dx. \\ -\frac{b_n}{n} &= \frac{1}{\pi} \int_0^{2\pi} \cos nx \left(\int_0^x f(t) dt \right) dx. \end{aligned}$$

Since we only know that $f(t)$ is integrable D , the evaluation of the preceding expressions requires some attention. Consider the first.

Writing

$$\phi(x) = \int_0^x f(t) dt,$$

we have, on integrating by parts (which is here permissible by the known properties of Denjoy integrals),

$$\begin{aligned} \frac{a_n}{n} &= \frac{1}{\pi} \int_0^{2\pi} \phi(x) \sin nx \, dx \\ &= \left[-\phi(x) \frac{\cos nx}{n\pi} \right]_0^{2\pi} + \frac{1}{\pi} \int_0^{2\pi} f(x) \frac{\cos nx}{n} \, dx \\ &= \frac{1}{n\pi} \int_0^{2\pi} f(x) \sin nx \, dx. \end{aligned}$$

Similarly,
$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx.$$

This proves the theorem.

6. We state as a lemma the following result which is implied in the proof of Theorem I.

LEMMA 5. *In any interval in which*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad [a_n = o(n), b_n = o(n)] \quad (16)$$

has its upper and lower Poisson sums finite and integrable D ,

$$F(x) = \frac{1}{4}a_0x^2 - \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2} \quad (17)$$

is continuous.

THEOREM II†. *If the trigonometric series (16) is summable (P) to zero almost everywhere, and if its upper and lower Poisson sums are finite except at a single point $x = x_0 \pmod{2\pi}$, then (16) is of the form*

$$\frac{a_0}{2} + a_0 \cos(x - x_0) + a_0 \cos 2(x - x_0) + \dots$$

We may suppose that $x_0 = 0$. By Lemma 5, $F(x)$ is continuous in any interval which does not contain $x = 0 \pmod{2\pi}$. By Lemma 1 we have,

† Cf. Zygmund, *loc. cit.*, 280.

in any interval $0 < c < d < 2\pi$,

$$\bar{D}^2 F(x) \geq f(x) \geq \underline{D}^2 F(x),$$

where $f(x)$ is zero almost everywhere, and finite everywhere in (c, d) . Hence, by Lemma 2,

$$F(x) = Ax + B \quad (0 < x < 2\pi).$$

It follows that

$$-\sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2}$$

is the Fourier series of the periodic function obtained by continuing

$$-\frac{a_0 x^2}{4} + Ax + B.$$

Consequently,

$$-\frac{a_n}{n^2} = \frac{1}{\pi} \int_0^{2\pi} \left(-\frac{a_0 x^2}{4} + Ax + B \right) \cos nx \, dx$$

$$-\frac{b_n}{n^2} = \frac{1}{\pi} \int_0^{2\pi} \left(-\frac{a_0 x^2}{4} + Ax + B \right) \sin nx \, dx,$$

i.e.
$$a_n = a_0, \quad b_n = \frac{n}{\pi} (2\pi A - a_0 \pi^2).$$

By hypothesis $b_n = o(n)$;

this requires $2\pi A - a_0 \pi^2 = 0$,

which proves the theorem.

7. THEOREM III. *If the upper and lower Poisson sums of*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad [a_n = o(n), \quad b_n = o(n)] \quad (18)$$

are zero almost everywhere, and finite except at a finite number of points $x_1, x_2, \dots, x_s$ ($0 \leq x_i < 2\pi$), then (18) is of the form

$$\sum_{i=1}^s B_i T(x - x_i),$$

where $T(x) = \frac{1}{2} + \cos x + \cos 2x + \dots,$

and B_i are constants.

We write, in $(0, 2\pi)$,

$$F(x) - \frac{1}{4}a_0x^2 \sim - \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2}. \quad (19)$$

In any interval (a, b) which does not contain any of the points x_i , $F(x)$ is continuous by Lemma 5, and so, by Lemmas 1 and 2, is linear. Hence the second member of (19) must converge at the points x_i . On the other hand, the coefficients in this series are $o(1/n)$, so that

$$\int_0^x \{F(t) - \frac{1}{4}a_0t^2\} dt$$

has a differential coefficient everywhere. It follows that $F(x)$ is continuous in $(0, 2\pi)$ and that $F(0) = F(2\pi)$. Since, further, $F'(x)$ has only a finite number of discontinuities, we have, in $(0, 2\pi)$,

$$F'(x) - \frac{1}{2}a_0x \sim \sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n},$$

where $F'(x)$ is not defined at $x = x_i$, and is constant in each of the intervals contiguous to the points x_i .

We suppose that

$$x_1 < x_2 < \dots < x_s,$$

and we write

$$\begin{aligned} F'(x) &= C_1 & (0 < x < x_1) \\ &= C_{i+1} & (x_i < x < x_{i+1}) \quad (i = 1, 2, \dots, s-1) \\ &= C_{s+1} & (x_s < x < 2\pi). \end{aligned}$$

Then

$$\begin{aligned} \frac{a_n}{n} &= \frac{1}{\pi} \int_0^{2\pi} \{F'(x) - \frac{1}{2}a_0x\} \sin nx \, dx, \\ -\frac{b_n}{n} &= \frac{1}{\pi} \int_0^{2\pi} \{F'(x) - \frac{1}{2}a_0x\} \cos nx \, dx. \end{aligned}$$

These give

$$\begin{aligned} a_n &= \sum_{i=1}^s \cos nx_i \frac{(C_{i+1} - C_i)}{\pi} + a_0 - \frac{C_{s+1} - C_1}{\pi}, \\ b_n &= \sum_{i=1}^s \sin nx_i \frac{(C_{i+1} - C_i)}{\pi}. \end{aligned}$$

Thus (18) is of the form

$$\frac{C_{s+1} - C_1}{2\pi} + a_0T(x) + \sum_{i=1}^s B'_i T(x - x_i).$$

In order that this should be summable (P) to zero almost everywhere, we must have

$$C_{s+1} - C_1 = 0.$$

If $x_1 = 0$, the series is of the required form. If $x_1 > 0$, we must have

$$a_0 = 0$$

in order that the origin should not be a point at which the Poisson sum is infinite. The series is again of the required form. This proves the theorem.

8. THEOREM IV. *If the upper and lower Poisson sums of*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad [a_n = o(n) \quad b_n = o(n)] \quad (20)$$

are finite and integrable L (i.e. in the sense of Lebesgue) in $a \leq x \leq b$, then the difference between (20) and the Fourier series of any function $f(x)$ which satisfies

$$\bar{P}(x) \geq f(x) \geq \underline{P}(x)$$

in (a, b) and is zero elsewhere in $(0, 2\pi)$, is uniformly summable $(C, 1)$ to zero in $(a+\epsilon, b-\epsilon)$; and the series conjugate to this difference is uniformly summable $(C, 1)$ in the same range.

When

$$a_n = o(n^\gamma) \quad b_n = o(n^\gamma) \quad (-1 < \gamma < 1)$$

there is uniform summability (C, γ) , as has been shown by Zygmund†.

We first observe that, by Lemma 5, $F(x)$ is continuous in (a, b) . If we write

$$\phi(x) = \text{Min} \{ \bar{D}^2 F(x), \bar{P}(x) \},$$

we have

$$F(x) = \int_a^x dy \int_a^y \phi(t) dt + Ax + B \quad (a < x < b).$$

It follows that, almost everywhere in (a, b) ,

$$D^2 F(x) = \phi(x).$$

Hence, by a known theorem of Fatou,

$$P(x) = \phi(x)$$

† Zygmund, *Math. Zeitschrift*, 24 (1926), 47-104 (93).

almost everywhere. Thus $\phi(x)$ is equivalent to $f(x)$ in (a, b) and

$$F(x) = \int_a^x dy \int_a^y f(t) dt + Ax + B \quad (a < x < b).$$

If, then,
$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + \beta_n \sin nx)$$

is the Fourier series of $f(x)$, it follows that

$$\frac{1}{4}(a_0 - a_0)x^2 - \sum_{n=1}^{\infty} \frac{(a_n - a_n) \cos nx + (b_n - \beta_n) \sin nx}{n^2}$$

converges in (a, b) to a linear function

$$A_1x + B_1.$$

Thus
$$- \sum_{n=1}^{\infty} \frac{(a_n - a_n) \cos nx + (b_n - \beta_n) \sin nx}{n^2} \quad (21)$$

is the Fourier series of a function $g(x)$ which in (a, b) has the form

$$A_1x + B_1 - \frac{1}{4}(a_0 - a_0)x^2. \quad (22)$$

If we use the equation

$$\begin{aligned} - \sum_{n=1}^{\infty} \frac{(a_n - a_n) \cos nx + (b_n - \beta_n) \sin nx}{n^2} r^n \\ = \frac{1}{\pi} \int_0^{2\pi} g(t) \frac{1 - r^2}{1 - 2r \cos(t-x) + r^2} dt \end{aligned}$$

and the expression (22), it follows that the series obtained by differentiating (21) term by term k times is uniformly summable (P) in $(a + \frac{1}{2}\epsilon, b - \frac{1}{2}\epsilon)$ where k is any positive integer.

In particular,

$$\sum_{n=1}^{\infty} (a_n - a_n) \cos nx + (b_n - \beta_n) \sin nx \quad (23)$$

is uniformly summable (P) to

$$-\frac{1}{2}(a_0 - a_0);$$

and
$$\sum_{n=1}^{\infty} -n(a_n - a_n) \sin nx + n(b_n - \beta_n) \cos nx \quad (24)$$

is uniformly summable (P) to zero. From this last fact we infer that

$$\sum_{n=1}^{\infty} -(a_n - a_n) \sin nx + (b_n - \beta_n) \cos nx \quad (25)$$

is uniformly summable (P) (not, in general, to zero).

Thus the series

$$\sum_{n=1}^{\infty} [(a_n - \alpha_n) - i(b_n - \beta_n)] z^n \quad (26)$$

tends uniformly to a limit as z tends radially to a point of the arc $(a + \frac{1}{2}\epsilon, b - \frac{1}{2}\epsilon)$ of the unit circle. It follows by a theorem of M. Riesz† that, since the coefficients in (26) are $o(n)$, the series

$$\sum_{n=1}^{\infty} [(a_n - \alpha_n) - i(b_n - \beta_n)] e^{in\theta}$$

is uniformly summable $(C, 1)$ for $a + \epsilon \leq \theta \leq b - \epsilon$. Hence the real and imaginary parts of the series have the same property. This is the required result.

9. We write

$$P(r, x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) r^n.$$

THEOREM V. *If the upper and lower Poisson sums of*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad [a_n = o(n), \quad b_n = o(n)] \quad (27)$$

are integrable D in $(0, 2\pi)$, and finite except at an enumerable set E , and if, at each point of E ,

$$\lim_{r \rightarrow 1} (1-r) |P(r, x)| = 0, \quad (28)$$

then (27) is a Fourier series.

The set E of points at which the upper and lower Poisson sums of (27) are not both finite, forms an ordinary inner limiting set. This set is enumerable; it therefore contains no component which is dense in itself. We can write

$$E \equiv E^1 = E_1 + E^2,$$

where E_1 is the set of isolated points of E^1 , and is not null. Similarly we can write, if E^2 is not null,

$$E^2 = E_2 + E^3,$$

where E_2 is the set of isolated points of E^2 , and is not null. In this manner we can define E_n for every finite n . If one of these sets is null,

† M. Riesz, *Journal für Math.*, 140 (1911), 89-99 (94).

all the sets of higher index are also null. If $E - \sum_{n < \omega} E_n$ is existent, we

write
$$E^\omega = E - \sum_{n < \omega} E_n,$$

and we can write E_ω as

$$E^\omega = E_\omega + E^{\omega+1},$$

where E_ω is the set of isolated points of E^ω , and is not null. Proceeding in this manner, we have, since E is enumerable,

$$E = \sum_{\beta < \gamma} E_\beta,$$

where γ is some ordinal of the second class.

We may suppose, without loss of generality, that, in (27),

$$a_0 = 0.$$

We denote the sum of

$$- \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2},$$

when existent, by $F(x)$.

Put
$$f(x) = \text{Min} \{ \bar{D}^2 F(x), \bar{P}(x) \} \quad (29)$$

wherever the second member has a meaning. In any interval (a, b) which contains no points of E ,

$$F(x) = - \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2} \quad (30)$$

is continuous, and

$$F(x) = \int_a^x dy \int_a^y f(t) dt + Ax + B \quad (a < x < b). \quad (31)$$

Consider a point x_0 of E_1 . There is an $\epsilon > 0$ such that $(x_0 - \epsilon, x_0 + \epsilon)$ contains no point of E other than x_0 . We then have

$$F(x) = \int_{x_0 - \epsilon}^x dy \int_{x_0 - \epsilon}^y f(t) dt + A_1 x + B_1 \quad (x_0 - \epsilon < x < x_0),$$

$$F(x) = \int_x^{x_0 + \epsilon} dy \int_y^{x_0 + \epsilon} f(t) dt + A_2 x + B_2 \quad (x_0 < x < x_0 + \epsilon).$$

Thus $F(x)$ is of bounded variation in $(x_0 - \epsilon, x_0 + \epsilon)$ if we neglect its value at x_0 . It follows that the second member of (30) converges at all points of this interval, and incidentally defines $F(x_0)$. Since its coefficients are $o(1/n)$, it is a differential coefficient. It follows that

$F(x)$ is continuous at x_0 . Further, (28) implies†

$$\lim_{h \rightarrow 0} \frac{F(x_0+h) + F(x_0-h) - 2F(x_0)}{h} = 0. \quad (32)$$

Thus $F(x)$ is continuous and (32) is satisfied at all points of E_1 . Hence in any interval (a, b) which contains no points of $E - E_1$, the equation (31) holds.

We can now complete the proof by transfinite induction. Let δ be an ordinal number, and suppose that it has been shown that $F(x)$ is continuous at all points of E_β for all $\beta < \delta$, and has the form (31) in any interval (a, b) which contains no point of

$$E^\delta = E - \sum_{\beta < \delta} E_\beta.$$

We shall show that $F(x)$ is continuous at all points of E_δ , and has the form (31) in any interval (a, b) which contains no point of $E^{\delta+1}$.

In any interval (a, b) which contains no point of E^δ , $F(x)$ is continuous and (31) holds. Let ξ be a point of E_δ . Then it is an isolated point of E^δ , and hence there is an $\epsilon > 0$ such that $(\xi - \epsilon, \xi + \epsilon)$ contains no point of E^δ except ξ . The argument then proceeds as for $\delta = 1$.

Having now proved that (31) holds in every interval, the result follows as in Theorem I.

10. The Tauberian theorem referred to in § 1 is as follows:—

THEOREM VI. *Let*

$$(i) \quad na_n > -C;$$

$$(ii) \quad a_n = o(n^2);$$

$$(iii) \quad \lim_{r \rightarrow 1} \sum_{n=1}^{\infty} a_n \frac{\sin n\theta}{n\theta} r^n = \phi(\theta) \quad (0 < |\theta| < a);$$

$$(iv) \quad \phi(\theta) = o(1) \quad \text{as } \theta \rightarrow 0.$$

Then

$$\sum_{n=1}^{\infty} a_n = 0.$$

This theorem was proved, but not published, by Hardy and Littlewood on the assumption

$$(ii') \quad a_n = o(n^{2-\epsilon}) \quad (\epsilon > 0).$$

† Cf. Zygmund, *Math. Zeitschrift*, 25 (1926), 274–290 (285).

The condition (ii) cannot be replaced by

$$a_n = O(n^2)$$

since

$$\lim_{\theta \rightarrow 1} \sum_{n=1}^{\infty} n^2 \frac{\sin n\theta}{n\theta} r^n = 0 \quad (|\theta| > 0).$$

Suppose that we can show that

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n} < \infty. \quad (33)$$

It will follow, by (iii) and (iv), that

$$\sum_{n=1}^{\infty} a_n \frac{\sin n\theta}{n\theta} \rightarrow 0$$

as $\theta \rightarrow 0$. By (33),

$$\sum_{n=1}^{\infty} \frac{a_n}{n} \sin nx \quad (34)$$

is the Fourier series of a continuous function, $F(x)$ say. Hence

$$\sum_{n=1}^{\infty} a_n \cos nx \frac{\sin n\theta}{n\theta} = \frac{F(x+\theta) - F(x-\theta)}{2\theta}.$$

At $x=0$ we have

$$\frac{F(\theta) - F(-\theta)}{2\theta} \rightarrow 0$$

as $\theta \rightarrow 0$. It follows, by a theorem due to Grosz†, that the series obtained by differentiating (34) is summable (P) at $x=0$ to zero; *i.e.*

$$\sum_{n=1}^{\infty} a_n = 0 \quad (P).$$

Since, however,

$$na_n > -C,$$

the required result follows in virtue of a well known theorem of Hardy and Littlewood.

It remains then to prove (33). The method which we shall use is due to Littlewood, who proved (33) on the assumption (ii'), using the theorem of Zygmund referred to in the proof of Theorem IV. By using Theorem IV, however, we are able to prove (33) on the assumption (ii). We require two lemmas.

† Cf. Rajchman and Zygmund, *loc. cit.*, 265.

LEMMA *a*. Suppose that

$$b_n = o(n),$$

$$\sum_{n=1}^{\infty} b_n \sin n\theta = \psi(\theta) \quad (P) \quad (0 \leq |\theta| \leq a),$$

and that $\psi(\theta)$ is finite and integrable. Let

$$\begin{aligned} \psi^*(\theta) &= \psi(\theta) \quad (0 \leq |\theta| \leq a), \\ &= 0 \quad (a < |\theta| < 2\pi), \end{aligned}$$

and let

$$\sum_{n=1}^{\infty} b_n^* \sin n\theta$$

be the Fourier series of ψ^* . Then

$$\sum_{n=1}^{\infty} (b_n - b_n^*)$$

exists (P).

This follows *a fortiori* from Theorem IV, which tells us that

$$\sum_{n=1}^{\infty} (b_n - b_n^*) \sin n\theta$$

and its conjugate are uniformly summable (C, 1) in $(0 \leq |\theta| < a - \epsilon)$.

LEMMA *\beta*. Under the conditions of the theorem

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

exists (P).

Let

$$b_n = \frac{a_n}{n} = o(n),$$

and

$$\psi(\theta) = \theta \phi(\theta).$$

Then, by Lemma *a*,

$$\sum_{n=1}^{\infty} (b_n - b_n^*)$$

exists (P). If

$$\chi(t) = \frac{\psi^*(t) - \psi^*(-t)}{t},$$

then $\chi(t)$ is absolutely integrable down to the origin; and hence

$$\sum_{n=1}^{\infty} b_n^*,$$

which is the value at $x=0$ of the series allied with the Fourier series

of $\psi^*(\theta)$, is convergent. It follows that

$$\sum_{n=1}^{\infty} b_n$$

exists (P).

We can now easily complete the proof of (33). If $a_n \geq 0$, we have

$$\frac{|a_n|}{n} r^n = \frac{a_n}{n} r^n.$$

If $a_n < 0$, we have

$$\frac{|a_n|}{n} r^n = \frac{a_n}{n} r^n - 2 \frac{a_n}{n} r^n < \frac{a_n}{n} r^n + \frac{2C}{n^2} r^n.$$

Hence

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n} r^n \leq \sum_{n=1}^{\infty} \frac{a_n}{n} r^n + 2C \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

The second member tends to a finite limit as $r \rightarrow 1$. Hence

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n}$$

is convergent.

THE GENERALIZED THIRD DERIVATIVE AND ITS APPLICATION TO THE THEORY OF TRIGONOMETRIC SERIES.

By S. VERBLUNSKY.

[Received 21 September, 1929.—Read 14 November, 1929.]

1. It is well known that, in the problem of determining under what conditions a trigonometric series of the form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1)$$

is a Fourier series, where

$$\Phi(x) = \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2} \quad (2)$$

converges everywhere to a continuous function, a fundamental part of the reasoning depends on the properties of the generalized second derivative of a continuous function. When we know only that

$$a_n = o(n), \quad b_n = o(n), \quad (3)$$

we cannot assert *a priori* that $\phi(x)$ exists, still less that it is continuous. However, the properties of the generalized third derivative of a continuous function may be employed without great difficulty. If $F(t)$ be a continuous function, we define the upper and lower limits as $h \rightarrow 0$ of

$$\frac{F(x+3h) - 3F(x+h) + 3F(x-h) - F(x-3h)}{8h^3}$$

to be the upper and lower generalized third derivatives of $F(t)$ at $t = x$. We say that (1) is summable $(R, 3)$ if

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \left(\frac{\sin n\theta}{n\theta} \right)^3$$

tends to a limit as $\theta \rightarrow 0$. If (3) holds, the summability $(R, 3)$ of (1) is equivalent to the existence of the generalized third derivative of

$$F(x) = \frac{a_0 x^3}{12} - \sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n^3}.$$

The theorem which we prove (Theorem IV, § 14) is as follows:—

If (1) satisfies (3), and if its upper and lower sums $(R, 3)$ are finite and equivalent to a function $f(x)$ which is integrable in $(0, 2\pi)$ in the sense of Denjoy-Perron, then (1) is the Fourier series of $f(x)$.

This is proved in §§ 6–14. In §§ 2–5 we give an investigation of some properties of the generalized third derivative. The main result in this connection is Theorem I. Though this theorem itself is not used in the application to trigonometric series, the reasoning on which it depends is used subsequently. As regards the proof of Theorem IV, it should be noticed that the idea of relatively uniform convergence plays an important part (*cf.* § 13).

2. LEMMA 1. *A function $f(x)$ is defined in the interval (a, b) with the property that if δ be (a, b) or any closed interval contained in (a, b) , then $\text{Max } f(x)$ in δ is attained at an end point of δ and at no interior point of δ . Then, either $f(x)$ is monotone in (a, b) or else there is a point ζ in (a, b) such that $f(x)$ is monotone in each of the intervals (a, ζ) , (ζ, b) .*

We shall say that δ is an r -interval if $\text{Max } f(x)$ in δ is attained at the right-hand end point; and that δ is an l -interval if $\text{Max } f(x)$ in δ is attained at the left-hand end point. Let n be a positive integer greater than unity. There are three cases.

(1) All pairs of abutting intervals of length less than $(b-a)/n$ are r -intervals. Then $f(x)$ is monotone increasing in (a, b) .

(2) All pairs of abutting intervals of length less than $(b-a)/n$ are l -intervals. Then $f(x)$ is monotone diminishing in (a, b) .

(3) There is a pair of abutting intervals of length less than $(b-a)/n$, one of which is an l -interval and the other an r -interval.

Let the intervals in question be (a, ζ_1) , (ζ_1, β) . Then (a, ζ_1) is an l -interval and (ζ_1, β) an r -interval, since otherwise $\text{Max } f(x)$ in (a, β) would be attained at ζ . If $\beta < b$, an interval (β, β') , where $\beta' > \beta$, is an r -interval. For, if (β, β') is an l -interval, $\text{Max } f(x)$ in (ζ, β') is attained

at β . Similarly, (β', β'') , where $\beta'' > \beta'$, is an r -interval. Hence $f(x)$ is monotone increasing in (β, b) . Similarly, it is monotone diminishing in (a, α) .

We now consider the interval (α, β) . We have three cases.

(1') All pairs of abutting intervals of length less than $(\beta - \alpha)/n$ are r -intervals. Then $f(x)$ is monotone increasing in (α, β) , and hence in (a, b) .

(2') All pairs of abutting intervals of length less than $(\beta - \alpha)/n$ are l -intervals. Then $f(x)$ is monotone diminishing in (α, β) , and hence in (a, β) .

(3') There is a pair of abutting intervals of length less than $(\beta - \alpha)/n$, one of which is an l -interval and the other an r -interval.

Let the intervals be (α_1, ζ_2) , (ζ_2, β_1) . Then, if $\beta_1 < \beta$, by the same argument as above, $f(x)$ is monotone increasing in (β_1, b) . If $\alpha_1 > \alpha$, $f(x)$ is monotone diminishing in (a, α_1) .

We repeat the argument with (α_1, β_1) and pairs of abutting intervals of length less than $(\beta_1 - \alpha_1)/n$, and so on. If the process stops after a finite number of steps, the lemma is proved. Otherwise, we obtain a sequence $\{(\alpha_r, \beta_r)\}$ of intervals each of which contains the succeeding ones, and such that $f(x)$ is monotone diminishing in (a, α_r) and monotone increasing in (β_r, b) . This sequence converges to a point ζ , since

$$\beta_{r+1} - \alpha_{r+1} < \frac{\beta_r - \alpha_r}{n} \quad (n > 1)$$

for all r ; and $f(x)$ is monotone diminishing in (a, ζ) and monotone increasing in (ζ, b) .

3. We write

$$\Delta^3 F(x, h) = F(x+3h) - 3F(x+h) + 3F(x-h) - F(x-3h),$$

$$D^3 F(x) = \lim_{h \rightarrow 0} \frac{\Delta^3 F(x, h)}{8h^3}.$$

LEMMA 2. Let $F(x)$ be a continuous function defined in an interval containing $(x_0 - 3h_0, x_0 + 3h_0)$. Let $F'(x)$ and $D^3 F(x)$ have finite values at each point of $(x_0 - 3h_0, x_0 + 3h_0)$. Let

$$\frac{\Delta^3 F(x_0, h_0)}{8h_0^3} = 0,$$

and suppose that, for $h \leq h_1$, $\Delta^3 F(x, h)/(8h^3)$ is of the same sign (≥ 0)

for all intervals $(x-3h, x+3h)$ strictly interior to (x_0-3h_0, x_0+3h_0) . Then there is a point ξ_1 in (x_0-3h_0, x_0+3h_0) such that

$$D^3F(\xi_1) = 0.$$

The expression $\Delta^3F(x, h)/(8h^3)$ will be called the third differential quotient of the interval $(x-3h, x+3h)$ with respect to $F(x)$. We may suppose, without loss of generality, that the third differential quotient with respect to $F(x)$ of all intervals $(x-3h, x+3h)$ strictly interior to (x_0-3h_0, x_0+3h_0) is positive; for if they were all negative for $F(x)$, they would be positive for $-F(x)$.

We have then

$$F(x+3h) - 3F(x+h) + 3F(x-h) - F(x-3h) > 0.$$

$$\begin{aligned} \text{Hence} \quad \frac{F(x+3h) - F(x-3h)}{6h} &> \frac{F(x+h) - F(x-h)}{2h} \\ &> \frac{F(x+h/3) - F(x-h/3)}{2 \cdot h/3} \\ &> \dots \\ &> F'(x). \end{aligned} \tag{4}$$

This applies to all points x of an interval d strictly interior to (x_0-3h_0, x_0+3h_0) .

Consider a diminishing sequence $\{h_n\}$ such that

$$h_n = 3h_{n+1}.$$

$$\text{Then, for all } x \text{ in } d, \quad \frac{F(x+h_n) - F(x-h_n)}{2h_n}$$

forms a diminishing sequence of continuous functions which tends to $F'(x)$. Hence $F'(x)$ is upper semi-continuous in d , and so attains its maximum in d .

If there is a point ξ interior to d at which $F'(x)$ is a maximum, then

$$\frac{F(\xi+h) - F(\xi-h)}{2h} \leq F'(\xi) \tag{5}$$

for all sufficiently small h . This contradicts (4). Hence $F'(x)$ possesses the property of Lemma 1, and so is either monotone in (x_0-3h_0, x_0+3h_0) , or else there is a point ξ interior to this interval such that $F'(x)$ is monotone diminishing in (x_0-3h_0, ξ) and monotone increasing in (ξ, x_0+3h_0) . But if a continuous function $F(x)$ has a finite differential coefficient $F'(x)$ at

every point of an interval, then $F'(x)$ cannot have an ordinary discontinuity. Hence $F'(x)$ is continuous in (x_0-3h_0, x_0+3h_0) .

Since the third differential quotient of (x_0-3h_0, x_0+3h_0) with respect to $F(x)$ is zero, there is a polynomial

$$P_2(x) = dx^2 + ex + f$$

such that

$$F_1(x) = F(x) - P_2(x)$$

vanishes at the four points

$$x_0-3h_0, \quad x_0-h_0, \quad x_0+h_0, \quad x_0+3h_0;$$

hence $F'_1(x)$ vanishes at three points interior to (x_0-3h_0, x_0+3h_0) . Since $F'_1(x)$ is continuous, it has at least one maximum ξ_1 , and one minimum ξ_2 , both interior to (x_0-3h_0, x_0+3h_0) . At ξ_1 we must have

$$\frac{\Delta^3 F_1(\xi_1, h)}{8h^3} \leq 0 \quad (6)$$

for a diminishing sequence of h . For if

$$\frac{\Delta^3 F_1(\xi_1, h)}{8h^3} > 0$$

for all sufficiently small h , then, as in (4),

$$\begin{aligned} \frac{F_1(\xi_1+3h) - F_1(\xi_1-3h)}{6h} &> \frac{F_1(\xi_1+h) - F_1(\xi_1-h)}{2h} \\ &> \dots \\ &> F'_1(\xi_1), \end{aligned}$$

which, as in (5), contradicts the fact that $F'_1(x)$ is a maximum at ξ_1 . Thus, by (6),

$$D^3 F(\xi_1) = D^3 F_1(\xi_1) \leq 0.$$

Since

$$\frac{\Delta^3 F(x, h)}{8h^3} > 0$$

for all intervals $(x-3h, x+3h)$, it follows that

$$D^3 F(\xi_1) \geq 0.$$

Hence the result.

4. Let $f(x, y)$ be a function of two variables defined in a closed rectangle whose sides are parallel to the axes. It has been shown by

Baire* that if

(i) $f(x, y)$ is everywhere continuous with respect to y ;

(ii) $f(x, y)$ is continuous with respect to x along straight lines parallel to the x -axis which meet the y -axis in an everywhere dense set of points; then on any curve $y = \phi(x)$, where $\phi(x)$ is a continuous function, there exists an everywhere dense set of points at which $f(x, y)$ is continuous in (x, y) . Further, if C denote the curve, and Π be any perfect set, there is an everywhere dense set of points on ΠC at which $f(x, y)$ is continuous on Π .

Let $F(x)$ be a continuous function defined in the closed interval $(a-3h_0, b+3h_0)$. We consider the function of x and h , $\Delta^3 F(x, h)/(8h^3)$ defined in $(a \leq x \leq b, -h_0 \leq h \leq h_0)$. We suppose that, for every x in (a, b) ,

$$D^3 F(x) = \lim_{h \rightarrow 0} \frac{\Delta^3 F(x, h)}{8h^3}$$

has a finite value, and we assign $D^3 F(x)$ as the value of $\Delta^3 F(x, h)/(8h^3)$ for $b = 0$.

The function of x and h , thus defined, is continuous with respect to h for every x , and thus satisfies (i). It is continuous with respect to x for h different from zero, and so satisfies (ii). We take $h = 0$ as the curve C in the above theorem of Baire.

Let P be any linear perfect set in (a, b) . Let δ be one of its contiguous intervals. We denote by $(\delta_l, 0)$, $(\delta_r, 0)$ the coordinates of the end points of δ . Let δ' denote the rectangle whose vertices are

$$(\delta_l, h_0), \quad (\delta_r, h_0), \quad (\delta_r, -h_0), \quad (\delta_l, -h_0).$$

The set of such rectangles is contiguous to a two-dimensional perfect set Π . If we consider the values of $\Delta^3 F(x, h)/(8h^3)$ on Π , and apply the above theorem of Baire, we have

LEMMA 3. *If $F(x)$ be a continuous function defined in $(a-3h_0, b-3h_0)$ such that $D^3 F(x)$ exists at every point of (a, b) , then, given any perfect set P in (a, b) , $\Delta^3 F(x, h)/(8h^3)$, considered as a function of x and h , possesses the following property: there is a set $E \subset P$, everywhere dense in P , at each point of which $\Delta^3 F(x, h)/(8h^3)$ is continuous in (x, h) , x being restricted to belong to P .*

* Baire, *Annali di Matematica* (3), 3 (1899), 1-122 (27-30).

5. We can now prove

THEOREM I. *Let $F(x)$ be a continuous function defined in an interval containing (x_0-3h_0, x_0+3h_0) . Let $F'(x)$ and $D^3F(x)$ have finite values at every point of this interval. Then*

$$\frac{\Delta^3 F(x_0, h_0)}{8h_0^3}$$

lies between the upper and lower bounds of $D^3F(x)$ in the interval (x_0-3h_0, x_0+3h_0) .

There is no difficulty in proving the result, and even a more general one, if we assume that $F'(x)$ is continuous (cf. § 6). If we merely assume that $F'(x)$ exists, it becomes necessary to employ a rather delicate argument.

We write
$$F_1(x) = F(x) - \frac{\Delta^3 F(x_0, h_0)}{8h_0^3} x^3,$$

so that
$$\frac{\Delta^3 F_1(x_0, h_0)}{8h_0^3} = 0.$$

There are two possibilities.

I. For $0 < h \leq h_1$ say, $\Delta^3 F_1(x, h)/(8h^3)$ is of the same sign (≥ 0) for all intervals $(x-3h, x+3h)$ strictly contained in (x_0-3h_0, x_0+3h_0) .

Then, by Lemma 2, there is a point ξ_1 interior to (x_0-3h_0, x_0+3h_0) , such that

$$D^3 F_1(\xi_1) = 0,$$

i.e.
$$D^3 F(\xi_1) = \Delta^3 F(x_0, h_0)/8h_0^3,$$

which proves the theorem.

II. There is a diminishing sequence $\{h_n\}$ such that, corresponding to each h_n , there are one or more points $x^{(n)}$ for which

$$\frac{\Delta^3 F_1(x^{(n)}, h_n)}{8h_n^3} = 0.$$

Consider such an interval $(x^{(n)}-3h_n, x^{(n)}+3h_n)$. If, for all intervals strictly interior to $(x^{(n)}-3h_n, x^{(n)}+3h_n)$, $\Delta^3 F_1(x, h)/(8h^3)$ were of the same sign (≥ 0) for $0 \leq h \leq h'_1$ say, the theorem would follow as for I. We may therefore suppose that $F_1(x)$ possesses the following property.

To each h_n of a diminishing sequence $\{h_n\}$ there corresponds at least one interval $(x^{(n)}-3h_n, x^{(n)}+3h_n)$ strictly interior to (x_0-3h_0, x_0+3h_0) and of zero third differential quotient with respect to $F_1(x)$. To each term h_{nr} of a diminishing sequence $\{h_{nr}\}$ there corresponds at least one interval

$(x^{(nr)} - 3h_{nr}, x^{(nr)} + 3h_{nr})$ strictly interior to $(x^{(n)} - 3h_n, x^{(n)} + 3h_n)$ and of zero third differential quotient with respect to $F_1(x)$; and so on.

There is thus an aggregate H of values of h such that, if $h \in H$, there is at least one interval $(x_h - 3h, x_h + 3h)$ of zero third differential quotient with respect to $F_1(x)$. Consider a fixed value of $h \in H$. To the points x_h we make correspond the open intervals $(x_h - 3h, x_h + 3h)$. These form an open set O'_h . We have such an open set for each $h \in H$. We denote by O_{h_1} the sum of all O'_h for $h < h_1$ when $h \in H$. If $h_2 < h_1$, then $O_{h_1} \supset O_{h_2}$. We thus have a family $\{O_h\}$ of open sets, each member of which contains all those of lesser index. Let E be the inner limiting set.

We first observe that E exists; for, in virtue of the property of $F_1(x)$ mentioned above, there is an interval, $(\xi_1 - 3h_1, \xi_1 + 3h_1)$ say, strictly interior to $(x_0 - 3h_0, x_0 + 3h_0)$ and of zero third differential quotient with respect to $F_1(x)$. There is an interval, $(\xi_2 - 3h_2, \xi_2 + 3h_2)$ say, strictly interior to $(\xi_1 - 3h_1, \xi_1 + 3h_1)$ and of zero third differential quotient with respect to $F_1(x)$; and so on. We may suppose that $h_n \rightarrow 0$. Then there is a single point ζ common to every member of the sequence of intervals, and ζ is a member of E .

There are now two cases.

(1) E is dense in itself. Since E is an ordinary inner limiting set which is dense in itself, it is unenumerable. It is further measurable (B). Hence, by a theorem due to Hausdorff and Alexandroff*, it contains a perfect set P .

Every point $x \in P$ has the property that, given δ , we can find an interval $d(x)$ containing x in its interior, of diameter less than δ and of zero third differential quotient with respect to $F_1(x)$.

This follows from the fact that x is interior to every O_h ; therefore interior to some O'_h for $h < \delta/6$; therefore interior to some interval of diameter less than δ and of zero third differential quotient with respect to $F_1(x)$, since O'_h is the sum of such intervals.

By Lemma 3, $\Delta^3 F_1(x, h)/(8h^3)$ is pointwise discontinuous on P as a function of (x, h) , where $x \in P$. Let $\bar{x}$ be a point of P at which it is continuous in (x, h) for $x \in P$. Then

$$D^3 F_1(\bar{x}) = 0.$$

For, if $D^3 F_1(\bar{x}) = c > 0$, there is an $\eta > 0$ such that

$$\frac{\Delta^3 F_1(x, h)}{8h^3} > \frac{c}{2} \quad (x \in P, |x - \bar{x}| < \eta, |h| < \eta).$$

* Cf. Sierpinski, *Fundamenta Math.*, 5 (1924), 166-171.

This contradicts the above mentioned property possessed by all points of P . Similarly, if $c < 0$. Hence

$$D^3F(\bar{x}) = \frac{\Delta^3F(x_0, h_0)}{8h_0^3},$$

which proves the theorem.

(2) E is not dense in itself. Let ξ be an isolated point of E . We can find an interval δ of zero third differential quotient with respect to $F_1(x)$ and containing ξ in its interior, but containing no other point of E . Let α and $\beta > \alpha$ be the end points of δ . Consider the interval (ξ, β) . Any interval d contained in (ξ, β) , and (ξ, β) itself, may be supposed to have a third differential quotient with respect to $F_1(x)$ which is different from zero. Otherwise we could apply to an interval d of zero third differential quotient the same argument as has been applied to $(x_0 - 3h_0, x_0 + 3h_0)$. It would then follow that either the theorem is true or else that (ξ, β) contains points of E in its interior. The latter conclusion would involve a contradiction.

Further, the third differential quotient of all such intervals d must be of the same sign; for the differential quotient is a continuous function of the end points of d , and so could not be of different sign for two intervals d_1, d_2 , without being zero for some interval d_3 .

It follows, by the reasoning of Lemma 2, that $F'_1(x)$ is continuous in (ξ, β) , and either monotone in this interval or else monotone in each of the intervals (ξ, ζ) , (ζ, β) , where $\xi < \zeta < \beta$. A similar result follows for the interval (α, ξ) . We thus deduce that $F'_1(x)$ is continuous in (α, β) . Since (α, β) is of zero third differential quotient with respect to $F_1(x)$, we can find a quadratic

$$P_2(x) = dx^2 + ex + f$$

such that

$$F_2(x) = F_1(x) - P_2(x)$$

vanishes at the four points

$$\alpha, \quad \alpha + (\beta - \alpha)/3, \quad \alpha + 2(\beta - \alpha)/3, \quad \beta;$$

hence $F'_2(x)$ vanishes at three points interior to (α, β) and, since it is continuous, has a maximum at ξ_1 , say, and a minimum at ξ_2 . We have, by an argument similar to that used in the proof of Lemma 2,

$$D^3F_2(\xi_1) \leq 0 \leq D^3F_2(\xi_2);$$

or, since $F_2(x)$ differs from $F_1(x)$ by a quadratic,

$$D^3F_1(\xi_1) \leq 0 \leq D^3F_1(\xi_2),$$

$$i.e. \quad D^3 F(\xi_1) \leq \frac{\Delta^3 F(x_0, h_0)}{8h_0^3} \leq D^3 F(\xi_2).$$

This proves the theorem.

6. LEMMA 4. If $\psi(x)$, $\psi'(x)$ are continuous in (x_0-3h_0, x_0+3h_0) , then

$$\frac{\Delta^3 \psi(x_0, h_0)}{8h_0^3}$$

lies between the upper and lower bounds of $\bar{D}^3 \psi(x)$, $\underline{D}^3 \psi(x)$, the upper and lower generalized third derivatives of $\psi(x)$, in the interval.

There is a polynomial

$$P_3(x) = \frac{\Delta^3 \psi(x_0, h_0)}{8h_0^3} \cdot \frac{x^3}{6} + ax^2 + bx + c,$$

which has the same value as $\psi(x)$ at the four points

$$x_0-3h_0, \quad x_0-h_0, \quad x_0+h_0, \quad x_0+3h_0.$$

If we write

$$\psi_1(x) = \psi(x) - P_3(x),$$

it follows that $\psi'_1(x)$ vanishes at three points interior to (x_0-3h_0, x_0+3h_0) . Since $\psi'_1(x)$ is continuous, it has a maximum at ξ_1 , say, and a minimum at ξ_2 . At ξ_1

$$\underline{D}^3 \psi_1(\xi_1) \leq 0,$$

as in the proof of Lemma 2; *i.e.*

$$\underline{D}^3 \psi(\xi_1) \leq \frac{\Delta^3 \psi(x_0, h_0)}{8h_0^3}.$$

$$\text{Similarly, at } \xi_2, \quad \bar{D}^3 \psi(\xi_2) \geq \frac{\Delta^3 \psi(x_0, h_0)}{8h_0^3}.$$

This proves the lemma.

7. We say that a finite single-valued function $F(x)$, defined in the interval (a, b) , possesses the property R if its points of discontinuity with respect to any perfect set $P \subset (a, b)$ are non-dense in P .

The following theorem is of fundamental importance in the subsequent reasoning.

THEOREM II. If $\psi(x)$ is continuous in (a, b) , $\psi'(x)$ possesses the property R , and $\underline{D}^3 \psi(x) > 0$ at every point of the interval, then $\psi'(x)$ is continuous.

The proof depends on five simple lemmas. Let h_ξ denote the greatest number such that

$$\Delta^3\psi(\xi, h) \geq 0 \quad (0 < h \leq h_\xi).$$

Let $\text{Min } h_x$ denote the minimum of h_x at x . Let $\text{Min}(h_x, \delta)$ denote the minimum of h_x in δ .

LEMMA α . *The points x at which $\text{Min } h_x = 0$ are non-dense in (a, b) .*

This is a consequence of the following three observations:—

(i) Since $\underline{D}^3\psi(x) > 0$ at every point, h_ξ is positive at every point.

(ii) h_ξ is upper semi-continuous. For otherwise, for suitable $\epsilon > 0$, there is a point ξ and a sequence $x_n \rightarrow \xi$ such that

$$h_{x_n} \geq h_\xi + \epsilon.$$

Hence, if H satisfies $0 < H \leq h_\xi + \epsilon$,

we have $\Delta^3\psi(x_n, H) \geq 0$

for all n . Hence $\Delta^3\psi(\xi, H) \geq 0$.

Since H may take all values in $(0, h_\xi + \epsilon)$, this contradicts the definition of h_ξ .

(iii) If $\phi(x) \geq 0$ be upper semi-continuous and $\text{Min } \phi(x) = 0$ at an everywhere dense set, then $\phi(x) = 0$ at an everywhere dense set. This is easily proved.

LEMMA β . *If $\psi'(x)$ be continuous in (a, b) and (α, β) be strictly interior to (a, b) then $\text{Min}\{h_x(\alpha, \beta)\} > 0$, provided that $\underline{D}^3\psi(x) > 0$ in (a, b) .*

If $(x-3h, x+3h)$ is strictly interior to (a, b) , $\Delta^3\psi(x, h)/(8h^3)$ lies between the upper and lower bounds of $\bar{D}^3\psi(x)$, $\underline{D}^3\psi(x)$ in $(x-3h, x+3h)$ by Lemma 4. Hence,

$$\Delta^3\psi(x, h) \geq 0.$$

Let 6η denote the smaller of $\alpha-a$, $b-\beta$. Then $h_x = \eta$ for every point of (α, β) .

Definition. A function will be said to be bi-monotone in (α, β) if it is not monotone in this interval, and if there is a $\gamma (\alpha < \gamma < \beta)$ such that the function is monotone in each of the intervals (α, γ) , (γ, β) , strictly diminishing in (α, γ) , and strictly increasing in (γ, β) .

LEMMA γ . If $\text{Min}(h_x, \delta) > 0$ and $D^3\psi(x) > 0$ at each point of δ , then $\psi'(x)$ is continuous in δ and either monotone or bi-monotone.

Let $\text{Min}(h_x, \delta) = h_\delta$. Then for $0 < h < h_\delta$, and $(x-3h, x+3h)$ in δ , $\Delta^3\psi(x, h) \geq 0$. By an argument similar to that used in the proof of Lemma 2, we infer that $\psi'(x)$ is upper semi-continuous in δ and that, if x is a point within δ , then

$$\frac{\psi(x+3h) - \psi(x-3h)}{6h} \geq \psi'(x) \quad (h < h_\delta).$$

Let a be any interval strictly interior to δ . If $\psi'(x)$ attains its maximum in d at an interior point of d , ξ say, then

$$\frac{\psi(\xi+h) - \psi(\xi-h)}{2h} \leq \psi'(\xi)$$

for sufficiently small h . It follows that, for sufficiently small h ,

$$\frac{\psi(\xi+h) - \psi(\xi-h)}{2h} = \psi'(\xi),$$

and so, for such h , $\Delta^3\psi(\xi, h) = 0$.

Hence $D^3\psi(\xi) = 0$,

which contradicts the hypothesis of the lemma. Thus $\psi'(x)$ attains its maximum in d at an end point of d . The remainder of the proof follows as in Lemma 2.

LEMMA δ . The set $\text{Min } h_x = 0$ is a non-dense perfect set II.

Since $\text{Min } h_x$ is a lower semi-continuous function, the set $\text{Min } h_x \leq 0$, i.e. the set $\text{Min } h_x = 0$, is closed. Further, this set cannot contain an isolated point. For otherwise there is, by Lemma γ , an interval on each side of the point in which $\psi'(x)$ is monotone and continuous. This implies that $\psi'(x)$ possesses an ordinary discontinuity (by Lemma β). Since this is impossible, the set $\text{Min } h_x = 0$ is perfect and, by Lemma α , non-dense.

LEMMA ϵ . The points of discontinuity of $\psi'(x)$ with respect to (a, b) are everywhere dense in II.

We first observe that, if $\text{Min } h_x > 0$, at each point of a closed interval δ , then an application of the Heine-Borel theorem shows that $\text{Min}(h_x, \delta) > 0$. It follows, by Lemma γ , that $\psi'(x)$ is continuous in

the intervals contiguous to Π . If now the points of discontinuity of $\psi'(x)$ with respect to (a, b) were not everywhere dense in Π , we could find an interval, containing points of Π in its interior, in which $\psi'(x)$ is continuous. This is impossible by Lemma β .

8. We can now prove Theorem II. There is an interval Δ , containing points of Π in its interior, such that $\psi'(x)$ is continuous on $\Pi\delta$. This follows from the hypothesis that $\psi'(x)$ possesses the property R . We choose Δ so that its end points are points of Π . Let p be a point of $\Pi\Delta$.

Since $\psi'(x)$ is continuous and monotone or bi-monotone in each of the contiguous intervals of Π , then, denoting such a contiguous interval by δ_n , we have

Max $\psi'(x)$ in δ_n is attained at an end point of δ_n (*i.e.* a point of Π).

Now p is a limiting point of end points of intervals δ_n , and $\psi'(x)$ is continuous at p on $\Pi\Delta$. Hence, if $\psi'(x)$ is not continuous at p on Δ , we have

$$\overline{\lim}_{x \rightarrow p} \psi'(x) = \psi'(p);$$

i.e. $\psi'(x)$ is upper semi-continuous at p . Hence $\psi'(x)$ is upper semi-continuous in Δ . It follows that in any interval $\delta \subset \Delta$, $\psi'(x)$ attains its maximum in δ . But since $\underline{D}^3\psi(x) > 0$ at every point, it follows by the argument of Lemma 2 that $\psi'(x)$ cannot attain its maximum at an interior point of δ . It therefore attains its maximum at an end point of δ , and so, by Lemma 1, is either monotone or bi-monotone in Δ , and, therefore, as a differential coefficient, continuous in Δ . This contradicts Lemma ϵ . It follows that Π is a null set; *i.e.* the set $\text{Min } h_x = 0$ is null. Hence, by an application of the Heine-Borel theorem, $\text{Min } (h_x, \delta) > 0$ for every interval δ strictly interior to (a, b) ; and so, by Lemma γ , $\psi'(x)$ is continuous in (a, b) (and monotone or bi-monotone). This proves the theorem.

9. LEMMA 5. If $\psi(x)$, $\psi'(x)$ be continuous in (a, b) and $\underline{D}^3\psi(x) > 0$ at each point of this interval, then

$$\psi'(x) < \psi'(a) + \frac{x-a}{b-a} \{ \psi'(b) - \psi'(a) \}.$$

$$\text{Write} \quad \Phi(x) = \psi'(x) - \psi'(a) - \frac{x-a}{b-a} \{ \psi'(b) - \psi'(a) \}.$$

$$\text{Then} \quad \Phi'(a) = \Phi'(b) = 0.$$

If, for some x interior to (a, b) ,

$$\Phi'(x) \geq 0,$$

then there is a point ξ at which $\Phi'(x)$ attains a maximum. Hence, as in the proof of Lemma 2,

$$\Delta^3 \Phi(\xi, h) \leq 0$$

for a diminishing sequence of h . Since

$$\Phi(x) - \psi(x)$$

is a quadratic,

$$\Delta^3 \Phi(x, h) = \Delta^3 \psi(x, h)$$

for all x and h . It follows that

$$\Delta^3 \psi(x, h) \leq 0$$

for a diminishing sequence of h , which contradicts $\underline{D}^3 \psi(x) > 0$. Hence

$$\Phi'(x) < 0 \quad (a < x < b),$$

which proves the lemma.

10. LEMMA 6. *If at a point x the function $F(x)$ has a continuous second derivative $F''(x)$, then $\bar{D}^3 F(x)$, $\underline{D}^3 F(x)$ lie between the greatest and the least of the four derivatives of $F''(x)$ at the point.*

It is easily verified that

$$\begin{aligned} \Delta^3 F(x, h) &= \int_{-h}^h dz \int_0^{2h} dy \{ F''(x+y+z) - F''(x-y+z) \} \\ &= \int_{-h}^h dz \int_0^{2h} [\{ F''(x+y+z) - F''(x) \} + \{ F''(x) - F''(x-y+z) \}] dy. \end{aligned}$$

We have

$$\int_{-h}^h dz \int_0^{2h} \{ F''(x+y+z) - F''(x) \} dy = \int_{-h}^h \frac{F''(x+y+z) - F''(x)}{y+z} (y+z) dy.$$

If we denote by $\underline{D}F''(x)$ the least of the four derivatives of $F''(x)$ at x , we have

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{1}{8h^3} \int_{-h}^h dz \int_0^{2h} \{ F''(x+y+z) - F''(x) \} dy \\ \geq \lim_{h \rightarrow 0} \underline{D}F''(x) \cdot \frac{1}{8h^3} \int_{-h}^h dz \int_0^{2h} (y+z) dy. \end{aligned}$$

Similarly

$$\lim_{h \rightarrow 0} \frac{1}{8h^3} \int_{-h}^h dz \int_0^{2h} \{F''(x) - F''(x - y + z)\} dy \\ \geq \lim_{h \rightarrow 0} \underline{D}F''(x) \cdot \frac{1}{8h^3} \int_{-h}^h dz \int_0^{2h} (y - z) dy.$$

By addition, $\underline{D}^3F(x) \geq \underline{D}F''(x)$;
and, similarly, $\bar{D}^3F(x) \leq \bar{D}F''(x)$.

11. In this section we establish two immediate extensions of some preceding results.

COROLLARY TO THEOREM II. *If $\psi(x)$ be continuous in (a, b) , $\psi'(x)$ possess the property R , and $\underline{D}^3\psi(x) \geq f(x)$ at every point of the interval, where $f(x)$ is a finite integrable function, then $\psi'(x)$ is continuous in (a, b) .*

Let $\phi_2(x)$ be a de la Vallée Poussin minor function of $f(x)$. Then

$$\underline{D}^3\left(\psi(x) - \int_a^x dy \int_a^y \phi_2(t) dt\right) > \underline{D}^3\psi(x) - \bar{D}^3 \int_a^x dy \int_a^y \phi_2(t) dt \\ > \underline{D}^3\psi(x) - f(x) \quad (\text{by Lemma 6}) \\ > 0.$$

Hence, by Theorem II, $\psi'(x) - \int_a^x \phi_2(t) dt$

is continuous. So, therefore, is $\psi'(x)$.

COROLLARY TO LEMMA 5. *If $\psi(x)$, $\psi'(x)$ be continuous in (a, b) and $\underline{D}^3\psi(x) \geq f(x)$ at every point of the interval, where $f(x)$ is finite and integrable, then*

$$\psi'(x) - \int_a^x dy \int_a^y f(t) dt \leq \psi'(a) + \frac{x-a}{b-a} \left[\psi'(b) - \psi'(a) - \int_a^b dy \int_a^y f(t) dt \right].$$

If $\phi_2(x)$ be a de la Vallée Poussin minor function of $f(x)$, we can put

$$\psi(x) - \int_a^x dy \int_a^y \phi_2(t) dt$$

for $\psi(x)$ in Lemma 5. We thus obtain

$$\psi'(x) - \int_a^x \phi_2(t) dt < \psi'(a) + \frac{x-a}{b-a} \left[\psi'(b) - \psi'(a) - \int_a^b \phi_2(t) dt \right].$$

We can select a sequence of minor functions $\phi_{2n}(x)$ which tend uniformly in (a, b) to

$$\int_a^x f(t) dt.$$

This proves the result.

12. THEOREM III. *If $\Phi(x)$ be continuous in (a, b) , $\Phi'(x)$ possess the property R , and $\bar{D}^3\Phi(x)$, $\underline{D}^3\Phi(x)$ be finite and equivalent to an integrable function $f(x)$, then*

$$\Phi'(x) = \int_a^x d\theta \int_a^\theta f(t) dt + Ax + B \quad (a < x < b),$$

where A and B are constants.

Since we can write

$$\underline{D}^3\Phi(x) \geq \underline{D}^3\Phi(x),$$

it follows, by the corollary to Theorem II, that $\Phi'(x)$ is continuous. Hence, by the corollary to Lemma 5,

$$\Phi'(x) - \int_a^x dy \int_a^y \underline{D}^3\Phi(t) dt \leq \Phi'(a) + \frac{x-a}{b-a} \left[\Phi'(b) - \Phi'(a) - \int_a^b dy \int_a^y \underline{D}^3\Phi(t) dt \right],$$

or, what is the same,

$$\Phi'(x) - \int_a^x dy \int_a^y f(t) dt \leq \Phi'(a) + \frac{x-a}{b-a} \left[\Phi'(b) - \Phi'(a) - \int_a^b dy \int_a^y f(t) dt \right].$$

In a similar manner we can show that

$$\Phi'(x) - \int_a^x dy \int_a^y \bar{D}^3\Phi(t) dt \geq \Phi'(a) + \frac{x-a}{b-a} \left[\Phi'(b) - \Phi'(a) - \int_a^b dy \int_a^y \bar{D}^3\Phi(t) dt \right].$$

This proves the theorem.

13. It is now necessary to introduce the idea of relatively uniform convergence. In accordance with the definition of E. H. Moore, we say that a sequence of finite single-valued functions $\{f_n(x)\}$, defined in an interval (a, b) , converges relatively uniformly to a function $f(x)$, defined in this interval, if there exists a finite function $\sigma(x)$ such that, given $\epsilon > 0$, there is an n_ϵ satisfying

$$|f(x) - f_n(x)| < \epsilon \sigma(x) \quad (n \geq n_\epsilon)$$

for all x in (a, b) .

We shall use the following result due to Chittenden* :—

If a sequence $\{f_n(x)\}$ of continuous functions converges relatively uniformly in (a, b) to a function $f(x)$, then the points of discontinuity of $f(x)$ with respect to any perfect set $P \subset (a, b)$ are non-dense in P ; i.e. $f(x)$ possesses the property R .

LEMMA 7. *If the upper and lower sums $(R, 3)$ of*

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \{a_n = o(n), b_n = o(n)\}$$

are everywhere finite, then

$$\sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2} \quad (7)$$

is everywhere convergent and possesses the property R .

We write

$$A_n x = a_n \cos nx + b_n \sin nx$$

and

$$F(\theta) = \sum_{n=1}^{\infty} A_n(x) \left(\frac{\sin n\theta}{n\theta} \right)^3.$$

Then, for each x , the upper and lower limits of $F(\theta)$ as $\theta \rightarrow 0$ are finite; while, for $\theta > \delta > 0$, $F(\theta)$ is continuous and $O(\theta^{-3})$. We have

$$\begin{aligned} \int_{\xi}^{\infty} F(\theta) d\theta &= \sum_{n=1}^{\infty} A_n(x) \int_{\xi}^{\infty} \left(\frac{\sin n\theta}{n\theta} \right)^3 d\theta \\ &= \sum_{n=1}^{\infty} \frac{A_n(x)}{n} \int_{n\xi}^{\infty} \left(\frac{\sin \theta}{\theta} \right)^3 d\theta \quad (\xi > 0). \end{aligned}$$

$$\begin{aligned} \text{Further,} \quad \int_{\eta}^{\infty} d\xi \int_{\xi}^{\infty} F(\theta) d\theta &= \sum_{n=1}^{\infty} \frac{A_n(x)}{n} \int_{\eta}^{\infty} d\xi \int_{n\xi}^{\infty} \left(\frac{\sin \theta}{\theta} \right)^3 \\ &= \sum_{n=1}^{\infty} \frac{A_n(x)}{n^2} \int_{n\eta}^{\infty} d\xi \int_{\xi}^{\infty} \left(\frac{\sin \theta}{\theta} \right)^3 d\theta \quad (\eta > 0). \end{aligned}$$

Hence, inverting the order of integration,

$$\int_{\eta}^{\infty} F(\theta) d\theta \int_{\eta}^{\theta} d\xi = \sum_{n=1}^{\infty} \frac{A_n(x)}{n^2} \int_{n\eta}^{\infty} \left(\frac{\sin \theta}{\theta} \right)^3 d\theta \int_{n\eta}^{\theta} d\xi.$$

The first member tends to a limit as $\eta \rightarrow 0$ in virtue of the above-mentioned properties of $F(\theta)$. Denoting this limit by $A(x)$, we have

$$\lim_{\eta \rightarrow 0} \sum_{n=1}^{\infty} \frac{A_n(x)}{n^2} \int_{n\eta}^{\infty} \left(\frac{\sin \theta}{\theta} \right)^3 d\theta \int_{n\eta}^{\theta} d\xi = A(x).$$

* Chittenden, *Trans. American Math. Soc.*, 20 (1919), 179-184.

We can now show that (7) is convergent. Let ν be an integer such that

$$|A_n(x)| < \epsilon n \quad (n > \nu),$$

where ϵ is an assigned positive number; and let N_η be the integer such that

$$\eta N_\eta \leq 1 < \eta(N_\eta + 1).$$

Then

$$\lim_{\eta \rightarrow 0} \sum_{n=1}^{\nu} \left\{ \frac{A_n(x)}{n^2} \int_0^\infty \left(\frac{\sin \theta}{\theta} \right)^3 d\theta \int_0^\theta d\xi - \frac{A_n(x)}{n^2} \int_{n\eta}^\infty \left(\frac{\sin \theta}{\theta} \right)^3 d\theta \int_{n\eta}^\theta d\xi \right\} = 0.$$

Next,

$$\begin{aligned} \int_0^\infty \left(\frac{\sin \theta}{\theta} \right)^3 d\theta \int_0^\theta d\xi - \int_{n\eta}^\infty \left(\frac{\sin \theta}{\theta} \right)^3 d\theta \int_{n\eta}^\theta d\xi \\ = \int_0^\infty \left(\frac{\sin \theta}{\theta} \right)^3 d\theta \int_0^{n\eta} d\xi - \int_0^{n\eta} \left(\frac{\sin \theta}{\theta} \right)^3 d\theta \int_0^{n\eta} d\xi, \end{aligned}$$

so that

$$\begin{aligned} \left| \sum_{n=\nu+1}^{N_\eta} \left\{ \frac{A_n(x)}{n^2} \int_0^\infty \left(\frac{\sin \theta}{\theta} \right)^3 d\theta \int_0^\theta d\xi - \frac{A_n(x)}{n^2} \int_{n\eta}^\infty \left(\frac{\sin \theta}{\theta} \right)^3 d\theta \int_{n\eta}^\theta d\xi \right\} \right| \\ < \epsilon \sum_{n=\nu+1}^{N_\eta} \frac{1}{n} \left[\left| \int_0^\infty \left(\frac{\sin \theta}{\theta} \right)^3 d\theta \int_0^{n\eta} d\xi \right| + \left| \int_0^{n\eta} \left(\frac{\sin \theta}{\theta} \right)^3 d\theta \int_0^{n\eta} d\xi \right| \right] \\ < K\epsilon\eta N_\eta, \end{aligned}$$

where K is a constant independent of ϵ and η .

Further,

$$\begin{aligned} \left| \sum_{n=N_\eta+1}^\infty \frac{A_n(x)}{n^2} \int_{n\eta}^\infty \left(\frac{\sin \theta}{\theta} \right)^3 d\theta \int_{n\eta}^\theta d\xi \right| \\ < \epsilon \sum_{n=N_\eta+1}^\infty \frac{1}{n} \cdot \frac{1}{n\eta} \\ < C\epsilon, \end{aligned}$$

where C is a constant independent of ϵ and η . Thus

$$\lim_{\eta \rightarrow 0} \left[\sum_{n=1}^{N_\eta} \frac{A_n(x)}{n^2} \int_0^\infty \left(\frac{\sin \theta}{\theta} \right)^3 d\theta \int_0^\theta d\xi - \sum_{n=1}^\infty \frac{A_n(x)}{n^2} \int_{n\eta}^\infty \left(\frac{\sin \theta}{\theta} \right)^3 d\theta \int_{n\eta}^\theta d\xi \right] = 0.$$

Since the second term of the first member tends to $A(x)$, this proves the convergence of (7).

We now write $F(\theta, x)$ for $F(\theta)$ to emphasize the dependence on x , and we have, in virtue of what precedes,

$$\lim_{\eta \rightarrow 0} \int_\eta^\infty F(\theta, x) d\theta \int_\eta^\theta d\xi = A(x),$$

or, what is the same, $\int_0^\infty F(\theta, x) d\theta \int_0^\theta d\xi = A(x).$

To show that the expression (7) possesses the property R , it is sufficient to show this for $A(x)$. The result will follow if we can prove that

$$\int_{\eta}^{\infty} F(\theta, x) d\theta \int_{\eta}^{\theta} d\xi$$

converges relatively uniformly to

$$\int_0^{\infty} F(\theta, x) d\theta \int_0^{\theta} d\xi$$

as $\eta \rightarrow 0$. Now

$$\begin{aligned} \int_0^{\infty} F(\theta, x) d\theta \int_0^{\theta} d\xi - \int_{\eta}^{\infty} F(\theta, x) d\theta \int_{\eta}^{\theta} d\xi \\ = \int_0^{\infty} F(\theta, x) d\theta \int_0^{\eta} d\xi - \int_0^{\eta} F(\theta, x) d\theta \int_0^{\eta} d\xi, \end{aligned}$$

which is less in absolute value than

$$2\eta \int_0^{\infty} |F(\theta, x)| d\theta,$$

the infinite integral being finite for each value of x . Hence the result.

14. We can now prove the main theorem.

THEOREM IV. *If the trigonometric series*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \{a_n = o(n), b_n = o(n)\} \quad (8)$$

have its upper and lower sums $(R, 3)$ finite and equivalent to a function $f(x)$ which is integrable in the sense of Denjoy-Perron in $(0, 2\pi)$, then (8) is the Fourier series of $f(x)$.

We may suppose, without loss of generality, that $a_0 = 0$. Write

$$\Phi(x) = - \sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n^3}. \quad (9)$$

$$\text{Then, by Lemma 7,} \quad - \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2} \quad (10)$$

is everywhere convergent and possesses the property R . Since the coefficients in (10) are $o(1/n)$ it follows from a well known theorem of Fatou that

$$\Phi'(x) = - \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2}. \quad (11)$$

Since the upper and lower sums $(R, 3)$ of (8) are $\bar{D}^3\Phi(x)$, $\underline{D}^3\Phi(x)$ respectively, it follows, by Theorem III, that

$$\Phi'(x) = \int_0^x dt \int_0^t f(\theta) d\theta + Ax + B, \quad (12)$$

where A and B are constant in an arbitrary interval $(0, X)$. From (11) and (12) it follows that

$$\sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n} \sim \int_0^x f(\theta) d\theta + A.$$

There is now no difficulty in showing that

$$\int_0^{2\pi} f(\theta) d\theta = 0,$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt dt, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt dt \quad (n = 1, 2, \dots).$$

A METHOD OF INTEGRATING THE GENERAL FORM OF A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER IN TWO DEPENDENT AND TWO INDEPENDENT VARIABLES

By M. KOURENSKY.

[Received 8 July, 1929.—Read 14 November, 1929.—Revised 12 February, 1930.]

1. With the given system of two equations of the first order

$$\left. \begin{aligned} F_1(x_1, x_2; z_1, z_2; p_1^1, p_2^1; p_1^2, p_2^2) &= 0 \\ F_2(x_1, x_2; z_1, z_2; p_1^1, p_2^1; p_1^2, p_2^2) &= 0 \end{aligned} \right\}, \quad (1)$$

containing all four derivatives p_i^k , where

$$p_i^k = \frac{\partial z_k}{\partial x_i} \quad (i, k = 1, 2),$$

we combine a third equation

$$\Phi(x_1, x_2; z_1, z_2; p_1^1, p_2^1; p_1^2, p_2^2) = c, \quad (2)$$

where c is a constant, such that the three equations (1) and (2) are compatible.

Writing

$$\begin{aligned} {}_gX_h &= \frac{\partial F_g}{\partial x_h} + \frac{\partial F_g}{\partial z_1} p_h^1 + \frac{\partial F_g}{\partial z_2} p_h^2, & {}_3X_h &= \frac{\partial \Phi}{\partial x_h} + \frac{\partial \Phi}{\partial z_1} p_h^1 + \frac{\partial \Phi}{\partial z_2} p_h^2, \\ {}_gP_h^k &= \frac{\partial F_g}{\partial p_h^k}, & {}_3P_h^k &= \frac{\partial \Phi}{\partial p_h^k} \quad (g, h, k = 1, 2), \end{aligned}$$

we obtain the conditions of compatibility by taking the matrix

$$\begin{vmatrix} {}_3X_1 & {}_3P_2^2 & {}_3P_2^1 & {}_3P_1^1 & {}_3P_1^2 & 0 & 0 \\ {}_1X_1 & {}_1P_2^2 & {}_1P_2^1 & {}_1P_1^1 & {}_1P_1^2 & 0 & 0 \\ {}_2X_1 & {}_2P_2^2 & {}_2P_2^1 & {}_2P_1^1 & {}_2P_1^2 & 0 & 0 \\ {}_1X_2 & {}_1P_1^2 & {}_1P_1^1 & 0 & 0 & {}_1P_2^1 & {}_1P_2^2 \\ {}_2X_2 & {}_2P_1^2 & {}_2P_1^1 & 0 & 0 & {}_2P_2^1 & {}_2P_2^2 \\ {}_3X_2 & {}_3P_1^2 & {}_3P_1^1 & 0 & 0 & {}_3P_2^1 & {}_3P_2^2 \end{vmatrix}$$

and selecting from it a non-vanishing determinant Δ of the fifth order and equating to zero the two determinants of the sixth order, obtained by adding the remaining row and each of the two remaining columns separately*.

For instance, if Δ is formed from the last five rows and columns of the matrix, we have the conditions

$$D\left(\frac{F_1, F_2}{p_1^1, p_1^2}\right) \neq 0, \quad D\left(\frac{F_1, F_2, \Phi}{p_1^1, p_1^2, p_2^2}\right) \neq 0, \quad (3)$$

where D denotes the Jacobian, while the unknown function Φ must satisfy the system of two equations

$$\left. \begin{aligned} D\left(\frac{F_1, F_2, \Phi}{p_2^1, p_2^2, x_2}\right) D\left(\frac{F_1, F_2, \Phi}{p_1^1, p_1^2, p_2^1}\right) &= D\left(\frac{F_1, F_2, \Phi}{p_1^1, p_1^2, x_1}\right) D\left(\frac{F_1, F_2, \Phi}{p_2^1, p_2^2, p_1^1}\right) \\ D\left(\frac{F_1, F_2, \Phi}{p_2^1, p_2^2, x_2}\right) D\left(\frac{F_1, F_2, \Phi}{p_1^1, p_1^2, p_2^2}\right) &= D\left(\frac{F_1, F_2, \Phi}{p_1^1, p_1^2, x_1}\right) D\left(\frac{F_1, F_2, \Phi}{p_2^1, p_2^2, p_1^2}\right) \end{aligned} \right\}, \quad (4)$$

so that

$$\left| \begin{aligned} &D\left(\frac{F_1, F_2, \Phi}{p_1^1, p_1^2, p_2^1}\right) D\left(\frac{F_1, F_2, \Phi}{p_2^1, p_2^2, p_1^1}\right) \\ &D\left(\frac{F_1, F_2, \Phi}{p_1^1, p_1^2, p_2^2}\right) D\left(\frac{F_1, F_2, \Phi}{p_2^1, p_2^2, p_1^2}\right) \end{aligned} \right| = 0.$$

If, however, we form Δ from the first five rows and the same columns as

* E. von Weber, *Math. Annalen*, 49 (1897), 444–450. M. Kourensky, “Sur l’intégration des équations différentielles aux dérivées partielles avec plusieurs variables dépendantes”, *Mem. de la Classe Phys.-Math., Acad. des Sci. de l’Ukraine* 5 (1927), No. 3, ch. 4.

before, we replace the inequalities (3) by

$$D\left(\frac{F_1}{p_1}, \frac{F_2}{p_2}\right) \neq 0, \quad D\left(\frac{F_1}{p_1}, \frac{F_2}{p_2}, \Phi\right) \neq 0. \quad (5)$$

The unknown function Φ is consequently defined by the system of two equations (4), of the first order, in one dependent variable Φ and eight independent variables

$$x_1, x_2; z_1, z_2; p_1^1, p_2^1; p_1^2, p_2^2.$$

If the system (4) can be integrated, then every integral of the system which is of the form $\Phi_1 = c_1$, where c_1 is an arbitrary constant, gives a third equation which is compatible with the given system (1).

2. For brevity we write

$$D\left(\frac{F_1}{p_i^k}, \frac{F_2}{p_j^l}\right) = D_{i,j}^{k,l}; \quad D\left(\frac{F_1}{p_i^k}, \frac{F_2}{x_j}\right) = D_{i,j}^{k,0} \quad (i, j = 1, 2; \quad k, l = 1, 2),$$

so that $D_{i,j}^{k,l} = -D_{j,i}^{l,k}, \quad D_{i,j}^{k,0} = -D_{i,0}^{0,k}.$

The system (4) is then expressible in the form

$$\begin{aligned} \frac{D_{22}^{20} \frac{\partial \Phi}{\partial p_2} + D_{22}^{01} \frac{\partial \Phi}{\partial p_2^2} + D_{22}^{12} \frac{d\Phi}{dx_2}}{D_{11}^{20} \frac{\partial \Phi}{\partial p_1} + D_{11}^{01} \frac{\partial \Phi}{\partial p_1^2} + D_{11}^{12} \frac{d\Phi}{dx_1}} &= \frac{D_{21}^{21} \frac{\partial \Phi}{\partial p_2} + D_{12}^{11} \frac{\partial \Phi}{\partial p_2^2} + D_{22}^{12} \frac{\partial \Phi}{\partial p_1}}{D_{12}^{21} \frac{\partial \Phi}{\partial p_1} + D_{21}^{11} \frac{\partial \Phi}{\partial p_1^2} + D_{11}^{12} \frac{\partial \Phi}{\partial p_2}} \\ &= \frac{D_{21}^{22} \frac{\partial \Phi}{\partial p_2} + D_{12}^{21} \frac{\partial \Phi}{\partial p_2^2} + D_{22}^{12} \frac{\partial \Phi}{\partial p_1^2}}{D_{12}^{22} \frac{\partial \Phi}{\partial p_1} + D_{21}^{21} \frac{\partial \Phi}{\partial p_1^2} + D_{11}^{12} \frac{\partial \Phi}{\partial p_2^2}} = \lambda, \end{aligned} \quad (6)$$

where $\frac{d\Phi}{dx_i} = \frac{\partial \Phi}{\partial x_i} + \frac{\partial \Phi}{\partial z_1} p_i^1 + \frac{\partial \Phi}{\partial z_2} p_i^2 \quad (i = 1, 2).$

The last two equations of the system (6) reduce to

$$\left. \begin{aligned} (\lambda D_{12}^{21} + D_{22}^{21}) \frac{\partial \Phi}{\partial p_1} + (\lambda D_{11}^{12} + D_{12}^{12}) \frac{\partial \Phi}{\partial p_2} + \lambda D_{21}^{11} \frac{\partial \Phi}{\partial p_1^2} + D_{21}^{11} \frac{\partial \Phi}{\partial p_2^2} &= 0 \\ \lambda D_{12}^{22} \frac{\partial \Phi}{\partial p_1} + D_{12}^{22} \frac{\partial \Phi}{\partial p_2} + (\lambda D_{21}^{21} + D_{22}^{21}) \frac{\partial \Phi}{\partial p_1^2} + (\lambda D_{11}^{12} + D_{21}^{12}) \frac{\partial \Phi}{\partial p_2^2} &= 0 \end{aligned} \right\} \quad (7)$$

The coefficients of $\frac{\partial \Phi}{\partial p_1}, \frac{\partial \Phi}{\partial p_2}, \frac{\partial \Phi}{\partial p_1^2}, \frac{\partial \Phi}{\partial p_2^2}$ in (7) being proportional, we see

that, *subject to the limitations*

$$D_{11}^{22} \neq 0; \quad D_{12}^{12} \neq 0; \quad D_{21}^{11} \neq 0, \quad (8)$$

we have the quadratic equation

$$D_{11}^{12} \lambda^2 + (D_{21}^{12} + D_{12}^{12}) \lambda + D_{22}^{12} = 0 \quad (9)$$

to be satisfied by λ .

Denoting the roots of this equation by λ_1 and λ_2 , we find, as a result of the first ratio in (6) combined with (7), that $\Phi_1 = c_1$ is an integral of one of the systems of two equations of the first order

$$\left. \begin{aligned} \lambda_1 D_{11}^{12} \left(\frac{\partial \Phi}{\partial x_1} + \frac{\partial \Phi}{\partial z_1} p_1^1 + \frac{\partial \Phi}{\partial z_2} p_1^2 \right) + D_{22}^{21} \left(\frac{\partial \Phi}{\partial x_2} + \frac{\partial \Phi}{\partial z_1} p_2^1 + \frac{\partial \Phi}{\partial z_2} p_2^2 \right) \\ + \lambda_1 D_{11}^{20} \frac{\partial \Phi}{\partial p_1^1} + D_{22}^{02} \frac{\partial \Phi}{\partial p_2^1} + \lambda_1 D_{11}^{01} \frac{\partial \Phi}{\partial p_1^2} + D_{22}^{10} \frac{\partial \Phi}{\partial p_2^2} = 0, \\ (\lambda_1 D_{12}^{21} + D_{22}^{21}) \frac{\partial \Phi}{\partial p_1^1} + (\lambda_1 D_{11}^{12} + D_{12}^{12}) \frac{\partial \Phi}{\partial p_2^1} + \lambda_1 D_{21}^{11} \frac{\partial \Phi}{\partial p_1^2} + D_{21}^{11} \frac{\partial \Phi}{\partial p_2^2} = 0. \end{aligned} \right\} \quad (I)$$

$$\left. \begin{aligned} \lambda_2 D_{11}^{12} \left(\frac{\partial \Phi}{\partial x_1} + \frac{\partial \Phi}{\partial z_1} p_1^1 + \frac{\partial \Phi}{\partial z_2} p_1^2 \right) + D_{22}^{21} \left(\frac{\partial \Phi}{\partial x_2} + \frac{\partial \Phi}{\partial z_1} p_2^1 + \frac{\partial \Phi}{\partial z_2} p_2^2 \right) \\ + \lambda_2 D_{11}^{20} \frac{\partial \Phi}{\partial p_1^1} + D_{22}^{02} \frac{\partial \Phi}{\partial p_2^1} + \lambda_2 D_{11}^{01} \frac{\partial \Phi}{\partial p_1^2} + D_{22}^{10} \frac{\partial \Phi}{\partial p_2^2} = 0, \\ \lambda_2 D_{12}^{22} \frac{\partial \Phi}{\partial p_1^1} + D_{12}^{22} \frac{\partial \Phi}{\partial p_2^1} + (\lambda_2 D_{21}^{21} + D_{22}^{21}) \frac{\partial \Phi}{\partial p_1^2} + (\lambda_2 D_{11}^{12} + D_{21}^{12}) \frac{\partial \Phi}{\partial p_2^2} = 0. \end{aligned} \right\} \quad (II)$$

If the roots of equation (9) are equal, the systems (I) and (II) coincide.

If it is possible to integrate even one of the systems* (I) and (II), then any integral of the form $\Phi_1 = c_1$ of the system (I) or (II) may be taken as equation (2), compatible with the given system (1). Any other integral $\Phi_2 = c_2$, derived either from the system (I) or from the system (II), gives a new equation which is compatible with the system (1) and with the equation $\Phi_1 = c_1$, since the system of two equations $\Phi_1 = c_1$, $\Phi_2 = c_2$ containing derivatives p_i^k of different variables z_1 and z_2 will be compatible†. If the three equations $F_1 = 0$, $\Phi_1 = c_1$, $\Phi_2 = c_2$ are compatible,

The Jacobi-Poisson conditions must be satisfied by every pair of equations.

† E. von Weber, *loc. cit.*, 449. M. Kourensky, *loc. cit.*, 72. M. Kourensky, "Die Grundformeln zur Integration . . .", *Sitzungsberichte der M.-N.-Är. S. Sewčenko Gesellsch.*, Lemberg, 1929, H.XI., 6.

and also the three equations $F_2 = 0$, $\Phi_1 = c_1$, $\Phi_2 = c_2$, then, by solving the four equations

$$F_1 = 0, \quad F_2 = 0; \quad \Phi_1 = c_1, \quad \Phi_2 = c_2$$

for the derivatives p_i^k , we can determine the unknowns z_1 and z_2 by quadratures from the equations

$$dz_k = p_1^k dx_1 + p_2^k dx_2 \quad (k = 1, 2),$$

where we shall have

$$\frac{dp_1^1}{dx_2} = \frac{dp_2^1}{dx_1}, \quad \frac{dp_1^2}{dx_2} = \frac{dp_2^2}{dx_1}.$$

3. If the conditions (8) are not satisfied, there are various possibilities to be considered. When we have

$$D_{12}^{22} \neq 0, \quad D_{12}^{12} \neq 0, \quad D_{21}^{11} = 0,$$

then

$$\lambda_3 = \frac{D_{22}^{12}}{D_{12}^{22}} = \frac{D_{21}^{21}}{D_{11}^{12}},$$

$$\left. \begin{aligned} &\lambda_3 D_{11}^{12} \frac{d\Phi}{dx_1} + D_{21}^{21} \frac{d\Phi}{dx_2} \\ &+ \lambda_3 D_{11}^{20} \frac{\partial \Phi}{\partial p_1^1} + D_{22}^{02} \frac{\partial \Phi}{\partial p_2^1} + \lambda_3 D_{11}^{01} \frac{\partial \Phi}{\partial p_1^2} + D_{22}^{10} \frac{\partial \Phi}{\partial p_2^2} = 0, \\ &\lambda_3 D_{12}^{22} \frac{\partial \Phi}{\partial p_1^1} + D_{12}^{22} \frac{\partial \Phi}{\partial p_2^1} + (\lambda_3 D_{21}^{21} + D_{22}^{21}) \frac{\partial \Phi}{\partial p_1^2} + (\lambda_3 D_{11}^{12} + D_{21}^{12}) \frac{\partial \Phi}{\partial p_2^2} = 0. \end{aligned} \right\} \quad \text{(III)}$$

When we have

$$D_{12}^{22} = 0, \quad D_{12}^{12} \neq 0, \quad D_{21}^{11} \neq 0,$$

then

$$\lambda_4 = \frac{D_{22}^{12}}{D_{21}^{21}} = \frac{D_{12}^{21}}{D_{11}^{12}},$$

$$\left. \begin{aligned} &\lambda_4 D_{11}^{12} \frac{d\Phi}{dx_1} + D_{22}^{21} \frac{d\Phi}{dx_2} \\ &+ \lambda_4 D_{11}^{20} \frac{\partial \Phi}{\partial p_1^1} + D_{22}^{02} \frac{\partial \Phi}{\partial p_2^1} + \lambda_4 D_{11}^{01} \frac{\partial \Phi}{\partial p_1^2} + D_{22}^{10} \frac{\partial \Phi}{\partial p_2^2} = 0, \\ &(\lambda_4 D_{12}^{21} + D_{22}^{21}) \frac{\partial \Phi}{\partial p_1^1} + (\lambda_4 D_{11}^{12} + D_{12}^{12}) \frac{\partial \Phi}{\partial p_2^1} + \lambda_4 D_{21}^{11} \frac{\partial \Phi}{\partial p_1^2} + D_{21}^{11} \frac{\partial \Phi}{\partial p_2^2} = 0. \end{aligned} \right\} \quad \text{(IV)}$$

When we have

$$D_{12}^{22} = 0, \quad D_{12}^{12} \neq 0, \quad D_{21}^{11} = 0,$$

then
$$\frac{D_{22}^{12}}{D_{12}^{21}} = \frac{D_{21}^{21}}{D_{11}^{12}}, \quad \frac{D_{22}^{12}}{D_{21}^{21}} = \frac{D_{12}^{21}}{D_{11}^{12}}.$$

If the condition $D_{21}^{21} \neq D_{12}^{21}$

is satisfied, we are led to the integration of one only of the systems

$$\left. \begin{aligned} \lambda_3 D_{11}^{12} \frac{d\Phi}{dx_1} + D_{22}^{21} \frac{d\Phi}{dx_2} + \lambda_3 D_{11}^{20} \frac{\partial \Phi}{\partial p_1} + D_{22}^{02} \frac{\partial \Phi}{\partial p_2} + \lambda_3 D_{11}^{01} \frac{\partial \Phi}{\partial p_1^2} + D_{22}^{10} \frac{\partial \Phi}{\partial p_2^2} &= 0, \\ D_{22}^{12} \frac{\partial \Phi}{\partial p_1^2} + D_{12}^{21} \frac{\partial \Phi}{\partial p_2^2} &= 0, \end{aligned} \right\} \quad (V)$$

$$\left. \begin{aligned} \lambda_4 D_{11}^{12} \frac{d\Phi}{dx_1} + D_{22}^{21} \frac{d\Phi}{dx_2} + \lambda_4 D_{11}^{20} \frac{\partial \Phi}{\partial p_1} + D_{22}^{02} \frac{\partial \Phi}{\partial p_2} + \lambda_4 D_{11}^{01} \frac{\partial \Phi}{\partial p_1^2} + D_{22}^{10} \frac{\partial \Phi}{\partial p_2^2} &= 0, \\ D_{22}^{12} \frac{\partial \Phi}{\partial p_1^2} + D_{21}^{21} \frac{\partial \Phi}{\partial p_2^2} &= 0. \end{aligned} \right\} \quad (VI)$$

If, however, $D_{21}^{21} = D_{12}^{21}$,

then we are led to the integration of the first equation of either system (V) or system (VI).

When we have

$$D_{12}^{22} \neq 0, \quad D_{12}^{12} = 0, \quad D_{21}^{11} \neq 0,$$

then the quadratic for λ assumes the form

$$D_{11}^{12} \lambda^2 + D_{21}^{12} \lambda + D_{22}^{12} = 0;$$

the coefficient of $\partial \Phi / \partial p_2^1$ in the second equation of (I) will be $\lambda_1 D_{11}^{12}$, and the coefficient of $\partial \Phi / \partial p_1^2$ in the second equation of (II) will be D_{22}^{21} .

When we have

$$D_{12}^{22} \neq 0, \quad D_{12}^{12} = 0, \quad D_{21}^{11} = 0,$$

then $D_{22}^{12} = 0$; corresponding to the root $\lambda = 0$ we have one of the two equivalent equations

$$D_{22}^{02} \frac{\partial \Phi}{\partial p_2^1} + D_{22}^{10} \frac{\partial \Phi}{\partial p_2^2} = 0, \quad D_{12}^{22} \frac{\partial \Phi}{\partial p_2^1} + D_{21}^{12} \frac{\partial \Phi}{\partial p_2^2} = 0, \quad (VII)$$

and, corresponding to the root

$$\lambda_5 = \frac{D_{12}^{21}}{D_{11}^{12}} \neq 0,$$

we are led to the integration of the system

$$\left. \begin{aligned} \lambda_5 D_{11}^{12} \frac{\partial \Phi}{\partial x_1} + \lambda_5 D_{11}^{20} \frac{\partial \Phi}{\partial p_1} + D_{22}^{02} \frac{\partial \Phi}{\partial p_2} + \lambda_5 D_{11}^{01} \frac{\partial \Phi}{\partial p_1^2} + D_{22}^{10} \frac{\partial \Phi}{\partial p_2^2} &= 0, \\ D_{12}^{21} \frac{\partial \Phi}{\partial p_1} + D_{11}^{12} \frac{\partial \Phi}{\partial p_2} &= 0. \end{aligned} \right\} \quad \text{(VIII)}$$

When we have

$$D_{12}^{22} = 0, \quad D_{12}^{12} = 0, \quad D_{21}^{11} \neq 0,$$

then $D_{11}^{21} = 0$; the equation for λ consequently becomes linear; we have only the root

$$\lambda_6 = \frac{D_{22}^{12}}{D_{12}^{21}},$$

and the corresponding system

$$\left. \begin{aligned} D_{22}^{21} \frac{\partial \Phi}{\partial x_2} + \lambda_6 D_{11}^{20} \frac{\partial \Phi}{\partial p_1} + D_{22}^{02} \frac{\partial \Phi}{\partial p_2} + \lambda_6 D_{11}^{01} \frac{\partial \Phi}{\partial p_1^2} + D_{22}^{10} \frac{\partial \Phi}{\partial p_2^2} &= 0, \\ D_{22}^{21} \frac{\partial \Phi}{\partial p_1} + D_{21}^{12} \frac{\partial \Phi}{\partial p_2} &= 0. \end{aligned} \right\} \quad \text{(IX)}$$

Lastly, when we have

$$D_{12}^{22} = 0, \quad D_{12}^{12} = 0, \quad D_{21}^{11} = 0,$$

$$\text{then} \quad D_{11}^{21} = 0, \quad D_{22}^{12} = 0, \quad D_{21}^{12} = 0.$$

The limitations of (3) and (5), which have not been violated until now for the systems (I)–(IX), no longer hold, and the method of integration is no longer valid. However, in this exceptional case, the system (1) contains the derivatives p_i^k , and we deduce from (1) an equation of the form

$$f(x_1, x_2, z_1, z_2) = 0, \quad \text{(X)}$$

independent of all the p_i^k , which enables us to express z_2 in terms of z_1, x_1, x_2 ; after substituting this value of z_2 in one of the equations (1), we are led to the integration of an equation of the first order in one dependent variable z_1 .

4. As a special case, if even one of the four derivatives p_i^k is absent from the given system (1), the problem of integration is reduced to the discovery of two integrals of the form $\Phi_1 = c_1, \Phi_2 = c_2$ of only one linear equation of the first order with a dependent variable Φ .

This is easily verified by selecting conditions of compatibility from the corresponding matrix. For example, if the system (1) does not contain the derivate p_2^2 , it is necessary to seek for integrals, not containing this

derivate p_2^3 , of the equation

$$D_{11}^{12} \left(\frac{\partial \Phi}{\partial x_1} + \frac{\partial \Phi}{\partial z_1} p_1 + \frac{\partial \Phi}{\partial z_2} p_1^2 \right) + D_{21}^{12} \left(\frac{\partial \Phi}{\partial x_2} + \frac{\partial \Phi}{\partial z_1} p_2 + \frac{\partial \Phi}{\partial z_2} p_2^2 \right) \\ + D_{11}^{20} \frac{\partial \Phi}{\partial p_1} + D_{12}^{20} \frac{\partial \Phi}{\partial p_2} + (D_{22}^{01} + D_{11}^{01}) \frac{\partial \Phi}{\partial p_1^2} = 0.$$

If the system (1) does not contain the two derivatives p_2^1, p_2^2 , it is necessary to seek for integrals, containing but one of these derivatives, of the equation

$$D_{11}^{12} \left(\frac{\partial \Phi}{\partial x_1} + \frac{\partial \Phi}{\partial z_1} p_1 + \frac{\partial \Phi}{\partial z_2} p_1^2 \right) + D_{11}^{20} \frac{\partial \Phi}{\partial p_1} + D_{12}^{20} \frac{\partial \Phi}{\partial p_2} + D_{11}^{01} \frac{\partial \Phi}{\partial p_1^2} + D_{21}^{01} \frac{\partial \Phi}{\partial p_2^2} = 0,$$

and so on. Auxiliary equations for all of the eleven possible cases will be investigated in a subsequent paper.

This method of integration is applicable when there are several dependent variables $z_1, z_2, \dots, z_n$ and several independent variables $x_1, x_2, \dots, x_m$, as is explained in my paper* "Die Grundformeln zur Integration der Gleichungen mit partiellen Ableitungen 1 und 2 Ordnung mit mehreren abhängigen und unabhängigen Variablen".

5. The following examples, illustrative of my method, may be of interest.

Example 1. The given system is

$$\left. \begin{aligned} (x_1^2 x_2^2 - 1) p_1^2 + 2x_1(1 - x_1 x_2) p_1 - (1 - x_1 x_2)^2 p_2^2 + (1 + x_1 x_2)(z_1 - x_2 z_2) &= 0, \\ (x_1^2 x_2^2 - 1) p_2^2 + 2x_2(1 - x_1 x_2) p_2 + (1 - x_1 x_2)^2 p_1^2 + (1 + x_1 x_2)(z_2 - x_1 z_1) &= 0. \end{aligned} \right\}$$

The quadratic for λ is

$$(1 - x_1 x_2)^3 (1 + x_1 x_2) (\lambda^2 - 1) = 0.$$

The root $\lambda_1 = 1$ gives the system

$$\left. \begin{aligned} (1 - x_1^2 x_2^2)(1 - x_1 x_2)^2 \frac{d\Phi}{dx_1} + 2x_2(1 - x_1^2 x_2^2)(1 - x_1 x_2) \frac{d\Phi}{dx_2} \\ + D_{11}^{20} \frac{\partial \Phi}{\partial p_1} + D_{22}^{02} \frac{\partial \Phi}{\partial p_2} + D_{11}^{01} \frac{\partial \Phi}{\partial p_1^2} + D_{22}^{10} \frac{\partial \Phi}{\partial p_2^2} &= 0, \\ \frac{\partial \Phi}{\partial p_1} + \frac{\partial \Phi}{\partial p_2} + x_1 \frac{\partial \Phi}{\partial p_1^2} + x_1 \frac{\partial \Phi}{\partial p_2^2} &= 0, \end{aligned} \right\}$$

* *Sitzungsberichte der M.-N.-Är. S. Šewčenko Gesellsch., Lemberg* (1929), H. XI.

of which integrals are

$$\frac{xp_1^1 - p_2^2}{1 + x_1 x_2} = c_2, \quad \frac{(1 + x_1 x_2)p_2^1 + (2x_1 x_2 - x_1^2 - 1)p_1^1 + (x_1 - 3x_2)p_2^2}{1 + x_1 x_2} = c_4,$$

while the root $\lambda_2 = -1$ gives the system

$$\left. \begin{aligned} (x_1^2 x_2^2 - 1)(1 - x_1 x_2)^2 \frac{d\Phi}{dx_1} + 2x_2(1 - x_1^2 x_2^2)(1 - x_1 x_2) \frac{d\Phi}{dx_2} \\ + D_{11}^{02} \frac{\partial \Phi}{\partial p_1^1} + D_{22}^{02} \frac{\partial \Phi}{\partial p_2^1} + D_{11}^{10} \frac{\partial \Phi}{\partial p_1^2} + D_{22}^{10} \frac{\partial \Phi}{\partial p_2^2} = 0, \\ - x_2 \frac{\partial \Phi}{\partial p_1^1} + x_2 \frac{\partial \Phi}{\partial p_2^1} - \frac{\partial \Phi}{\partial p_1^2} + \frac{\partial \Phi}{\partial p_2^2} = 0, \end{aligned} \right\}$$

of which integrals are

$$\frac{x_2 p_2^2 + p_1^1}{1 + x_1 x_2} = c_1, \quad \frac{(1 + x_1 x_2)p_1^2 + (1 - 2x_1 x_2 - x_2^2)p_2^2 - (x_2 + 3x_1)p_1^1}{1 + x_1 x_2} = c_3.$$

Hence
$$\begin{aligned} p_1^1 &= c_1 + c_2 x_2, & p_2^1 &= c_1 + c_2(x_1 - x_2) - c_2 x_2 + c_4, \\ p_2^2 &= -c_2 + c_1 x_1, & p_1^2 &= c_2 + c_1(x_1 + x_2) + c_1 x_1 + c_3, \end{aligned}$$

and from these we obtain the complete integral

$$\left. \begin{aligned} z_1 &= c_1(x_1 + x_2) + c_2 x_2(x_1 - x_2) + c_4 x_2 + c_3, \\ z_2 &= c_2(x_1 - x_2) + c_1 x_1(x_1 + x_2) + c_3 x_1 + c_4, \end{aligned} \right\}$$

and the general integral

$$\left. \begin{aligned} z_1 &= \phi_1(x_1 + x_2) + x_2 \phi_2(x_1 - x_2), \\ z_2 &= \phi_2(x_1 - x_2) + x_1 \phi_1(x_1 + x_2). \end{aligned} \right\}$$

Example 2. The system is

$$\left. \begin{aligned} x_1 \frac{\partial z_1}{\partial x_1} + \frac{\partial z_2}{\partial x_1} &= z_1, \\ \frac{\partial z_1}{\partial x_1} + x_1 \frac{\partial z_2}{\partial x_1} &= z_2. \end{aligned} \right\}$$

This is a linear system, but Hamburger's method is not immediately applicable, since Hamburger's auxiliary quadratic equation vanishes identically. My methods give

$$\frac{dx_1}{1 - x_1^2} = \frac{dz_1}{p_1^1(1 - x_1^2)} = \frac{dz_2}{p_1^2(1 - x_1^2)} = \frac{dp_2^1}{p_2^2 - x_1 p_2^1} = \frac{dp_2^2}{p_2^1 - x_1 p_2^2} = \frac{d\Phi}{0},$$

whence

$$\frac{dp_2^1 - x_1 dp_2^2}{p_2^2 - 2x_1 p_2^1 + x_1^2 p_2^2} = \frac{dx_1}{1 - x_1^2}, \quad \frac{(1 - x_1^2) d(p_2^1 - x_1 p_2^2) - (p_2^1 - x_1 p_2^2) d(1 - x_1^2)}{(1 - x_1^2)^2} = 0,$$

and so
$$\frac{p_2^1 - x_1 p_2^2}{1 - x_1^2} = c_1, \quad \frac{p_2^2 - x_1 p_2^1}{1 - x_1^2} = c_2,$$

Hence

$$p_2^2 = c_1 x_1 + c_2, \quad p_2^1 = c_2 x_1 + c_1 = \frac{\partial z_1}{\partial x_2} = \frac{\partial p_2^2}{\partial x_1} + x_1 \frac{\partial p_2^1}{\partial x_2} = c_1 + x_1 \frac{\partial p_2^1}{\partial x_2},$$

so that

$$p_2^1 = c_2 x_2 + c_3,$$

and the complete integral is

$$\left. \begin{aligned} z_1 &= c_1 x_2 + c_2 x_1 x_2 + c_3 x_1 + c_4, \\ z_2 &= c_4 x_1 + c_1 x_1 x_2 + c_2 x_2 + c_3. \end{aligned} \right\}$$

SUR LES SOLUTIONS DE L'ÉQUATION DIFFÉRENTIELLE LINEAIRE DU SECOND ORDRE À COEFFICIENT PÉRIODIQUE

By Ž. MARKOVIĆ.

[Received 11 May, 1929—Read 16 May, 1929.]

1. Dans ce qui suit, nous considérons l'équation différentielle linéaire du second ordre à coefficient périodique sous des conditions générales, et surtout ses solutions périodiques. Nous établissons des relations qui nous permettent d'une part de simplifier les calculs à effectuer avec ces fonctions, de rendre plus maniables les expressions où elles entrent et de mettre en évidence, dans la mesure du possible, l'analogie de ces fonctions avec des fonctions circulaires, auxquelles d'ailleurs elles se réduisent dans un cas limite, et d'autre part de déduire quelques faits concernant les solutions périodiques d'une manière simple et directe. Les solutions périodiques sont étudiées à partir d'une équation intégrale homogène à noyau symétrique qui se présente naturellement et la question de l'existence simultanée de deux solutions périodiques, linéairement indépendantes, de même période est discutée dans le cas général ainsi que dans le cas où le coefficient de l'équation différentielle est une fonction paire. Quelques remarques de nature qualitative, déduites des formules générales, laissent entrevoir les circonstances qui se présentent dans l'allure des courbes représentatives des solutions périodiques.

Soit donnée l'équation différentielle

$$(1.1) \quad \frac{d^2 x}{dt^2} = \psi(t)x(t),$$

la fonction réelle $\psi(t)$ étant bornée, sommable et périodique de période π . Si $\psi(t)$ est paire, c'est l'équation de Hill; si ses coefficients de Fourier se réduisent aux deux premiers, c'est l'équation de Mathieu.

Considérons deux solutions particulières $s(t)$ et $c(t)$ de l'équation (1.1) satisfaisant aux conditions initiales

$$(1.2) \quad \begin{cases} s(0) = 0, & c(0) = 1, \\ \dot{s}(0) = 1, & \dot{c}(0) = 0, \end{cases}$$

et formant un système fondamental. On a entre ces fonctions et leurs dérivées l'identité

$$(1.3) \quad c(t)\dot{s}(t) - s(t)\dot{c}(t) = 1.$$

Les fonctions $s(t)$ et $c(t)$, ainsi que leurs dérivées $\dot{s}(t)$ et $\dot{c}(t)$, sont des fonctions absolument continues.

2. L'équation (1.1) restant inaltérée par l'introduction de la variable $t + n\pi$ ou $t - n\pi$ au lieu de t ($n = 1, 2, \dots$), les fonctions $s(t + n\pi)$, $c(t + n\pi)$; $s(t - n\pi)$, $c(t - n\pi)$ en sont aussi des solutions; en les exprimant donc comme combinaisons linéaires à coefficients constants des fonctions $s(t)$ et $c(t)$, on a les formules

$$(2.1) \quad \begin{cases} s(t + n\pi) = s(t)\dot{s}(n\pi) + c(t)s(n\pi), \\ c(t + n\pi) = c(t)c(n\pi) + s(t)\dot{c}(n\pi), \end{cases}$$

et celles qui en résultent en les différentiant

$$(2.2) \quad \begin{cases} \dot{s}(t + n\pi) = \dot{s}(t)\dot{s}(n\pi) + \dot{c}(t)s(n\pi), \\ \dot{c}(t + n\pi) = \dot{c}(t)c(n\pi) + \dot{s}(t)\dot{c}(n\pi), \end{cases}$$

et qui nous donnent pour $t = -n\pi$ et en tenant compte de (1.3)

$$(2.3) \quad \begin{cases} s(-n\pi) = -s(n\pi), & c(-n\pi) = \dot{s}(n\pi), \\ \dot{s}(-n\pi) = c(n\pi), & \dot{c}(-n\pi) = -\dot{c}(n\pi); \end{cases}$$

d'autre part les relations qui ayant égard à (2.2) s'écrivent

$$(2.4) \quad \begin{cases} s(t - n\pi) = s(t)c(n\pi) - c(t)s(n\pi), \\ c(t - n\pi) = c(t)\dot{s}(n\pi) - s(t)\dot{c}(n\pi). \end{cases}$$

Si la fonction $\psi(t)$ est paire, $s(t)$ est une fonction impaire, $c(t)$ une fonction paire et l'on a la relation $\dot{s}(n\pi) = c(n\pi)$ qui rend les calculs plus simples dans ce cas.

En résolvant le système (2.1) ou (2.4) par rapport à $s(t)$ et $c(t)$, on a

$$(2.5) \quad \begin{cases} s(t) = s(t + n\pi)c(n\pi) - c(t + n\pi)s(n\pi) \\ \quad = s(t - n\pi)\dot{s}(n\pi) + c(t - n\pi)s(n\pi), \\ \text{et} \\ c(t) = c(t + n\pi)\dot{s}(n\pi) - s(t + n\pi)\dot{c}(n\pi) \\ \quad = c(t - n\pi)c(n\pi) + s(t - n\pi)\dot{c}(n\pi). \end{cases}$$

D'une manière analogue on obtient les identités

$$(2.6) \quad \left\{ \begin{aligned} s(n\pi) &= s(t+n\pi) \dot{s}(t) - \dot{s}(t+n\pi) s(t) \\ &= s(t) \dot{s}(t-n\pi) - \dot{s}(t) s(t-n\pi), \\ \dot{s}(n\pi) &= c(t) \dot{s}(t+n\pi) - s(t+n\pi) \dot{c}(t) \\ &= c(t-n\pi) \dot{s}(t) - s(t) \dot{c}(t-n\pi), \\ c(n\pi) &= c(t+n\pi) \dot{s}(t) - s(t) \dot{c}(t+n\pi) \\ &= c(t) \dot{s}(t-n\pi) - \dot{c}(t) s(t-n\pi), \\ \dot{c}(n\pi) &= c(t) \dot{c}(t+n\pi) - c(t+n\pi) \dot{c}(t) \\ &= c(t-n\pi) \dot{c}(t) - c(t) \dot{c}(t-n\pi). \end{aligned} \right.$$

Toutes ces relations sont d'un emploi commode dans les calculs, en particulier dans ceux de la fonction de Green correspondant aux différentes conditions aux limites. Il en sera de même des relations suivantes obtenues en combinant les relations ci-dessus :

$$(2.7) \quad \left\{ \begin{aligned} s(t) c(\xi-n\pi) - c(t) s(\xi-n\pi) &= s(t+n\pi) c(\xi) - c(t+n\pi) s(\xi), \\ s(t) c(\xi+n\pi) - c(t) s(\xi+n\pi) &= s(t-n\pi) c(\xi) - c(t-n\pi) s(\xi). \end{aligned} \right.$$

3. Supposons que la fonction $\psi(t)$ se présente sous forme

$$\psi(t) = \lambda + \psi_1(t),$$

λ étant un paramètre réel, et que l'on a, ce qu'on peut toujours admettre,

$\int_0^\pi \psi_1 dt = 0$, et considérons en même temps que (1.1) l'équation

$$(3.1) \quad \frac{d^2 x}{dt^2} = \psi_1(t) x(t).$$

Soient $\sigma(t)$ et $\gamma(t)$ deux solutions particulières de cette équation, définies par les conditions initiales

$$(3.2) \quad \left\{ \begin{aligned} \sigma(0) &= 0, & \gamma(0) &= 1, \\ \dot{\sigma}(0) &= 1, & \dot{\gamma}(0) &= 0. \end{aligned} \right.$$

En faisant la combinaison intégrable d'une solution $x(t)$ de l'équation (1.1) et d'une solution $x_1(t)$ de l'équation (3.1), on trouve

$$x_1 \dot{x} - x \dot{x}_1 = \lambda \int_0^t x(\xi) x_1(\xi) d\xi + C;$$

on a donc en particulier les relations

$$(3.3) \quad \begin{cases} \sigma(t) \dot{s}(t) - s(t) \dot{\sigma}(t) = \lambda \int_0^t \sigma(\xi) s(\xi) d\xi, \\ \gamma(t) \dot{s}(t) - s(t) \dot{\gamma}(t) = 1 + \lambda \int_0^t \gamma(\xi) s(\xi) d\xi, \end{cases}$$

et

$$(3.4) \quad \begin{cases} \sigma(t) \dot{c}(t) - \dot{\sigma}(t) c(t) = -1 + \lambda \int_0^t \sigma(\xi) c(\xi) d\xi, \\ \gamma(t) \dot{c}(t) - \dot{\gamma}(t) c(t) = \lambda \int_0^t \gamma(\xi) c(\xi) d\xi. \end{cases}$$

Des équations (3.3) et (3.4), linéaires en $s(t)$ et $c(t)$, on tire en les résolvant les expressions

$$(3.5) \quad \begin{cases} s(t) = \sigma(t) + \lambda \int_0^t [\sigma(t) \gamma(\xi) - \gamma(t) \sigma(\xi)] s(\xi) d\xi, \\ c(t) = \gamma(t) + \lambda \int_0^t [\sigma(t) \gamma(\xi) - \gamma(t) \sigma(\xi)] c(\xi) d\xi, \end{cases}$$

qui constituent deux équations intégrales de seconde espèce du type de Volterra pour les fonctions $s(t)$ et $c(t)$, au noyau $\sigma(t) \gamma(\xi) - \gamma(t) \sigma(\xi)$, les termes connus étant $\sigma(t)$ et $\gamma(t)$. Résolues par les méthodes de Volterra, elles nous donnent les fonctions $s(t)$ et $c(t)$ comme fonctions entières du paramètre λ , se réduisant aux fonctions $\sigma(t)$ et $\gamma(t)$ pour $\lambda = 0$. Elles vont nous permettre d'écrire d'une manière simple l'équation intégrale des solutions périodiques.

4. On obtient des formules que nous allons utiliser par la suite en faisant des combinaisons intégrables de deux solutions $x_1(t)$ et $x_2(t)$ de l'équation (1.1). Multiplions l'équation $\ddot{x}_1 = \psi(t) x_1$ par $\dot{x}_2$ et l'équation $\ddot{x}_2 = \psi(t) x_2$ par $\dot{x}_1$ et intégrons la somme des produits obtenus. On aura

$$(4.1) \quad \dot{x}_1 \dot{x}_2 \Big|_0^t = \int_0^t \psi(\xi) d(x_1 x_2)$$

et pour $x_1 = x_2$,

$$(4.2) \quad \dot{x}_1^2 \Big|_0^t = \int_0^t \psi(\xi) d(x_1^2).$$

D'autre part, en multipliant l'équation $\ddot{x} = \psi(t) x$ par x et en intégrant, on a, après une intégration par parties effectuée au premier membre,

$$(4.3) \quad x \dot{x} \Big|_0^t - \int_0^t \dot{x}^2 d\xi = \int_0^t \psi(\xi) x^2(\xi) d\xi.$$

Dans les cas où la fonction $\psi(t)$ est absolument continue, il résulte de (4.1), en intégrant par parties,

$$(4.4) \quad \dot{x}_1 \dot{x}_2 \Big|_0^t = \psi(t) x_1(t) x_2(t) \Big|_0^t - \int_0^t \dot{\psi}(\xi) x_1(\xi) x_2(\xi) d\xi,$$

et de (4.2)

$$(4.5) \quad \dot{x}_1^2 \Big|_0^t = \psi(t) x_1^2(t) \Big|_0^t - \int_0^t \dot{\psi}(\xi) x_1^2(\xi) d\xi.$$

Notons encore que le produit $y(t) = x_1(t) x_2(t)$ de deux solutions de l'équation (1.1) satisfait à l'équation intégral-différentielle suivante :

$$\ddot{y} = 2\psi y + 2 \int_0^t \psi(\xi) \dot{y}(\xi) d\xi,$$

obtenue à l'aide de la relation (4.1) et qui, en introduisant la fonction $\dot{y}(t) = \phi(t)$, se ramène à l'équation intégrale de Volterra de seconde espèce

$$(4.6) \quad \phi(t) = 1 + \int_0^t K(t, \xi) \phi(\xi) d\xi,$$

avec le noyau $K(t, \xi) = 2 \left[\int_\xi^t \psi(\xi_1) d\xi_1 + (t - \xi) \psi(\xi) \right]$, pourvu que les conditions initiales pour la fonction $y(t)$ soient $y(0) = 0$, $\dot{y}(0) = 1$, $\ddot{y}(0) = 0$, comme le sera dans la suite.

5. Les relations qui existent entre les fonctions $s(t)$ et $c(t)$ et leurs dérivées sont données par les formules

$$(5.1) \quad \begin{cases} \dot{s}(t) = 1 + \int_0^t \psi(\xi) s(\xi) d\xi, \\ \dot{c}(t) = \int_0^t \psi(\xi) c(\xi) d\xi. \end{cases}$$

Pour avoir les relations qui existent entre $\dot{c}(t)$ et $s(t)$ d'une part et $\dot{s}(t)$ et $c(t)$ d'autre part, multiplions (1.3) par $\dot{c}(t)$ et $\dot{s}(t)$ respectivement, en nous servant des relations (4.1) et (4.2); on trouve ainsi les formules

$$(5.2) \quad \dot{c}(t) = \lambda s(t) + \int_0^t \psi_1(\xi) d[\{c(t) s(\xi) - s(t) c(\xi)\} c(\xi)]$$

et

$$(5.3) \quad \dot{s}(t) = c(t) + \int_0^t \psi_1(\xi) d[\{c(t) s(\xi) - s(t) c(\xi)\} s(\xi)].$$

En substituant (5.2) et (5.3) dans (1.3), on a l'identité

$$(5.4) \quad c^2(t) - \lambda s^2(t) + \int_0^t \psi_1(\xi) d[c(t)s(\xi) - s(t)c(\xi)]^2 = 1.$$

Si $\psi_1(t) \equiv 0$ et $\lambda < 0$, les formules (5.2)–(5.4) se réduisent aux formules correspondantes pour les fonctions circulaires $(\sin \lambda^{\frac{1}{2}}t)/\lambda^{\frac{1}{2}}$ et $\cos \lambda^{\frac{1}{2}}t$, définies par les mêmes conditions initiales que les fonctions $s(t)$ et $c(t)$, et si $\lambda > 0$ elles se réduisent aux relations analogues pour les fonctions hyperboliques correspondantes.

6. Les formules (2.1) et (2.4) donnent pour la valeur π l'analogie des théorèmes d'addition pour les fonctions circulaires. Pour les valeurs quelconques de l'autre variable h on peut trouver une expression de $x(t+h)$ se réduisant pour $\psi_1(t) \equiv 0$ au théorème d'addition pour les fonctions circulaires.

En nous guidant par ce qu'il y a lieu dans le cas de l'équation de Mathieu, où $\psi(t)$ est une fonction analytique de t et où l'on obtient l'expression de $x(t+h)$ par le développement en série de Taylor, posons

$$(6.1) \quad x(t+h) = x(t)c(t, h) + \dot{x}(t)s(t, h),$$

où les fonctions à déterminer sont les deux fonctions $c(t, h)$ et $s(t, h)$, indépendantes de la solution particulière $x(t)$ de l'équation (1.1).

L'intégrale générale de l'équation (1.1) se présentant sous la forme

$$x(t) = x(0)c(t) + \dot{x}(0)s(t),$$

on a de (6.1) pour $t = 0$

$$c(0, h) = c(h), \quad s(0, h) = s(h),$$

$$\text{et pour } h = 0 \quad c(t, 0) = 1, \quad s(t, 0) = 0,$$

$$\dot{c}_h(t, 0) = 0, \quad \dot{s}_h(t, 0) = 1.$$

La fonction $x(t+h)$ étant une fonction symétrique de t et h , on a

$$\frac{\partial x(t+h)}{\partial t} = \frac{\partial x(t+h)}{\partial h},$$

d'où l'on déduit les relations suivantes

$$(6.2) \quad \begin{cases} \frac{\partial c}{\partial h} - \frac{\partial c}{\partial t} = \psi(t)s(t, h), \\ \frac{\partial s}{\partial h} - \frac{\partial s}{\partial t} = c(t, h), \end{cases}$$

formant un système d'équations linéaires aux dérivées partielles pour les fonctions $s(t, h)$ et $c(t, h)$.

Puisqu'on a

$$\ddot{x}(t+h) = \psi(t+h) x(t+h),$$

et que

$$\ddot{x}(t+h) = \frac{\partial^2 x(t+h)}{\partial t^2} = \frac{\partial^2 x(t+h)}{\partial h^2},$$

on a de (6.1)

$$\begin{aligned} \ddot{x}(t+h) &= x(t) \psi(t+h) c(t, h) + \dot{x}(t) \psi(t+h) s(t, h) \\ &= x(t) \frac{\partial^2 c(t, h)}{\partial h^2} + \dot{x}(t) \frac{\partial^2 s(t, h)}{\partial h^2}, \end{aligned}$$

d'où, en identifiant,

$$\frac{\partial^2 s(t, h)}{\partial h^2} = \psi(t+h) s(t, h),$$

$$\frac{\partial^2 c(t, h)}{\partial h^2} = \psi(t+h) c(t, h).$$

En ayant égard aux conditions initiales, on trouve que les fonctions $s(t, h)$ et $c(t, h)$ sont des solutions des équations intégrales linéaires du type de Volterra

$$(6.3) \quad \begin{cases} s(t, h) = h + \int_0^h (h-h_1) \psi(t+h_1) s(t, h_1) dh_1, \\ c(t, h) = 1 + \int_0^h (h-h_1) \psi(t+h_1) c(t, h_1) dh_1, \end{cases}$$

au même noyau $-(h-h_1)\psi(t+h_1)$, qui se réduisent pour $t=0$ aux équations intégrales des fonctions $s(h)$ et $c(h)$, conformément aux conditions initiales pour $t=0$. En résolvant donc les équations (6.3) par les méthodes de Volterra, on obtient l'expression explicite des fonctions cherchées. On vérifie sans peine que les fonctions ainsi obtenues satisfont aux équations différentielles (6.2).

Si la fonction $\psi(t)$ se réduit à une constante négative $-q^2$, les équations (6.3) montrent que les fonctions $s(t, h)$ et $c(t, h)$ ne dépendent plus de t et qu'elles se réduisent à $(\sin qh)/q$ et $\cos qh$ respectivement, donnant d'après (6.1) le théorème d'addition pour les fonctions circulaires.

On trouverait d'une manière analogue pour les fonctions $s(h, t)$ et $c(h, t)$ les équations différentielles

$$\frac{\partial^2 s(h, t)}{\partial t^2} = \psi(t+h) s(h, t),$$

$$\frac{\partial^2 c(h, t)}{\partial t^2} = \psi(t+h) c(h, t),$$

et les équations intégrales correspondantes.

Des équations (6.3) il résulte, en introduisant $t+\pi$ au lieu de t , la périodicité des fonctions $s(t, h)$ et $c(t, h)$ par rapport à t ,

$$c(t+\pi, h) = c(t, h),$$

$$s(t+\pi, h) = s(t, h).$$

Si l'on écrit la relation (6.1) en particulier pour les fonctions $s(t)$ et $c(t)$ et qu'on en calcule les fonctions $s(t, h)$ et $c(t, h)$, on trouve

$$s(t, h) = c(t) s(t+h) - s(t) c(t+h),$$

$$c(t, h) = \dot{s}(t) c(t+h) - \dot{c}(t) s(t+h).$$

7. L'équation (1.1), étant une équation à coefficient périodique, peut admettre des solutions périodiques de période π ou d'un multiple de π . L'existence de ces solutions sera établie en partant d'une équation intégrale homogène du type de Fredholm à noyau symétrique que nous allons écrire.

Mais on peut indiquer à priori deux cas où l'existence de telles solutions est impossible. Pour que la solution $x(t)$ soit périodique de période π , il faut et il suffit que l'on ait $x(\pi) = x(0)$, $\dot{x}(\pi) = \dot{x}(0)$. Mais alors la relation (4.3) donne

$$\int_0^\pi \psi(\xi) x^2(\xi) d\xi = - \int_0^\pi \dot{x}^2(\xi) d\xi;$$

ainsi dans le cas de périodicité la constante $\int_0^\pi \psi(\xi) x^2(\xi) d\xi$ doit être négative. Par suite, si dans l'intervalle $(0, \pi)$ on a $\psi(t) \geq 0$, l'équation (1.1) n'admet pas des solutions périodiques.

Si d'autre part $\psi(t)$ est absolument continue, il résulte de la relation (4.5) que l'on a dans le cas de $x(t)$ périodique de période π

$$\int_0^\pi \dot{\psi}(\xi) x^2(\xi) d\xi = 0.$$

Donc si $\psi(t) \geq 0$ presque partout dans $(0, \pi)$, l'équation (1.1) ne saurait admettre des solutions périodiques.

Pour construire l'équation intégrale des solutions périodiques de l'équation (1.1), écrivons sa solution générale en faisant la combinaison linéaire à coefficients constants quelconques C_0 et C_1 des expressions (3.5); on a ainsi

$$x(t) = C_0 \gamma(t) + C_1 \sigma(t) + \lambda \int_0^t [\sigma(t) \gamma(\xi) - \gamma(t) \sigma(\xi)] x(\xi) d\xi.$$

La condition nécessaire et suffisante pour que $x(t)$ soit périodique de période π étant $x(0) = x(\pi)$, $\dot{x}(0) = \dot{x}(\pi)$, on a, en exprimant d'abord la

première condition et à l'aide de la relation (2.4), pour les solutions satisfaisant à cette condition l'équation intégrale

$$(7.1) \quad x(t) = C_0 \frac{\sigma(t) - \sigma(t - \pi)}{\sigma(\pi)} + \lambda \int_0^\pi G_1(t, \xi) x(\xi) d\xi,$$

à noyau

$$G_1(t, \xi) = \begin{cases} \frac{\sigma(t) \sigma(\xi - \pi)}{\sigma(\pi)} & (t \leq \xi), \\ \frac{\sigma(t - \pi) \sigma(\xi)}{\sigma(\pi)} & (t \geq \xi), \end{cases}$$

C_0 étant quelconque. Cette équation constitue la généralisation naturelle de l'équation intégrale correspondant au cas des fonctions circulaires, à laquelle elle se réduit pour $\psi_1(t) \equiv 0$, $\lambda < 0$.

En différentiant l'équation (7.1) et en exprimant la seconde condition $\dot{x}(0) = \dot{x}(\pi)$, on détermine C_0 qui porté dans (7.1) conduit, moyennant les relations (2.1), (2.5), (2.7), à l'équation intégrale homogène

$$(7.2) \quad x(t) = -\lambda \int_0^\pi G(t, \xi) x(\xi) d\xi,$$

où le noyau $G(t, \xi)$, qui est la fonction de Green de l'expression $x - \psi_1(t)x$ pour les conditions de périodicité, est donné par l'expression

$$(7.3) \quad G(t, \xi) = \begin{cases} \frac{[\sigma(\pi + t) - \sigma(t)] \gamma(\xi) - [\gamma(\pi + t) - \gamma(t)] \sigma(\xi)}{[\gamma(\pi) + \dot{\sigma}(\pi) - 2]} & (t \leq \xi), \\ \frac{\gamma(t) [\sigma(\pi + \xi) - \sigma(\xi)] - \sigma(t) [\gamma(\pi + \xi) - \gamma(\xi)]}{[\gamma(\pi) + \dot{\sigma}(\pi) - 2]} & (t \geq \xi). \end{cases}$$

Le noyau étant symétrique et fermé, il y a une infinité de valeurs caractéristiques réelles λ_n , ayant leur seul point d'accumulation à l'infini, auxquelles correspondent les fonctions caractéristiques $x_n(t)$ qui sont les solutions périodiques de l'équation (1.1), formant un système complet et liées par les relations d'orthogonalité

$$\int_0^\pi x_m(t) x_n(t) dt = 0 \quad (m \neq n).$$

La résolvante $\Gamma(t, \xi; \lambda)$ du noyau $G(t, \xi)$, d'après le théorème de Hilbert, étant la fonction de Green de l'expression $\ddot{x} - \psi(t)x$ pour les mêmes conditions de périodicité, a pour l'expression

$$(7.4) \quad \Gamma(t, \xi; \lambda) = \begin{cases} \frac{[s(\pi + t) - s(t)] c(\xi) - [c(\pi + t) - c(t)] s(\xi)}{[c(\pi) + \dot{s}(\pi) - 2]} & (t \leq \xi), \\ \frac{c(t) [s(\pi + \xi) - s(\xi)] - s(t) [c(\pi + \xi) - c(\xi)]}{[c(\pi) + \dot{s}(\pi) - 2]} & (t \geq \xi). \end{cases}$$

Il résulte de cette expression de la résolvante que le déterminant de Fredholm $D(\lambda)$ du noyau $G(t, \xi)$ est donné par

$$D(\lambda) = c(\pi) + \dot{s}(\pi) - 2,$$

dont les zéros sont les valeurs caractéristiques λ_n du noyau $G(t, \xi)$; tandis que le premier mineur de Fredholm est donné par

$$(7.5) \quad \Delta(t, \xi; \lambda) = \begin{cases} [s(\pi+t) - s(t)] c(\xi) - [c(\pi+t) - c(t)] s(\xi) & (t \leq \xi), \\ c(t) [s(\pi+\xi) - s(\xi)] - s(t) [c(\pi+\xi) - c(\xi)] & (t \geq \xi). \end{cases}$$

La formule
$$\frac{dD(\lambda)}{d\lambda} = - \int_0^\pi \Delta(t, t; \lambda) dt$$

s'écrit d'après (7.5)

$$\frac{dD(\lambda)}{d\lambda} = \int_0^\pi c(\pi+t) s(t) dt - \int_0^\pi s(\pi+t) c(t) dt,$$

ou bien, en vertu de (2.1)

$$\frac{dD(\lambda)}{d\lambda} = \int_0^\pi \{s(t) c(t) [c(\pi) - \dot{s}(\pi)] + s^2(t) \dot{c}(\pi) - c^2(t) s(\pi)\} dt.$$

Si la fonction $\psi(t)$ est paire, on a $c(\pi) = \dot{s}(\pi)$ (voir le no. 2) et l'on a ainsi

$$\frac{dD(\lambda)}{d\lambda} = \dot{c}(\pi) \int_0^\pi s^2(t) dt - s(\pi) \int_0^\pi c^2(t) dt.$$

Et comme $s(\pi)$ et $\dot{c}(\pi)$ ne peuvent pas s'annuler simultanément (voir le no. 9), on a $dD(\lambda)/d\lambda \neq 0$, et les racines λ_n sont toutes simples.

Le mineur $\Delta(t, \xi; \lambda)$ étant une solution de l'équation homogène (7.2), on peut vérifier que les solutions périodiques $s(t)$ et $c(t)$ satisfont à l'équation (7.2). Si $s(t)$ est périodique de période π , on a $c(\pi) = 1$ et alors la relation (2.1) ($n = 1$) nous donne

$$c(\pi+t) - c(t) = s(t) \dot{c}(\pi) \quad \text{et} \quad \Delta(t, \xi; \lambda) = -s(t) s(\xi) \dot{c}(\pi),$$

c'est à dire $s(t)$ multipliée par la constante $-s(\xi) \dot{c}(\pi)$ inessentielle. Pour $c(t)$ périodique, on a $\dot{s}(\pi) = 1$ et $s(\pi+t) - s(t) = c(t) s(\pi)$, par conséquent $\Delta(t, \xi; \lambda) = c(t) c(\xi) s(\pi)$, c'est à dire $c(t)$ multipliée par la constante $c(\xi) s(\pi)$.

Quant au signe des valeurs caractéristiques λ_n , il résulte de la remarque faite au commencement du no. 7 sur le signe de la fonction $\psi(t)$ que, même dans le cas où $\psi_1(t) \leq 0$ dans $(0, \pi)$, le noyau $G(t, \xi)$ n'a qu'un nombre fini de valeurs caractéristiques positives, toutes les autres étant négatives. Si $\psi_1(t) \geq 0$, toutes les valeurs caractéristiques sont négatives.

De ce qui a été dit précédemment, on voit que parmi les fonctions $\lambda + \psi_1(t)$, $\psi_1(t)$ restant le même, les seules qui admettent des solutions périodiques de période π sont celles où λ égale une valeur caractéristique λ_n du noyau $G(t, \xi)$. Et comme λ est proportionnel à $\int_0^\pi \psi dt$, en vertu de la supposition faite sur la fonction $\psi_1(t)$ au début du no. 3, c'est la valeur de cette intégrale qui décide de l'existence des solutions périodiques pour l'équation (1.1) correspondante.

On peut faire rentrer dans le cas précédent le cas où $\lambda = 0$ et $\psi(t) \equiv \psi_1(t)$, c'est à dire le cas de l'équation (3.1). Posons $\psi_1(t) = \psi(t) - \lambda$, λ étant une constante quelconque; l'équation (3.1) s'écrit

$$\ddot{x} - \psi(t)x = -\lambda x.$$

En procédant comme dans le cas de l'équation (1.1), on voit que les fonctions $G(t, \xi)$ et $\Gamma(t, \xi)$ ont changé de rôle de sorte que l'équation intégrale des solutions périodiques s'écrit dans ce cas

$$x(t) = \int_0^\pi \lambda \Gamma(t, \xi) x(\xi) d\xi,$$

$\Gamma(t, \xi)$ étant donné par l'expression (7.4), tandis que la résolvante sera la fonction $G(t, \xi)$ donnée par (7.3) et où λ n'entre plus. L'équation (3.1) admettra donc des solutions périodiques de période π si $t = \pi$ est une racine de l'équation $\gamma(t) + \sigma(t) - 2 = 0$.

8. Nous allons montrer que dans le cas de la fonction $\psi(t)$ paire les valeurs caractéristiques du noyau $G(t, \xi)$ sont toutes simples, c'est à dire l'impossibilité de coexistence, pour la même valeur de λ , de deux fonctions périodiques $s(t)$ et $c(t)$ de période π linéairement indépendantes. L'équation de Mathieu rentre dans ce cas. M. Ince* a démontré le premier l'impossibilité en question en supposant la fonction $\psi(t)$ continue et paire. La démonstration qui suit s'applique aussi bien au cas de l'équation de Mathieu qu'au cas plus général considéré ici†.

Supposons donc la fonction $\psi(t)$ paire, les autres suppositions du no. 1 étant les mêmes. Soit $c(t)$ périodique de période π et soient $c_0, c_1, \dots, c_n, \dots$ ses constantes de Fourier; la série $\sum_{n=0}^{\infty} c_n^2$ est donc convergente. Après

* E. L. Ince, "Periodic solutions of a linear differential equation of the second order with periodic coefficients", *Proc. Camb. Phil. Soc.*, 23 (1927), 44-46.

† Pour une autre démonstration dans le cas de l'équation de Mathieu voir Ž. Marković, "Sur la non-existence simultanée de deux fonctions de Mathieu", *Proc. Camb. Phil. Soc.*, 23 (1927), 203-205.

deux intégrations par parties effectuées dans l'hypothèse de $c(t)$ périodique et en vertu de l'équation (1.1), on trouve

$$(8.1) \quad \begin{cases} c_n = -\frac{1}{4n^2} \frac{2}{\pi} \int_0^\pi \psi(t) c(t) \cos 2nt \, dt & (n \neq 0), \\ \int_0^\pi \psi(t) c(t) \, dt = 0 & (n = 0). \end{cases}$$

Soient $a_1, a_2, \dots, a_n, \dots$ les constantes de Fourier de la fonction $\psi_1(t)$. En vertu des formules donnant les constantes de Fourier du produit de deux fonctions sommables bornées, les relations (8.1) s'écrivent

$$\begin{aligned} -4n^2 c_n &= a_n c_0 + \frac{1}{2} \sum_{i=1}^{\infty} c_i (a_{i+n} + a_{i-n}) \quad (a_{-i} = a_i), \\ \lambda c_0 + \frac{1}{2} \sum_{i=1}^{\infty} c_i a_i &= 0. \end{aligned}$$

Les constantes de Fourier de la fonction $c(t)$ satisfont donc au système suivant d'équations linéaires à une infinité d'inconnues

$$(8.2) \quad \left\{ \begin{aligned} &\lambda c_0 + \frac{a_1}{2} c_1 + \frac{a_2}{2} c_2 + \dots + \frac{a_n}{2} c_n + \dots = 0, \\ &\frac{1}{4} a_1 c_0 + \left[1 + \frac{\lambda}{4} + \frac{a_2}{8} \right] c_1 + \left[\frac{a_1}{8} + \frac{a_3}{8} \right] c_2 \\ &\quad \quad \quad + \dots + \left[\frac{a_{n-1}}{8} + \frac{a_{n+1}}{8} \right] c_n + \dots = 0, \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ &\frac{1}{4n^2} a_n c_0 + \left[\frac{a_{n-1}}{8n^2} + \frac{a_{n+1}}{8n^2} \right] c_1 + \left[\frac{a_{n-2}}{8n^2} + \frac{a_{n+2}}{8n^2} \right] c_2 \\ &\quad \quad \quad + \dots + \left[1 + \frac{\lambda}{4n^2} + \frac{a_{2n}}{8n^2} \right] c_n + \dots = 0, \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \end{aligned} \right.$$

dont le déterminant

$$\Delta = \begin{vmatrix} 1 + (\lambda - 1), & \frac{a_1}{2}, & \frac{a_2}{2}, & \dots, & \frac{a_n}{2}, & \dots \\ \frac{1}{4} a_1, & 1 + \frac{\lambda}{4} + \frac{a_2}{8}, & \frac{a_1}{8} + \frac{a_3}{8}, & \dots, & \frac{a_{n-1}}{8} + \frac{a_{n+1}}{8}, & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{1}{4n^2} a_n, & \frac{a_{n-1}}{8n^2} + \frac{a_{n+1}}{8n^2}, & \frac{a_{n-2}}{8n^2} + \frac{a_{n+2}}{8n^2}, & \dots, & 1 + \frac{\lambda}{4n^2} + \frac{a_{2n}}{8n^2}, & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} = 0$$

donne les valeurs λ_n correspondant aux fonctions périodiques $c_n(t)$.

Le déterminant Δ est absolument convergent. Soit en effet $1+c_{ii}$ l'élément diagonal, c_{ik} l'élément de la ligne i et de la colonne k . Les constantes $|\alpha_i|$ étant bornées dans leur ensemble, on a $|\alpha_i| < N$; de plus

$\sum_{n=1}^{\infty} \alpha_n^2$ converge et λ est fini. On a donc

$$|c_{ii}| < \frac{N_1}{8(i-1)^2},$$

N_1 étant une constante indépendante de i ; ainsi la série $\sum_{i=1}^{\infty} |c_{ii}|$ converge, et on verrait aussi que la série $\sum_{i,k} |c_{ik}|^2$ est convergente.

Puisque $\dot{c}(\pi) = 0$ et que $\dot{c}(t)$ est une fonction impaire, la fonction la plus générale satisfaisant à cette condition ne diffère de $c(t)$ que nous considérons que par un facteur constant; les constantes c_n ne comportent donc d'autre indétermination qu'un facteur arbitraire; le rang du déterminant Δ est donc 1. Désignons par $\begin{pmatrix} i \\ k \end{pmatrix}$ le mineur de Δ obtenu en y supprimant la ligne i et la colonne k . On peut montrer qu'on a toujours $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq 0$.

Remarquons d'abord à cet effet que la première des équations (8.2) est une conséquence des autres, puisque la relation (8.1) a été établie en supposant $\dot{c}(\pi) = 0$ et que la première équation, d'après (5.1), l'exprime justement. Donc au moins un des mineurs $\begin{pmatrix} 1 \\ k \end{pmatrix}$ est différent de zéro et la résolution du système (8.2) est donnée par les formules

$$c_0 = \frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ k \end{pmatrix}} \rho_k, \dots, c_i = \frac{\begin{pmatrix} 1 \\ i \end{pmatrix}}{\begin{pmatrix} 1 \\ k \end{pmatrix}} \rho_k, \dots,$$

ρ_k étant une constante arbitraire. Si l'on avait $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$, on aurait $c_0 = 0$; mais alors le déterminant du système (8.2) correspondant à ce cas, c'est à dire dépourvu de la première colonne, ne saurait être de rang 1. Il aurait la première ligne qui est une conséquence des autres, et encore une autre, puisque le déterminant du système restant est $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ et il est supposé nul.

Le mineur $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ étant différent de zéro, les constantes c_i s'expriment

Soit a_i ($i = 1, 2, \dots$) la valeur commune de deux membres de chaque ligne, supposée existante. Le déterminant Δ_1 du système linéaire au premier membre est le mineur $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ de Δ . En multipliant les équations (8.4) par $\mu_2, \mu_3, \dots$ et faisant la somme, on a, en vertu de (8.3) et pour les premiers membres seulement,

$$\sum_{k=1}^{\infty} b_k a_k = \sum_{k=1}^{\infty} \mu_{k+1} a_k ;$$

mais la fonction $s(t)$ étant périodique et impaire, on a

$$\sum_{k=1}^{\infty} b_k a_k = \int_0^{\pi} \psi_1(t) s(t) dt = \int_0^{\pi} \psi(t) s(t) dt = 0,$$

donc

$$\sum_{k=1}^{\infty} \mu_{k+1} a_k = 0.$$

Si $a_k \neq 0$ ($k = 1, 2, \dots$), il y aurait une relation entre les éléments d'une colonne quelconque du déterminant $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ des premiers membres du système (8.4), ce qui est incompatible avec $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq 0$. On a par suite $a_1 = a_2 = \dots = a_k = \dots = 0$. Mais alors le système d'équations composé des premiers membres de (8.4) est homogène et à déterminant différent de zéro; sa solution unique est la solution triviale $b_i = 0$ ($i = 1, 2, \dots$); donc $s(t) \equiv 0$. L'impossibilité de coexistence de deux fonctions périodiques de période π , l'une impaire, l'autre paire, pour la même valeur du paramètre λ , est ainsi démontrée.

Le cas des fonctions périodiques de période 2π se ramenant à celui des fonctions périodiques de période π , dans ce cas particulier comme dans le cas général, on verra la démonstration à la fin du no. 9.

9. Reprenons maintenant la fonction $\psi(t)$ du no. 1. Les fonctions $s(t)$ et $c(t)$ n'étant plus l'une impaire, l'autre paire, admettent l'une et l'autre le développement en série de Fourier général. Soient c_n, d_n les constantes de Fourier de $c(t)$; c'_n, d'_n celles de $s(t)$. En procédant comme au no. 8, on trouve que les quantités c'_n, d'_n satisfont au même système d'équations linéaires que les quantités c_n, d_n . Or, en effectuant les intégrations par parties, on constate que dans le calcul des constantes c_n et c'_n entrent seules les conditions de périodicité $\dot{c}(\pi) = 0, \dot{s}(\pi) - 1 = 0$ respectivement, tandis que dans celui de d_n et d'_n n'entrent que les conditions $c(\pi) - 1 = 0, s(\pi) = 0$ qui sont indépendantes de celles relatives aux dérivées. Le déterminant du système linéaire ne peut être de rang 1, car alors une des

équations serait une conséquence de toutes les autres, ce qui est impossible, les deux conditions qui ont donné le système d'équations linéaires homogènes étant indépendantes. Et comme il ne peut être plus grand que 2, le rang du système d'équations est 2, introduisant linéairement deux constantes arbitraires dans les coefficients de Fourier cherchés et exprimant les fonctions $c(t)$ et $s(t)$ qui en résultent comme combinaisons linéaires de deux mêmes fonctions linéairement indépendantes, ces fonctions n'étant plus assujetties à la condition d'être paires ou impaires. Par un choix convenable de deux constantes arbitraires on arrive à satisfaire simultanément aux conditions de périodicité pour les fonctions $c(t)$ et $s(t)$ et l'on a ainsi une paire de fonctions linéairement indépendantes et périodiques simultanément de période π .

Pour traiter de l'existence simultanée des fonctions périodiques $s(t)$ et $c(t)$ de période 2π , supposons la fonction $s(t)$ périodique, c'est à dire $s(2\pi) = 0$, $\dot{s}(2\pi) = 1$. La relation $s(2\pi + t) = s(t)$ qui s'écrit aussi $s(t + \pi) = s(t - \pi)$ donne pour $t = 0$ et à l'aide de (2.3) ($n = 1$) $s(\pi) = s(-\pi) = -s(\pi)$: on a donc

$$(9.1) \quad s(\pi) = 0,$$

et alors (1.3) donne $c(\pi) \dot{s}(\pi) = 1$.

Mais de la relation $\dot{s}(t + \pi) = \dot{s}(t - \pi)$ il résulte d'une manière analogue $\dot{s}(\pi) = \dot{s}(-\pi) = c(\pi)$ et par conséquent $\dot{s}^2(\pi) = 1$, d'où l'on conclut

$$(9.2) \quad \dot{s}(\pi) = c(\pi) = -1,$$

puisque l'hypothèse $\dot{s}(\pi) = 1$ avec $s(\pi) = 0$ définirait la fonction $s(t)$ comme fonction périodique de période π .

La première relation (2.1) pour $n = 1$ donne pour la fonction $s(t)$ de période 2π

$$(9.3) \quad s(t + \pi) = -s(t).$$

Si $c(t)$ est périodique de période 2π , on trouve en procédant de même,

$$(9.4) \quad \begin{cases} \dot{c}(\pi) = 0, \\ c(\pi) = \dot{s}(\pi) = -1, \end{cases}$$

et de la relation (2.1)

$$(9.5) \quad c(\pi + t) = -c(t).$$

Si la fonction $\psi(t)$ est paire, on trouve encore

$$\dot{s}(\tfrac{1}{2}\pi) = c(\tfrac{1}{2}\pi) = 0.$$

Puisque l'on a pour la fonction périodique $s(t)$, $s(2\pi) = 0$, et que en vertu de (1.3) $\dot{s}(2\pi) = c(2\pi) = 1$, la condition de coexistence de la fonction périodique $c(t)$ de la même période est $\dot{c}(2\pi) = 0$, ou bien d'après la formule (4.1)

$$\int_0^{2\pi} \psi_1(t) \dot{y}(t) dt = 0,$$

en posant $y(t) = s(t) c(t)$. Mais il résulte de (9.3) et (9.5) que

$$y(\pi + t) = y(t);$$

dans ce cas aussi, la fonction $y(t)$ est périodique de période π satisfaisant à la même équation intégrale (4.6) comme dans le cas des fonctions périodiques de période π et, en vertu de (9.1) et (9.4), aux mêmes conditions initiales que dans le cas de période π . En introduisant dans l'intervalle $(\pi, 2\pi)$ la variable $\pi + t$ au lieu de t , on trouve

$$\int_0^{2\pi} \psi_1(t) \dot{y}(t) dt = 2 \int_0^{\pi} \psi_1(t) \dot{y}(t) dt.$$

La question de la possibilité ou de l'impossibilité de coexistence de deux solutions périodiques de période 2π , linéairement indépendantes, est ramenée par là au cas, traité précédemment, des fonctions périodiques de période π .

10. Passons au cas* des fonctions périodiques de période 3π et considérons la fonction périodique $s(t)$. La relation $s(3\pi + t) = s(t)$ s'écrit aussi sous la forme

$$s(2\pi + t) = s(t - \pi),$$

d'où pour $t = 0$ et en vertu de (2.3)

$$(10.1) \quad s(2\pi) = s(-\pi) = -s(\pi),$$

et d'une manière analogue $\dot{s}(2\pi) = \dot{s}(-\pi) = c(\pi)$.

D'autre part, en vertu de (2.1), on a

$$s(3\pi + t) = s(2\pi + t) \dot{s}(\pi) + c(2\pi + t) s(\pi) = s(t);$$

on en obtient pour $t = 0$

$$s(2\pi) \dot{s}(\pi) + c(2\pi) s(\pi) = 0,$$

* Voir au sujet de l'existence de paires des solutions périodiques de période $n\pi$, $n \geq 3$, E. G. C. Poole, "On certain classes of Mathieu functions", *Proc. London Math. Soc.* (2), 20 (1922), 374; E. Hille, "On the zeros of Mathieu functions", *Proc. London Math. Soc.* (2), 23 (1925), 185; et E. L. Ince, "Periodic solutions of a linear differential equation of the second order with periodic coefficients", *Proc. Camb. Phil. Soc.*, 23 (1927), 44-46.

et, en différentiant la relation précédente, on a pour $t = 0$

$$\dot{s}(2\pi)\dot{s}(\pi) + \dot{c}(2\pi)s(\pi) = 1.$$

On en tire, puisque $s(\pi) \neq 0$, à l'aide des relations (10.1) et de (1.3),

$$(10.2) \quad \begin{cases} c(2\pi) = \dot{s}(\pi), \\ \dot{c}(2\pi) = -\dot{c}(\pi). \end{cases}$$

En décomposant, à l'aide des formules (2.1), $s(2\pi+t)$ et $c(2\pi+t)$ plus loin, on trouve la relation

$$(10.3) \quad c(\pi) + \dot{s}(\pi) = -1.$$

Calculons maintenant la valeur $c(3\pi+t)$ à l'aide des relations (2.1) et (10.2).

On a

$$\begin{aligned} c(3\pi+t) &= c[2\pi+(\pi+t)] = c(\pi+t)c(2\pi) + s(\pi+t)\dot{c}(2\pi) \\ &= c(\pi+t)\dot{s}(\pi) - s(\pi+t)\dot{c}(\pi), \end{aligned}$$

et en vertu du deuxième groupe des formules (2.5)

$$c(3\pi+t) = c(t).$$

Ainsi est démontrée l'existence simultanée des fonctions périodiques $s(t)$ et $c(t)$ de période 3π .

Si la fonction $\psi(t)$ est paire, on a, pour la fonction périodique $s(t)$, $s(\frac{3}{2}\pi) = 0$, et pour $c(t)$ périodique on a $\dot{c}(\frac{3}{2}\pi) = 0$. De plus

$$c(\pi) = \dot{s}(\pi) = -\frac{1}{2}.$$

11. Soit maintenant $s(t)$ périodique de période 4π . La relation $s(4\pi+t) = s(t)$ s'écrivant aussi

$$s(2\pi+t) = s(t-2\pi),$$

on a, pour $t = 0$ et à l'aide de (2.3) ($n = 2$),

$$(11.1) \quad s(2\pi) = 0,$$

et d'une manière analogue

$$\dot{s}(2\pi) = s(-2\pi) = c(2\pi);$$

mais la relation (1.3) nous donne alors $c^2(2\pi) = 1$; donc

$$(11.2) \quad \dot{s}(2\pi) = c(2\pi) = -1,$$

puisque l'hypothèse $\dot{s}(2\pi) = 1$ avec $s(2\pi) = 0$ donnerait pour $s(t)$ une fonction périodique de période 2π .

D'après (2.1) ($n = 2$), on peut écrire

$$s(4\pi + t) = s(2\pi + t) \dot{s}(2\pi) + c(2\pi + t) s(2\pi);$$

on a donc, moyennant (11.1) et (11.2),

$$s(4\pi + t) = -s(2\pi + t).$$

D'autre part, de la même formule (2.1) ($n = 2$) et des (11.1) et (11.2), il s'ensuit

$$s(2\pi + t) = -s(t),$$

ce qu'on écrit aussi $s(\pi + t) = -s(t - \pi)$,

d'où, en passant aux dérivées, et d'après (2.3)

$$(11.3) \quad \dot{s}(\pi) = -\dot{s}(-\pi) = -c(\pi).$$

Mais la relation (2.2) ($n = 1$) donne

$$\dot{c}(2\pi) = \dot{c}(\pi) [c(\pi) + \dot{s}(\pi)],$$

et comme $\dot{c}(\pi)$, d'après (1.3) et (11.3), ne peut être nul, on a

$$(11.4) \quad c(2\pi) = 0.$$

Pour la fonction périodique $s(t)$ de période 4π on trouve encore facilement la relation

$$s(3\pi + t) = s(t - \pi),$$

d'où $\dot{s}(3\pi) = c(\pi)$, $s(3\pi) = -s(\pi)$,

et pour la fonction périodique $c(t)$,

$$c(3\pi) = \dot{s}(\pi), \quad \dot{c}(3\pi) = -\dot{c}(\pi).$$

Pour calculer $c(4\pi + t)$, on trouve à l'aide de (2.1), $n = 2$, en décomposant

$$c(4\pi + t) = c(2\pi + t) c(2\pi) + s(2\pi + t) \dot{c}(2\pi),$$

et d'après (11.2) et (11.4)

$$c(4\pi + t) = -c(2\pi + t),$$

et en vertu des mêmes relations

$$c(4\pi + t) = c(t),$$

ce qui prouve la possibilité de coexistence de deux fonctions périodiques $s(t)$ et $c(t)$ de période 4π .

Avec des raisonnements analogues on démontre le même fait pour les fonctions périodiques de période $n\pi$, n entier.

On voit que pour $n \geq 3$ donné c'est le nombre toujours plus grand des relations (2.1)–(2.4) valables pour ce n en même temps que pour tous les n moindres qui permet que ce fait se réalise.

12. La généralité des suppositions sur la fonction $\psi(t)$ ne permet pas de poursuivre l'étude qualitative des solutions de l'équation (1.1) si loin comme dans le cas de l'équation de Mathieu*. Mais on peut faire quelques remarques générales aussi dans le cas (1.1) qui nous laissent entrevoir quelques-unes des propriétés de ses solutions. Nous nous bornerons au cas des fonctions périodiques de période π .

Il résulte d'abord de la relation (1.3) que les zéros de la fonction $c(t)$ et les extrema de $s(t)$ ne peuvent coïncider avec les zéros de $s(t)$ ou les extrema de $c(t)$. L'une et l'autre de ces fonctions traverse donc l'axe des t sous un angle différent de zéro.

Quant aux zéros des fonctions $s(t)$ et $c(t)$, on voit facilement que la fonction $s(t)$ de période π possède à cause de $s(\pi) = 0$, $\dot{s}(\pi) = 1$, à l'intérieur de l'intervalle $(0, \pi)$ au moins un zéro et alors la fonction $c(t)$, correspondant à la même valeur de λ et qui n'est pas périodique si $\psi(t)$ est une fonction paire, possède à l'intérieur du même intervalle, d'après le théorème de Sturm, au moins deux zéros avec $c(\pi) = 1$ mais $\dot{c}(\pi) \neq 0$. Si la fonction périodique $c(t)$ possède de zéros, auquel cas elle en possède au moins deux, la fonction $s(t)$ correspondant à la même valeur de λ en possède au moins un à l'intérieur de $(0, \pi)$ avec $\dot{s}(\pi) = 1$.

On obtient des renseignements sur les valeurs extrema des fonctions $s(t)$ et $c(t)$ à l'aide de la relation (4.2).

Soit t_0 une valeur pour laquelle la fonction $x_1(t) = s(t)$ atteint l'extremum. On a de cette relation

$$0 = 1 + \lambda s^2(t_0) + 2 \int_0^{t_0} \psi_1(\xi) s(\xi) \dot{s}(\xi) d\xi.$$

Si $\lambda > 0$, on doit avoir $\int_0^{t_0} \psi_1 s \dot{s} d\xi < 0$, et alors on a

$$s^2(t_0) = \frac{2 \left| \int_0^{t_0} \psi_1 s \dot{s} d\xi \right| - 1}{\lambda}.$$

Si $\lambda < 0$, on a $s^2(t_0) = \frac{1 + 2 \int_0^{t_0} \psi_1 s \dot{s} d\xi}{|\lambda|}$; si de plus $\int_0^{t_0} \psi_1 s \dot{s} d\xi > 0$, on conclut $s^2(t_0) > \frac{1}{|\lambda|}$; si $\int_0^{t_0} \psi_1 s \dot{s} d\xi < 0$, $s^2(t_0) < \frac{1}{|\lambda|}$.

* Voir à ce sujet Ž. Marković, "Sur les fonctions de Mathieu de période π ", *Bull. de la classe des Sciences mathématiques de l'Académie des Sciences de Zagreb*, 21 (1925), 34–58.

Dans le cas $\lambda < 0$, qui constitue la règle, on obtient des évaluations plus précises. Soit t_0 l'argument du premier extremum de la fonction $s(t)$ à partir de $t = 0$; on a dans $(0, t_0)$ $\dot{s}s \geq 0$. Soit M la borne supérieure de $\psi(t)$ dans $(0, \pi)$. On a de la relation ci-dessus

$$|\lambda| s^2(t_0) \leq 1 + M s^2(t_0).$$

On en conclut
$$s^2(t_0) \leq \frac{1}{|\lambda| - M},$$

dès que $|\lambda| > M$.

De la relation (4.2) on a pour $x_1(t) = c(t)$ et pour ses valeurs extrema, si $\lambda > 0$,

$$c^2(t_0) = 1 - \frac{2 \int_0^{t_0} \psi_1 c \dot{c} d\xi}{\lambda};$$

donc $c^2(t_0) \geq 1$ suivant que $\int_0^{t_0} \psi_1 c \dot{c} d\xi \leq 0$; et pour $\lambda < 0$,

$$c^2(t_0) = 1 + \frac{2 \int_0^{t_0} \psi_1 c \dot{c} d\xi}{|\lambda|};$$

donc $c^2(t_0) \leq 1$ suivant que $\int_0^{t_0} \psi_1 c \dot{c} d\xi \geq 0$.

D'autre part, de la relation (4.3) il résulte qu'au zéro ou au point d'extremum de la fonction $x(t) = s(t)$ on doit avoir $\int_0^t \psi s^2 d\xi < 0$; donc si en particulier la fonction $\psi(t)$ est d'abord positive, il ne peut y avoir dans l'intervalle respectif ni de zéro ni de point d'extremum de cette fonction, avec une conclusion analogue pour la fonction $c(t)$.

On peut se faire une idée sur l'allure des solutions périodiques de l'équation (1.1) en faisant la discussion d'après le signe de la fonction $\psi(t)$. Pour chaque type nous ne considérons que le cas le plus simple. Supposons d'abord $\psi(t) < 0$ dans $(0, \pi)$; c'est le cas qui se présente finalement, même si la fonction $\psi(t)$ ne satisfait d'abord à cette condition, $\psi(t)$ étant bornée et $\lim_{n \rightarrow \infty} \lambda_n = -\infty$. Pour la fonction $s(t)$ on a $s(0) = 0$, $\dot{s}(0) = 1$, $\ddot{s}(0) = 0$; l'origine est un point d'inflexion. On a $\ddot{s}(t) < 0$ jusqu'au zéro de $s(t)$; $\dot{s}(t)$ décroît jusqu'au minimum négatif au zéro de $s(t)$, qui est un point d'inflexion. À partir de ce point, $\ddot{s}(t) > 0$, $\dot{s}(t)$ croît de la valeur négative minima par la valeur zéro jusqu'à $+1$ pour $t = \pi$. La courbe tourne toujours sa concavité vers l'axe des t . Pour la fonction $c(t)$ on a $c(0) = 1$, $\dot{c}(0) = 0$, $\ddot{c}(0) = \psi(0) < 0$. Par conséquent $c(t)$ a une valeur maxima pour $t = 0$; $\dot{c}(t)$ décroît jusqu'au minimum négatif au zéro de $c(t)$, puis croît par la valeur zéro jusqu'à une valeur maxima au second

zéro de $c(t)$, en décroissant ensuite jusqu'à zéro pour $t = \pi$. La courbe tourne sa concavité vers l'axe des t .

Supposons maintenant $\psi(t) > 0$ au commencement de l'intervalle $(0, \pi)$. Le point $t = 0$ est un point d'inflexion pour $s(t)$ et minimum pour $c(t)$. Pour $t > 0$, $\dot{s}(t)$ et $\dot{c}(t)$ sont des fonctions croissantes jusqu'au point où la fonction $\psi(t)$ change de signe et qui est un point d'inflexion pour l'une et l'autre courbe, les deux courbes tournant leur convexité vers l'axe des t . L'allure des courbes dans le reste de l'intervalle dépend des changements de signe de la fonction $\psi(t)$ introduisant de nouveaux points d'inflexion.

Si $\psi(t) < 0$ au commencement de l'intervalle, l'allure des courbes $s(t)$ et $c(t)$ ressemble à celle du premier cas considéré, le cours ultérieur dépendant des changements de signe de $\psi(t)$ qui amènent pour les courbes des passages de la convexité à la concavité et inversement.

A CONVERGENCE THEOREM FOR NON-ABSOLUTELY
CONVERGENT INTEGRALS

By M. E. GRIMSHAW.

[Received 28 October, 1929.—Read 14 November, 1929.]

1. This note is concerned with the determination of sufficient conditions for

$$(1) \quad \lim_{n \rightarrow \infty} \int_a^b f(t) k_n(t) dt = 0,$$

where $f(t)$ is integrable in the general Denjoy sense. By the use of the idea of uniform integrability, the conditions are made to depend upon those already known to be sufficient in the case of absolute integrability*. A solution of the problem has been given by Burkill in a paper recently published†. Similar results are obtained here as an application of the main theorem.

2. A sequence of conditionally integrable functions $\{\phi_n(t)\}$ is said to be *uniformly integrable*‡ in (a, b) if each perfect subset Π of (a, b) contains a portion ϖ which is such that

$$\int_{\varpi} |\phi_n(t)| dt < \infty$$

for all n , and, if (a_ν, b_ν) denote a typical complementary interval of ϖ ,

$$\sum_{\nu} \left| \int_{a_\nu}^{b_\nu} \phi_n(t) dt \right|$$

is uniformly convergent.

* E. W. Hobson, *Theory of functions of a real variable*, 2 (1926), 422.

† J. C. Burkill, *Journal London Math. Soc.*, 4 (1929), 127. Although that paper is confined to the special Denjoy integral, the proofs of Theorems I and II apply equally to the general Denjoy integral. I am indebted to Mr. Burkill for this remark.

‡ I wish to thank Mr. Pollard for the introduction of the idea of uniform integrability.

In accordance with Denjoy's theory the function $\phi_n(t)$ is integrable in (a, b) in the general Denjoy sense if each perfect subset Π of (a, b) contains a portion ϖ_n such that

$$(1) \quad \int_{\varpi_n} |\phi_n(t)| dt < \infty, \\ \sum_v \left| \int_{a_{n,v}}^{b_{n,v}} \phi_n(t) dt \right| < \infty,$$

where $(a_{n,v}, b_{n,v})$ denotes a complementary interval of ϖ_n . Uniform integrability is equivalent to the existence of portions ϖ_n having a common portion ϖ , together with the uniform convergence of (1) when ϖ_n is replaced by ϖ .

3. THEOREM. *In order that*

$$(1) \quad \lim_{n \rightarrow \infty} \int_a^b f(t) k_n(t) dt = 0,$$

where (a, b) is a finite interval, it is sufficient that

$$\lim_{n \rightarrow \infty} \int_a^b g(t) k_n(t) dt = 0$$

for any absolutely integrable function $g(t)$, that the continuity in x of the integrals

$$(2) \quad \int_a^x f(t) k_n(t) dt$$

be uniform in n for each x in $a \leq x \leq b$ (to the right at a and to the left at b), and that the functions

$$\phi_n(t) = f(t) k_n(t)$$

be uniformly integrable in (a, b) .

Suppose first that $f(t)$, $k_n(t)$ are defined for all values of t . We shall say that the integral

$$(3) \quad \int f(t) k_n(t) dt$$

converges to zero in a given open interval if (1) is true whenever a and b lie in the interval, and that it converges to zero at a point if it is convergent to zero in some open interval surrounding the point. If (3) does not converge to zero at a point, then every open interval surrounding the point contains at least one range for which (1) is not true.

The set of points at which (3) converges to zero forms an open set such that (1) is true whenever the closed interval (a, b) belongs to the set. If (1) is not true for a particular interval (a, b) , then this interval, taken closed, must contain at least one point at which (3) does not converge to zero. The set Π of such points is obtained as the common part of two closed sets, and is closed. If the continuity of the integrals (2) is uniform in n for all x , then this set Π is also perfect. For, in this case, if (3) converges to zero in two adjacent open intervals, it converges to zero in the open interval formed by their sum and the common end point, so that there can be no isolated points of Π .

Now let $f(t)$, $k_n(t)$ be considered only in the range (a, b) . Define

$$\begin{aligned} f_1(t) &= f(t) \text{ in } (a, b), \\ &= 0 \text{ elsewhere.} \end{aligned}$$

The integral

$$(4) \quad \int f_1(t) k_n(t) dt$$

converges to zero for all points outside (a, b) . The intervals and points where (3) converges to zero are defined as the intervals and points lying in the closed interval (a, b) , where (4) converges to zero. If (1) is not true for (a, b) , the set of points of non-convergence of (3) is a closed subset Π of (a, b) . If the continuity of the integrals (2) is uniform in n for each x in $a \leq x \leq b$ (to the right at a and to the left at b), then Π is perfect.

By the uniform integrability, Π contains a portion ϖ such that

$$\int_{\varpi} |f(t) k_n(t)| dt < \infty$$

and

$$\sum_{\nu} \left| \int_{a_{\nu}}^{b_{\nu}} f(t) k_n(t) dt \right|$$

is uniformly convergent, where (a_{ν}, b_{ν}) is a complementary interval of ϖ . Since $f(t)$ is integrable in the general Denjoy sense in (a, b) , ϖ can be so chosen that

$$\int_{\varpi} |f(t)| dt < \infty.$$

By the definition of Π ,

$$\lim_{n \rightarrow \infty} \int_{a_{\nu}}^{\beta_{\nu}} f(t) k_n(t) dt = 0$$

when (a_{ν}, β_{ν}) is any interval $a_{\nu} < a_{\nu} < \beta_{\nu} < b_{\nu}$, and by the uniform

continuity
$$\lim_{n \rightarrow \infty} \int_{a_\nu}^{b_\nu} f(t) k_n(t) dt = 0.$$

The uniform convergence now gives

$$(5) \quad \lim_{n \rightarrow \infty} \sum_{\nu} \int_{a_\nu}^{b_\nu} f(t) k_n(t) dt = 0.$$

Take $g(t)$ to be the function which coincides with $f(t)$ on ϖ and vanishes elsewhere. It is absolutely integrable, and so

$$\lim_{n \rightarrow \infty} \int_a^b g(t) k_n(t) dt = 0,$$

which is

$$(6) \quad \lim_{n \rightarrow \infty} \int_{\varpi} f(t) k_n(t) dt = 0.$$

By addition of (5) and (6)

$$\lim_{n \rightarrow \infty} \int_I f(t) k_n(t) dt = 0,$$

where I is the interval between the end points of ϖ .

A similar argument will show that, if J is any interval contained in I ,

$$(7) \quad \lim_{n \rightarrow \infty} \int_J f(t) k_n(t) dt = 0.$$

Now, unless Π is empty, there is at least one interval J for which (7) is not true. We conclude that Π is empty, and the theorem follows.

4. We proceed to consider a special case of the theorem. Let $f(t)$ be integrable in the general Denjoy sense in (a, b) ,

$$(A) \quad \lim_{n \rightarrow \infty} \int_a^\beta k_n(t) dt = 0,$$

whenever $a \leq a \leq \beta \leq b$, and

$$(B) \quad \int_a^b |dk_n(t)| \leq K$$

for all n , where K is constant.

Then, as Burkill shows,

$$(C) \quad \overline{\lim}_{n \rightarrow \infty} |k_n(t)| \leq K.$$

Since $k_n(t)$ is of bounded variation,

$$f(t) k_n(t)$$

is integrable in the general Denjoy sense in (a, b) . Also, for sufficiently large n ,

$$\left| \int_x^{x+\delta} f(t) k_n(t) dt \right| \leq (K+\epsilon) |F(x+\delta)| + W \int_x^{x+\delta} |dk_n(t)| \\ \leq W(2K+\epsilon),$$

where

$$F(t) = \int_x^t f(u) du,$$

and W is the upper bound of $|F(t)|$ in $(x, x+\delta)$, and tends to zero with δ . Thus

$$\int_a^x f(t) k_n(t) dt$$

is continuous at x , uniformly in n .

Again, every perfect subset of (a, b) contains a portion ϖ for the complementary intervals of which

$$(1) \quad \sum_{\nu} \left| \int_{a_{\nu}}^{b_{\nu}} f(t) dt \right| < \infty,$$

and over which $f(t)$ is integrable (L). Therefore $f(t)k_n(t)$ is integrable (L) over ϖ . Also, for sufficiently large n ,

$$(2) \quad \left| \int_{a_{\nu}}^{b_{\nu}} f(t) k_n(t) dt \right| \leq (K+\epsilon) \left| \int_{a_{\nu}}^{b_{\nu}} f(t) dt \right| + W_{\nu} \int_{a_{\nu}}^{b_{\nu}} |dk_n(t)|,$$

where W_{ν} is the upper bound of $\left| \int_{a_{\nu}}^t f(u) du \right|$ in (a_{ν}, b_{ν}) . Now W_{ν} tends to zero as $\nu \rightarrow \infty$, and, for all n ,

$$\sum_{\nu} \int_{a_{\nu}}^{b_{\nu}} |dk_n(t)| \leq K,$$

so that

$$\sum_{\nu} W_{\nu} \int_{a_{\nu}}^{b_{\nu}} |dk_n(t)|$$

is uniformly convergent. Together with (1) and (2) this shows that

$$\sum_{\nu} \left| \int_{a_{\nu}}^{b_{\nu}} f(t) k_n(t) dt \right|$$

is uniformly convergent. Thus the functions $\{f(t)k_n(t)\}$ are uniformly integrable in (a, b) . Moreover, (A), (B), (C) are sufficient for

$$\lim_{n \rightarrow \infty} \int_a^b g(t) k_n(t) dt = 0,$$

when $g(t)$ is absolutely integrable in (a, b) , so that the conditions of the theorem are fulfilled. It follows that

$$\lim_{n \rightarrow \infty} \int_a^b f(t) k_n(t) dt = 0,$$

whenever $f(t)$ is integrable in the general Denjoy sense in (a, b) provided that (A) and (B) are satisfied.

The result for the infinite range is deduced without difficulty; namely
If $f(t)$ is integrable in the general Denjoy sense in (a, ∞) ,

$$\lim_{n \rightarrow \infty} \int_a^\beta k_n(t) dt = 0$$

for every finite interval $a \leq \alpha \leq \beta$,

$$\int_a^\infty |dk_n(t)| \leq K$$

for each n , then

$$\lim_{n \rightarrow \infty} \int_a^\infty f(t) k_n(t) dt = 0.$$

ON DARBOUX LINES

By V. C. MORTON.

[Received 20 September, 1929.—Read 14 November, 1929.]

1. The Darboux lines on the quadric, the cyclide*, the general cylinder†, the general surface of revolution‡, the general cone and the right helicoid§, have been found, together with several of their properties. Their differential equation of the second order, even when the surface is referred to its lines of curvature, has been used in a form somewhat inconvenient when the intrinsic nature of the surface is known. In this paper a simple form for the equation is given with the two parameters of the surface as the only variables involved||. It is clearly seen from the equation why solutions are to be expected in the cases given above. I give solutions for all isothermic surfaces which are the envelopes of spheres, and for a more general type of helicoid. I give a first integral for the general helicoid. I also investigate loxodromic¶. Darboux lines on Weingarten surfaces.

I denote by dashes differentiation with respect to the parameter u of the surface, or in the case of a function of a single variable differentiation with respect to that variable; by suffixes 1 and 2 partial differentiation with respect to the parameters u and v ; and by α and β the principal curvatures of the surface along the lines of curvature $v = \text{constant}$ and $u = \text{constant}$ respectively.

If s , ν , κ , τ are respectively the arc, normal curvature, geodesic curvature, and geodesic torsion of a curve on a surface, then the classical

* Darboux, *Comptes rendus*, 73 (1871), 732.

† Enneper, *Göttinger Nachrichten* (1871), 577.

‡ R. E. Colomb, *Journal London Math. Soc.*, 1 (1926), 106.

§ H. Hilton, *Proc. London Math. Soc.* (2), 29 (1929), 517.

|| One form of equation with the parameter s of the curve eliminated is given by J. G. Hardy, *American J. of Math.*, 20 (1898), 283.

¶ A curve will be called loxodromic if it meets the lines of curvature at a constant angle.

equation of a Darboux line is

$$\frac{d\nu}{ds} + \tau\kappa = 0. \quad (1)$$

If we take the lines of curvature as parametric, and denote by θ the angle between the Darboux line and the parametric family associated with the principal curvature α , and by s_1, s_2 the arcs of the lines of curvature, then we may write*

$$\tau = -(a - \beta) \sin \theta \cos \theta, \quad (2)$$

$$\frac{d\nu}{ds} - 2\kappa\tau = \frac{\partial\alpha}{\partial s_1} \cos^3 \theta + 3 \frac{\partial\alpha}{\partial s_2} \cos^2 \theta \sin \theta + 3 \frac{\partial\beta}{\partial s_1} \cos \theta \sin^2 \theta + \frac{\partial\beta}{\partial s_2} \sin^3 \theta. \quad (3)$$

Also we have the equations

$$\kappa = \frac{d\theta}{ds} + \kappa_\alpha \cos \theta + \kappa_\beta \sin \theta, \quad (4)$$

$$(a - \beta) \kappa_\alpha = \frac{\partial\alpha}{\partial s_2}, \quad (a - \beta) \kappa_\beta = \frac{\partial\beta}{\partial s_1}, \quad (5)$$

where κ_α and κ_β are the geodesic curvatures of the lines of curvature.

From (1), (2), (3), (4), and (5) we find that an intrinsic form of equation for a Darboux line is

$$3(a - \beta) \frac{d\theta}{ds} \cos \theta \sin \theta = \frac{\partial\alpha}{\partial s_1} \cos^3 \theta + \frac{\partial\beta}{\partial s_2} \sin^3 \theta. \quad (6)$$

If parameters u, v are chosen along the lines of curvature, (6) may be written†

$$3\theta'(a - \beta) \tan \theta = \sqrt{\frac{E}{G}} \beta_2 \tan^3 \theta + a_1 \quad (7)$$

or, alternatively,

$$3(a - \beta) \frac{d}{du} \log \left(\frac{G}{E} v'^2 + 1 \right) = 2 \left(\frac{G\beta_2}{E} v'^3 + a_1 \right), \quad (8)$$

a simple form of equation when E, G, α, β are known functions of u and v .

For an isothermic surface, referred to isothermic parameters along its lines of curvature, the equation is still further simplified, since then

* Darboux, *Surfaces*, Livre V, Ch. II. Tableau V.

† Originally I had used rectangular axes to derive (7) from the equation $\Sigma X (d^3x/ds^3) = 0$, which is given by the definition of a Darboux line. The above direct investigation was suggested to me by a referee.

$G = E$, and, moreover, a function P can be found such that*

$$\alpha_1 = -(\alpha - \beta)P_1, \quad \beta_2 = (\alpha - \beta)P_2.$$

Equation (7) then takes the form

$$3\theta' \tan \theta = P_2 \tan^3 \theta - P_1, \quad (9)$$

and the investigation depends on the form of the single function P .

For example, if a surface of constant mean curvature $2c$ is referred to isothermic parameters along its lines of curvature, the equation of its Darboux lines is

$$3 \frac{d}{du} \log(1+v'^2) = \frac{\partial}{\partial u} \log(\alpha - c) - v'^3 \frac{\partial}{\partial v} \log(\alpha - c). \quad (10)$$

2. Since G/E , $\alpha_1/(\alpha - \beta)$, and $\beta_2/(\alpha - \beta)$ are unaltered by inversion, which process also preserves the lines of curvature, equation (8) gives the result: *The Darboux lines on two inverse surfaces correspond*†. Thus, if the lines are known on any surface, then they are also known on the inverse surface with respect to any point.

3. Except for the right helicoid, all the surfaces already investigated form a single class, namely those surfaces for which (8) has an integral of the form

$$\frac{\phi(u, v) + cf(u, v)}{\psi(u, v) + c\chi(u, v)} = v'^2.$$

It can be shown, by comparison with (8), that, for this class, the solution must be of the restricted form

$$E(U - k) + Gv'^2(V - k) = 0.$$

If neither U nor V is a constant, parameters U and V may be taken along the lines of curvature, and the solution is then

$$E(u - k) + Gv'^2(v - k) = 0,$$

where E and G are now the coefficients of the new line element.

* The condition for integrability is that the Codazzi equations

$$\alpha_2 = -(\alpha - \beta)\lambda_2, \quad \beta_1 = (\alpha - \beta)\lambda_1$$

should be consistent, where $\lambda = \frac{1}{2} \log E$. P is in fact $-\lambda - \log(\alpha - \beta)$.

† This result, which is to be expected from the definition of the lines, seems not to have been noticed.

The surfaces must then satisfy the equations*

$$2(u-v)a_1 = 2(u-v)\beta_2 = 3(a-\beta).$$

It follows that the surfaces must be isothermic. The quadric and cyclide belong to this class, and together are self-inverse.

The case for V constant occurs when the right-hand side of (8), divided by $a-\beta$, is a perfect differential. Then β_2 must be zero, while $a_1/(a-\beta)$ must be a function of u alone. The surfaces must then be tubular surfaces of a special type. They are, in fact, just the isothermic tubular surfaces (including the surface of revolution, the general cone, and cylinder as special cases), as can be seen as follows.

$$\text{We may write} \quad \beta = U, \quad \frac{a_1}{a-\beta} = -\frac{\bar{U}'}{\bar{U}}$$

$$\text{giving} \quad \bar{U}a = \int U\bar{U}' du + V.$$

It follows, from the Codazzi equations, that the surface is isothermic.

Also, if a tubular surface is isothermic, then, since in that case $\beta = U$ and

$$\frac{\partial}{\partial u} \frac{\beta_2}{a-\beta} + \frac{\partial}{\partial v} \frac{a_1}{a-\beta} = 0,$$

$a_1/(a-\beta)$ must be a function of u alone and (8) may be integrated twice, since the variables are separable after the first integration. The complete solution is, in fact,

$$\int \mu G^{\frac{1}{2}} dv = \int \mu E^{\frac{1}{2}} (aT-1)^{\frac{1}{2}} du + b,$$

where

$$3 \log T = 2 \int \frac{a_1}{a-\beta} du,$$

and μ is the function of u and v which makes $\mu G^{\frac{1}{2}}$ and $\mu E^{\frac{1}{2}}$ functions of v and u respectively. For example, the general cone is given by

$$\beta = 0, \quad a = Ue^{-ev}, \quad E = G = e^{2cv},$$

* The equation

$$2(u-v)R_{12} = 3(R_2 - R_1), \quad \text{where} \quad R_2 = a, \quad R_1 = \beta,$$

may be substituted for this pair. The Codazzi and Gauss equations then give a single further partial equation of the third order in R containing two arbitrary functions of u and v respectively. It is doubtful, however, whether any solutions exist other than those leading to the quadric and cyclide.

and the Darboux lines are*

$$v = \int (aU^{\frac{2}{3}} - 1)^{\frac{1}{2}} du + b.$$

For the general cylinder, choosing u to be the arc of a fixed right section, whose radius of curvature is ρ , and v to be the length along a generator, we have

$$\beta = 0, \quad \alpha = 1/\rho, \quad E = G = 1,$$

and the Darboux lines are†

$$v = \int \left(\frac{\alpha}{\rho^2} - 1 \right)^{\frac{1}{2}} du + b.$$

The type of solution, in the general case, for V constant is to be expected, since the inverse of a surface of revolution is an isothermic tubular surface.

The case for U constant leads again to the same class of surface.

The simplest surface of the class is the Dupin cyclide, with anchor ring, circular cone, and right cylinder as special cases. For this surface α and β are functions of v and u respectively, and equation (7) gives the Darboux lines to be all the loxodromic lines on the surface; it is the only surface with this property‡. It is, moreover, the only surface on which there can exist two different loxodromic families of Darboux lines§, as can be seen at once from (7)||.

4. There is, however, another simple class of surface on which the lines can be found when the surface is referred to its lines of curvature, the general helicoid of constant mean curvature $2c$.

I shall need here and later the following result which has an interest in itself.

If the usual parameters are taken along the lines of curvature of a Weingarten surface, giving

$$\log E = -2 \int \frac{d\alpha}{\alpha - \beta}, \quad \log G = 2 \int \frac{d\beta}{\alpha - \beta},$$

* Cf. Hilton, *loc. cit.*, 521. The cone and cylinder are the only isothermic developable surfaces.

† Cf. Enneper, *loc. cit.*, where this result is obtained by a different method.

‡ This result is due to Hilton, *loc. cit.*, 523-525.

§ Excluding the unreal solutions of the equation $\tan^3 \theta = \tan^3 \epsilon$.

|| Cf. Hilton, *loc. cit.*, 523, where the families are taken to be lines of torsion.

then the necessary and sufficient condition that the surface is helicoidal is that a is a function of $u+av$, where a is a constant; the helices are then the curves $u+av = \text{constant}$.

For first suppose that a is a function of $u+av$. The orthogonal trajectories of the curves $a = \text{constant}$ are the curves $Eadu - Gdv = 0$. Since E and G are functions of $u+av$ an integrating factor is $1/(a^2E + G)$, so we may put

$$dp = du + av,$$

$$(a^2E + G)dq = Eadu - Gdv.$$

The two new quadratic forms for the surface with parameters p and q will now be such that all the coefficients are functions of p , and the surface must be a helicoid with $p = \text{constant}$ for helices.

Suppose now that the surface is helicoidal. The quadratic forms are $Edu^2 + Gdv^2$, $aEdu^2 + \beta Gdv^2$. If we choose, as new parametric curves, the helices $a = \text{constant}$ and the orthogonal trajectories $q = \text{constant}$, we may put

$$da = a_1 du + a_2 dv,$$

$$\lambda dq = Ea_2 du - Ga_1 dv.$$

Using the fact that the coefficients of da^2 in both the new quadratic forms must be functions of a , we find that both

$$\frac{a_1^2}{E} + \frac{a_2^2}{G} \quad \text{and} \quad \frac{a a_1^2}{E} + \frac{\beta a_2^2}{G}$$

must be functions of a , and it follows that, since a_1 and a_2 must then be functions of a , then a must be a function of $u+av$.

Now a helicoid of constant mean curvature is isothermic. If isothermic parameters are chosen along the lines of curvature, then α and β must have the forms $c + f(u+av)$ and $c - f(u+av)$ respectively by the above result.

Equation (8) then becomes

$$\frac{3(d/du)(v'^2)}{(1+v'^2)(1-av'^3)} = \frac{f'(u+av)}{f(u+av)},$$

and, on putting p for $u+av$, we get

$$\left(\frac{6v'}{1+v'^2} + \frac{6av'^2}{1-av'^3} \right) dv' = \frac{f'(p)}{f(p)} dp,$$

which leads to the integral

$$f(u+av) = \frac{k(1+v'^2)^3}{(1-av'^3)^2}, \quad (11)$$

$$\text{or} \quad (\cos^3 \theta - a \sin^3 \theta)^2 f(u+av) = k, \quad (12)$$

$$\text{or} \quad (\cos^3 \theta - a \sin^3 \theta)(a-c)^{\frac{1}{2}} = \text{constant},$$

an invariant form for change of parameters along the lines of curvature.

The system of curves $\cot^3 \theta = a$ is a family of loxodromic Darboux lines.

Using the Codazzi and Gauss equations we find that the function $f(p)$ must satisfy an equation

$$ff'' - f'^2 - \frac{1}{2}bkf(c^2 - f^2) = 0,$$

that is, on integration, an equation

$$f'^2 = df^2 - bc^2kf - bkf^3. \quad (13)$$

Equation (12) may now be transformed into an equation of the first degree with variables separable as follows: Differentiating (12) and rejecting the factor $1+a \tan \theta$, whose vanishing does not give a solution of the original equation of the second order, we have

$$f' = \frac{6k \cos^2 \theta \sin \theta \theta'}{(\cos^3 \theta - a \sin^3 \theta)^3},$$

whence, using (13), we find

$$6 \cos^2 \theta \sin \theta \theta' = \{d(\cos^3 \theta - a \sin^3 \theta)^2 - bc^2(\cos^3 \theta - a \sin^3 \theta)^4 - bk^2\}^{\frac{1}{2}}.$$

This equation, together with (12), gives the Darboux lines, the constants b, c, d being constants of the surface under consideration.

5. The existence of loxodromic Darboux lines on a surface depends on the form of the right-hand side of (7). There are two possibilities:

(i) A family of Darboux lines may exist cutting the lines of curvature $v = \text{constant}$ at a given angle ϵ .

(ii) Isolated lines may have this property.

In the first case the magnitudes for the surface must satisfy identically

$$\sqrt{\frac{E}{G}} \beta_2 \tan^3 \epsilon + \alpha_1 = 0. \quad (14)$$

In the second case (14) itself gives the line or lines, but the equation must then give for the line, as a consequence*,

$$\tan \epsilon = \sqrt{\frac{G'}{E}} v'.$$

I investigate here the following problem :

To find all the Weingarten surfaces on which there exists a family of loxodromic Darboux lines.

For a Weingarten surface, for which $\beta = f(a)$,

$$\log E = 2 \int \frac{da}{f-a}, \quad \log G = -2 \int \frac{f' da}{f-a}.$$

Equation (14) then gives

$$ra_2 + a_1 = 0, \quad (15)$$

where $\log r = \log a + \log f' + \int \frac{1+f'}{f-a} da \quad (\tan^2 \epsilon = a). \quad (16)$

Since r is a function of a , the general solution of (15) is

$$v = ru + \psi(a) \quad \text{where } \psi \text{ is arbitrary.} \quad (17)$$

whence

$$a_1 = -\frac{r}{ur' + \psi'},$$

$$a_2 = \frac{1}{ur' + \psi'},$$

$$a_{11} = \frac{2urr'^2 - ur^2r'' + 2rr'\psi' - r^2\psi''}{(ur' + \psi')^3},$$

$$a_{22} = -\frac{ur'' + \psi''}{(ur' + \psi')^3}.$$

The Gauss equation is

$$E^{-1} \left\{ \frac{2f'^2 a_1^2}{(f-a)^2} - \frac{f'' a_1^2}{f-a} - \frac{f' a_{11}}{f-a} \right\} + G^{-1} \left\{ \frac{2a_2^2}{(f-a)^2} + \frac{a_{22}}{f-a} \right\} + af = 0,$$

which may thus be written

$$\begin{aligned} E^{-1} \left\{ r^2(ur' + \psi') \left(\frac{2f'^2}{(f-a)^2} - \frac{f''}{f-a} \right) - \frac{rf'}{f-a} (2ur'^2 - urr'' + 2r'\psi' - r\psi'') \right\} \\ + G^{-1} \left\{ \frac{2(ur' + \psi')}{(f-a)^2} - \frac{ur'' + \psi''}{f-a} \right\} + a(ur' + \psi')^3 f = 0. \end{aligned}$$

* Cf. Hilton, *loc. cit.*, 522.

If we exclude the surface of revolution and developable, which (except for the anchor ring, circular cone, and cylinder) obviously cannot satisfy (14), we must have $r' = 0$, since u is the only variable present which is not a function of a .

Equation (17) then gives a to be a function of $v - bu$, say, where $r = b$. The value $b = 0$ merely gives the surfaces of revolution of the Dupin cyclide type.

Now (16) becomes

$$\log f' + \int \frac{1+f'}{f-a} da = \log \frac{b}{a},$$

whence

$$\frac{1+f'}{f-a} + \frac{f''}{f'} = 0,$$

and, on integration,

$$(f+a-2c)^2 = 4k(f-c).$$

The only Weingarten surfaces, therefore, on which a system of loxodromic Darboux lines can exist (with the exceptions given above) are those helicoids for which

$$(a+\beta-2c)^2 = 4k(\beta-c).$$

This class includes, for $k = 0$, the helicoids of constant mean curvature investigated in § 4.

6. The equation of the Darboux lines admits a first integral for the general helicoid when (as is always possible) the surface is referred to parameters along its lines of curvature such that

$$a = f(u+av), \quad \beta = \psi(u+av),$$

$$\frac{1}{2} \log E = - \int \frac{da}{a-\beta}, \quad \frac{1}{2} \log G = \int \frac{d\beta}{a-\beta}.$$

E and G are thus functions of p , where p is written for $u+av$.

Equation (7) becomes

$$3 \tan \theta \frac{d\theta}{du} = \frac{1}{2} a \sqrt{\frac{E}{G} \frac{G'}{G}} \tan^3 \theta - \frac{1}{2} \frac{E'}{E},$$

giving

$$6 \sin \theta \cos^2 \theta \left(1 + a \frac{dv}{du} \right) \frac{d\theta}{dp} = \sqrt{E} \left(\frac{aG'}{G^{\frac{3}{2}}} \sin^3 \theta - \frac{E'}{E^{\frac{3}{2}}} \cos^3 \theta \right),$$

which, on putting $\sqrt{(E/G)} \tan \theta$ for dv/du , gives

$$\frac{aG'}{G^{\frac{3}{2}}} \sin^3 \theta - \frac{6a \sin^2 \theta \cos \theta (d\theta/dp)}{\sqrt{G}} = \frac{E' \cos^3 \theta}{E^{\frac{3}{2}}} + \frac{6 \sin \theta \cos^2 \theta (d\theta/dp)}{\sqrt{E}}.$$

A first integral is thus

$$\frac{a \sin^3 \theta}{\sqrt{G}} - \frac{\cos^3 \theta}{\sqrt{E}} = b. \quad (18)$$

In the case of the special class, for which

$$(a + \beta - 2c)^2 = 4k(\beta - c),$$

we may write

$$E = m^{-2}, \quad G = (m-1)^{-2}, \quad a = 2km - km^2 + c, \quad \beta = km^2 + c,$$

where m is a function of $u + av$.

It is found that the Gauss equation is an ordinary differential equation of the second order, and can be integrated to give the form of m . The result may be used, as in § 4, to give the complete integral of the Darboux lines. The result, however, is lengthy.

AN EXTENSION OF THE THEOREM OF THE FIFTH ASSOCIATED LINE

By T. G. ROOM.

[Received 8 October, 1929.—Read 14 November, 1929.]

The theorem which it is sought to extend is: "If a, b, c, d are four arbitrary lines in $[4]$, and a', b', c', d' are the transversal lines of the sets of three of them, then the four $[3]$'s, $[a, a']$, etc., have a common line, the fifth associated line of a, b, c, d . These five lines form a symmetrical system, and any plane which meets four of them meets the fifth". The dual of this theorem is: "If $\alpha, \beta, \gamma, \delta$ are four arbitrary planes in $[4]$, and $\alpha', \beta', \gamma', \delta'$ are the four planes which meet sets of three of them in lines, then the four points, (α, α') , etc., lie in a plane, the associated plane of $\alpha, \beta, \gamma, \delta$. The five planes form a symmetrical system, and any line which meets four of them meets the fifth". An extension, of the type to be considered, of this dual theorem is already known†. It is obtained as follows. In $[6]$ take five arbitrary $[4]$'s. Any pair of these has a common plane, so that we have in each of them four planes, which will define a fifth associated plane. These five derived associated planes, one in each of the $[4]$'s, lie in a $[5]$. This $[5]$ is therefore uniquely defined by the five $[4]$'s, but, being of different dimension from them, cannot, of course, form with them a symmetrical configuration. It has, however, the "incidence" property in a weakened form, namely, that lying in it there are ∞^1 of the planes which exist meeting in lines the five $[4]$'s, whereas no other $[5]$ contains more than two of these, and a general one does not contain any.

The main chain of theorems with which this paper is concerned is that derived from a figure in $[7]$; it may be summarized as follows.

(i) Three planes in $[7]$ have a unique transversal line.

(ii) If the transversal lines of the sets of three out of four planes in $[7]$ meet one of the planes in three collinear points, then they meet

† Room, *Proc. London Math. Soc.* (2), 28 (1927), 336.

each of the planes in collinear points, and there is a [4] which contains the four transversal lines, and meets all the planes in lines. (This imposes one condition on the planes.) The converse is also true.

(iii) If there are [4]'s which meet in lines four of the sets of four out of five planes in [7], then there is a [4] which meets in lines the last set of four. These five planes and five [4]'s are said to form a double-five of [2]'s and [4]'s.

(iv) Two [4]'s in [7] have a common line; thus each [4] of a double-five contains four lines, its intersections with the others, and these four will define a unique fifth associated line. The five such fifth associated lines lie in a [4] and are themselves associated.

(v) The [4]'s of a double-five and the [4] which contains the fifth associated lines form a symmetrical system, and are called six associated [4]'s.

(vi) Dually, the five planes of a double-five define a sixth associated plane, forming with them a symmetrical system.

(vii) There is a plane which meets in lines each four [4]'s of an associated set. The fifteen planes obtained in this way form six sets of five which belong to double-fives, and to each of these sets, therefore, a sixth associated plane can be added. These six sixth associated planes are themselves associated, and there is a quadric with respect to which they are the polars of the six original [4]'s.

(viii) Any five planes in [7] are met by six planes. The six planes which meet five planes of an associated set of six also meet the sixth plane, and are themselves associated.

The six [4]'s of (v) form a system strictly analogous to a set of five associated lines in [4] in that any five members of the set define uniquely the sixth, and that—result (viii)—all spaces having certain intersections with five of them have a similar intersection with the sixth.

All the results are obtained by the method of "symbols of points"[†], and, in using this method, simplification of ideas can frequently be effected by ascribing to "syzygies" their geometric significance instead of leaving them as mere algebraic relations. For example, the theorems (i) to (v) of the chain are most easily proved by working with nine base points, which, since they lie in [7], must be connected by a syzygy, say $W = 0$, where W is the sum of algebraic multiples of the points. This is

[†] Cf. Baker, *Principles of geometry*, 1 (1922), 70.

equivalent to regarding the figure as primarily one in [8] (the space defined by the nine points), which has been projected from the point of that space whose symbol is W . Further, it so happens that for theorems (ii) to (v)—once the existence of the required lines is granted—the relation $W = 0$ is never required, so that the theorems, with slight modifications of wording, will be true in [8] as well as in [7]. The modified theorems are :

(ii)' If every set of three out of four planes in [8] has a transversal line, then the lines lie in a [4], and the points in which they meet any of the planes are collinear.

(iii)' Precisely the same as (iii), with [8] substituted for [7].

(iv)' Two [4]'s of a double-five in [8] have a line common, viz., the transversal of one of the sets of three planes. Thence as (iv).

(v)' Precisely the same as (v).

This is a situation very similar to that which arises in the proof of the ordinary Desargues theorem†, and, in fact, just as the plane Desargues theorem, from one point of view, is a theorem for [3] which has been projected on to a plane, so the theorems (ii) to (v) are theorems for [8] which can be projected into theorems of [7].

Thus in this paper we start in § 1 by obtaining the eight-dimensional theorems (ii)' to (v)'; this enables us in § 2 to write down easily the corresponding theorems for [7], and then, taking account of the syzygy $W = 0$, we proceed to complete the chain.

1. A figure defined by five [4]'s in [8].

Let $K_1, \dots, K_5$ be five [4]'s in [8] such that every pair has a common line. The space $[K_\alpha, K_\beta]$ joining K_α and K_β is a [7], and the six [7]'s $[K_\alpha, K_\beta]$, where $\alpha, \beta = 1, \dots, 4$ will have a common plane. This plane lies in the [5] ($[K_1, K_2], [K_1, K_3], [K_1, K_4]$), which contains K_1 , and therefore it meets K_1 in a line. Similarly, it meets K_2, K_3, K_4 in lines. To prevent confusion, and with an eye to subsequent developments, this plane will be called h_{56} . We obtain in this way a double-five of [4]'s $K_1, \dots, K_5$ and [2]'s $h_{15}, \dots, h_{56}$, in which a [2] h_{a6} meets a [4] K_β in a line if $\alpha \neq \beta$, but does not meet K_α at all.

Since h_{56} meets K_1 and K_2 both in lines, it must meet the line of intersection of these [4]'s in a point—the point of intersection of the two

† Cf., e.g., Baker, *ibid.*, 73.

lines. Thus the two sets of four lines (K_1, K_a) and (K_1, h_{a6}) in K_1 form a double-four. The associated line of either of these sets of four is the common intersection of the four [3]'s $[(K_1, K_a), (K_1, h_{a6})]$. If we denote temporarily this line by a_1 , and the other associated lines similarly by $a_2, \dots, a_5$, we see that the [7] $[K_1, h_{16}]$ contains all five lines a_a . It contains a_1 because it contains K_1 , and a_2 because it contains the [3] $[(K_2, K_1), (K_2, h_{26})]$. Thus the five lines a_a lie in each of the five [7]'s $[K_a, h_{a6}]$, and therefore in the space of intersection of these. This space of intersection cannot be a [3] (this can be seen, for example, by projecting from one of the lines a_a), nor of dimension greater than four. Call this [4] K_6 . Without much difficulty it can be shown that the five lines (K_a, K_6) —formerly a_a —are associated, and that any four of the [4]'s $K_1, \dots, K_6$ have a plane which meets them all in lines, and that if h_{56} meets $K_1, \dots, K_4$ in lines, and h_{12} meets $K_3, \dots, K_6$ in lines, then h_{56} and h_{12} meet the line (K_3, K_4) in the same point. We call this point A_{56}^{12} , where this is to be understood to be the same point as A_{12}^{56} and A_{56}^{21+} .

The figure now consists of sixty points A_{cd}^{ab} , which lie by sets of fifteen in six [4]'s $K_1, \dots, K_6$ — K_a contains all the points without suffix a —and form in each of these the usual fifteen point and fifteen line figure†, and by sets of six in fifteen planes $h_{a\beta}$ — $h_{a\beta}$ contains all points with upper (or lower) suffixes $a\beta$ —forming in these the vertices of quadrilaterals.

THEOREM 1.1. *If $K_1, \dots, K_5$ are five [4]'s in [8], every pair of which has a common line, and a_1 is the line in K_1 associated with the lines (K_1, K_a) , etc., then $a_1, \dots, a_5$ lie in a [4] K_6 and are themselves associated.*

$K_1, \dots, K_6$ will be called six associated [4]'s.

THEOREM 1.2. *Any four [4]'s of an associated set of six are met by a unique plane in lines.*

THEOREM 1.3. *Any [5] which meets in planes five of a set of six associated [4]'s, and does not lie in a [7] joining two of them, meets the sixth also in a plane§.*

Turning now to the algebraic point of view, we take the [7] $[K_a, K_\beta]$ to be

$$x_{a\beta}(=x_{\beta a})=0 \quad (\alpha, \beta=1, \dots, 5; \alpha \neq \beta).$$

† At best the geometrical arguments producing these results are the verbal equivalents of the algebra below, so that it does not seem necessary to give them.

‡ See, e.g., Baker, *Principles of geometry*, 4 (1925), 113.

§ The dual of this is considered at greater length in Theorem 3.2.

Then we can treat the $x_{\alpha\beta}$ as coordinates in [9], the figure in [8] being a prime section, say, by the prime

$$\sum_{\alpha, \beta=1}^5 x_{\alpha\beta} = 0.$$

Then the equations of the various spaces are :

$$\left. \begin{aligned} K_1: & x_{12} = x_{13} = x_{14} = x_{15} = 0, \quad x_{34} + x_{42} + x_{23} + x_{25} + x_{35} + x_{45} = 0, \\ h_{16}: & x_{34} = x_{42} = x_{23} = x_{25} = x_{35} = x_{45} = 0, \quad x_{12} + x_{13} + x_{14} + x_{15} = 0, \\ [K_1, h_{16}]: & x_{34} + x_{42} + x_{23} + x_{25} + x_{35} + x_{45} = x_{12} + x_{13} + x_{14} + x_{15} = 0, \\ \text{and therefore} & \\ K_6: & \sum_{a=1}^5 x_{aa} = 0 \quad (a = 1, \dots, 5). \end{aligned} \right\} \quad (1.1)$$

K_6 is thus represented by five independent equations in [9], and this constitutes the algebraic proof that it is a [4].

Now adjoin to the scheme the further coordinates $x_{a6} = x_{6a}$, defined by the equations

$$\sum_{a=1}^6 x_{aa} = 0 \quad (a = 1, \dots, 5; a \neq a).$$

From a geometrical point of view the $x_{\alpha\beta}$ are now best treated as coordinates in [14], the figure under consideration being that which lies in the [8].

$$\sum_{a=1}^6 x_{aa} = 0 \quad (a = 1, \dots, 6; a \neq a). \quad (1.2)$$

In this scheme the equations of the spaces are :

$$\left. \begin{aligned} K_1: & x_{12} = x_{13} = x_{14} = x_{15} = x_{16} = 0 \quad (\text{with } \sum_a x_{aa} = 0), \\ h_{16}: & x_{16} = x_{34} = x_{42} = x_{23} = x_{25} = x_{35} = x_{45} = 0 \quad (\text{with } \sum_a x_{aa} = 0). \end{aligned} \right\} \quad (1.3)$$

Collecting all the equations for A_{cd}^{ab} , we find that they reduce to

$$x_{ac} = x_{bd} = -x_{ad} = -x_{bc}, \quad \text{all other } x_{\alpha\beta} \text{ zero.}$$

Thus if we represent a general point of the [14] by $\sum_{\alpha\beta} x_{\alpha\beta} X_{\alpha\beta}$, we get

$$A_{cd}^{ab} = X_{ac} + X_{bd} - X_{ad} - X_{bc}. \quad (1.4)$$

This gives the syzygies which determine the whole configuration, viz.

$$\begin{aligned} A_{cd}^{ab} &= A_{ab}^{cd} = -A_{cd}^{ba}, \\ A_{cd}^{ab} + A_{db}^{ac} + A_{bc}^{ad} &= 0 \end{aligned} \quad (1.5)$$

—these are points of the line (K_e, K_f) , where a, b, c, d, e, f are all

different—and

$$A_{cd}^{ab} + A_{de}^{ab} + A_{ec}^{ab} = 0 \quad (1.6)$$

—these are points of the line (K_f, h_{ab}) .

The analogue of this notation for the ordinary fifth associated line theorem of [4] may be helpful. This we can obtain by considering that part of the figure which lies in K_6 . The five associated lines are (K_6, K_a) , and the transversals of sets of three of them are $(K_6, h_{a\beta})$. In one of the usual notations for the figure in [4] the whole set of fifteen lines is taken together, and the lines are denoted by dual symbols consisting of pairs from six digits†. We may take $p_{a6} = p_{6a}$ to be the line (K_6, K_a) , and $p_{a\beta}$ to be $(K_6, h_{a\beta})$. In the ordinary notation we have the point common to $p_{a\delta}, p_{\beta\gamma}, p_{\delta\epsilon}$ denoted by $a\delta.\beta\gamma.\delta\epsilon$, whereas in the notation used here it is $A_{\delta\epsilon}^{\beta\gamma}$, a symbol which destroys the symmetry of the figure. The symmetrical notation, however, clearly cannot be applied to any figure of the family other than that in [4]‡, because only in that case are the “associated” spaces and the spaces “transversal” to them of the same dimension.

We can, for the four-dimensional figure, write

$$A_{cd}^{ab} = X_{ac} + X_{bd} - X_{ad} - X_{bc} \quad (a, b, c, d = 1, \dots, 5)$$

(relations which will give all the required syzygies between the fifteen points $A_{\gamma\delta}^{a\beta}$), and so obtain it as the [4]-section of a simplex in [9], just as we can obtain the eight-dimensional figure as the section of a simplex in [14].

The six [4]’s $K_1, \dots, K_6$ are part of a symmetrical system of sixteen [4]’s, related to them in the same way as the sixteen planes of a Kummer figure are related to certain sets of six of the planes§. The remaining ten [4]’s are typified by $K_{123} = K_{456}$ (where the order of the suffixes is not significant), with equations

$$x_{23} = x_{31} = x_{12} = x_{56} = x_{64} = x_{45} \quad (\text{with } \sum_a x_{aa} = 0). \quad (1.7)$$

K_{123} contains fifteen points made up of a set of nine like A_{cd}^{ab} (where $a, b = 1, 2, 3; c, d = 4, 5, 6$), which lie by threes on two sets of three generators of a quadric in [3], these being the lines in which it meets $K_1, K_2, K_3; K_4, K_5, K_6$; and of the six points with symbols

$$5(X_{ad} + X_{be} + X_{cf}) - \sum_{a\beta} X_{a\beta}$$

(where $a, b, c = 1, 2, 3; d, e, f = 4, 5, 6$).

† Cf. Baker, *Principles of geometry*, 4 (1925), 115.

‡ See foot-note on p. 474 regarding the general figure of the family of which the four- and eight-dimensional figures discussed above are members.

§ See, e.g., Hudson, *Kummer’s quartic surface* (1905), 45.

These are such that the difference of two of them, which have one common point $X_{\alpha\beta}$ among those that occur in their symbols, is a point $A_{\alpha\beta}^{ab}$. For example

$$\{5(X_{15} + X_{26} + X_{34}) - \Sigma X_{\alpha\beta}\} - \{5(X_{16} + X_{25} + X_{34}) - \Sigma X_{\alpha\beta}\} = 5A_{56}^{12}. \quad (1.8)$$

These three points lie in both K_{123} and K_{124} , so that these two [4]'s have a common line.

THEOREM 1.4. *Five [4]'s in [8], every pair of which has a common line, define a set of sixteen [4]'s K_α , $K_{\alpha\beta\gamma}$, with the properties:*

(1) *Any pair intersects in a line.*

(2) *The fifteen lines in any one of them form the usual fifteen line fifteen point figure in [4].*

(3) *The set $K_1, \dots, K_6$ is associated†, and the five sets like $K_1, K_2, K_{123}, \dots, K_{126}$ are associated.*

(4) *h_{12} meets $K_{12\alpha}$ and K_α in the same line.*

(5) *The plane which meets $K_1, K_{123}, K_{124}, K_{125}$ in lines meets $K_6, K_{623}, K_{624}, K_{625}$ in the same lines‡.*

2. The double-five of planes and [4]'s in [7]§.

The figure which is to be considered in this section is actually the projection from an arbitrary point of the double-five of [4]'s and [2]'s discussed at the beginning of § 1, but its construction, since it involves Theorem 2.1 (below), is best effected independently. It has the important property of being self-dual, so that, in addition to a sixth associated [4] of the five [4]'s (which exists in consequence of its projection relation to the figure in [8]), there is a sixth associated plane.

To construct the figure from arbitrary elements, we take a plane h_{56} , and in it a quadrilateral whose vertices are the six points $A_{\alpha\beta}^{56}$ ($\alpha, \beta = 1, \dots, 4$)||, these being connected by the four relations

$$A_{\alpha\beta}^{56} + A_{\beta\gamma}^{56} + A_{\gamma\alpha}^{56} = 0.$$

† Theorem 1.1, p. 458.

‡ The equations of the plane are

$$-4x_{16} = x_{12} = x_{13} = x_{14} = x_{15} = x_{62} = x_{63} = x_{64} = x_{65} \quad (\text{with } \Sigma_\alpha x_{\alpha\alpha} = 0).$$

§ For the algebra which leads to the results of this section, see Appendix to § 2, which follows immediately, pp. 467-473.

|| It is convenient to use this notation, although nothing is implied by it except the usual convention $A_{\gamma\delta}^{\alpha\beta} = A_{\alpha\beta}^{\gamma\delta} = -A_{\gamma\delta}^{\beta\alpha}$. The only relations that are assumed are those given in the text.

Through each of these points draw an arbitrary line, and take on the line through $A_{\alpha\beta}^{56}$ two arbitrary points $A_{\beta\beta}^{5a}$ and $A_{\beta\beta}^{6a}$. We may suppose that

$$A_{\alpha\beta}^{56} + A_{\beta\beta}^{6a} + A_{\beta\beta}^{a5} = 0.$$

Denote by h_{a6} the plane $[A_{\beta\beta}^{a6}, A_{\gamma\beta}^{a6}, A_{\delta\beta}^{a6}]$. These five planes $h_{16}, \dots, h_{56}$ are then such that any set of four of them which includes h_{56} has a [4] which meets them all in lines. From the ten relations between the points which we have so far assumed, we can deduce four others such as

$$(A_{52}^{16} + A_{35}^{16}) + (A_{53}^{26} + A_{15}^{26}) + (A_{51}^{36} + A_{25}^{36}) = 0.$$

This typical relation is one connecting points of h_{16}, h_{26}, h_{36} respectively, so that these are the points of the transversal line of the three planes. The points in which the transversals from h_{16} to pairs of the other three planes meet h_{16} are thus $A_{53}^{16} + A_{45}^{16}$, $A_{54}^{16} + A_{25}^{16}$, $A_{52}^{16} + A_{35}^{16}$, and these are collinear. Thus the four transversal lines are such that each set of three of them has a transversal line, and they therefore lie in a [4] which meets $h_{16}, h_{26}, h_{36}, h_{46}$ each in a line.

The construction does not explicitly assume any particular number of dimensions for the space in which the figure lies, but since the figure is built up from a plane and six lines which meet it, it can lie in a space of dimension eight at most, but could equally well lie in a [7]. In [7] it is one restriction on the freedom of four planes that they should have a [4] meeting them all in lines, a condition which is equivalent to making collinear the points in which the transversals from any one of the planes to the pairs of the others meet the plane. Thus we have

THEOREM 2.1. *If five [2]'s in [7] are such that four of the sets of four of them each have a [4] meeting them in lines, then the same is true of the fifth set of four.*

We shall call this figure also a double-five of planes and [4]'s in [7]. It is necessarily the projection of a double-five of planes and [4]'s in [8]. The condition can be expressed more neatly in its dual form, viz., the necessary and sufficient condition that five planes in [7] should belong to a double-five is that they should be the sections of five [3]'s in [8], every pair of which has a common point†.

† Such a set of five [3]'s in [8] is defined by the ten points of intersection, and therefore has freedom 80; so that a general [7] section of the figure has freedom 71. Five planes in [7] have freedom 75, and that each set of four should have a "transversal [4]" imposes apparently five conditions. That only four conditions are actually imposed indicates that the existence of four of the [4]'s implies the existence of the last.

If the figure is taken to lie in [8], it is identical with that considered in § 1, and the scheme of points $A_{\gamma\delta}^{a\beta}$ can be completed accordingly [cf. equations (1.4), (1.5), (1.6)]. The figure in [7] will have exactly the same scheme of points, that it should lie in [7] being expressed analytically by there being one other syzygy connecting them. This is most conveniently written in terms of the points $X_{a\beta}$ in [14] which were introduced, say

$$W = \sum_{a\beta} p_{a\beta} X_{a\beta} = 0 \quad (p_{a\beta} = p_{\beta a}; \alpha, \beta = 1, \dots, 6).$$

Since this is equivalent to assuming that the figure in [7] is the projection from the point $\Sigma p_{a\beta} X_{a\beta}$ of the figure in the [8] whose equations are

$$\sum_a x_{aa} = 0, \quad (1.2)$$

i.e. since $\Sigma p_{a\beta} X_{a\beta}$ lies in the [8], we must have

$$\sum_a p_{aa} = 0 \quad (\text{six equations}).$$

The associated [4] in [7] of $K_1, \dots, K_5$ is constructed in exactly the same way as is the corresponding [4] in [8]: each of the [4]'s K_a is met by the others in lines, and the five fifth associated lines of these sets of four lie in the sixth associated [4]. The algebra which shows the existence of this [4] is precisely the same as that used in [8], and the new syzygy $W = 0$ is not required; but the sixth associated plane of $h_{16}, \dots, h_{56}$ is not definable in [8] except in terms of the point of projection—it is, in fact, derived from one member of an ∞^5 family of [3]'s.

To obtain this sixth associated plane geometrically, the figure in [7] is projected from one of the planes, say h_{56} , on to a [4]. We get from $h_{16}, \dots, h_{46}$ and $K_1, \dots, K_4$ a double-four of planes, which defines a fifth associated plane in the [4] in the usual way. This is joined to h_{56} , giving a [5]. Projecting from each of the planes h_{a6} in turn, we get five [5]'s, whose intersection is the required sixth associated plane. That they do intersect in a plane follows by duality from the existence of K_6 given $K_1, \dots, K_5$.

THEOREM 2.2. *Five planes which belong to a double-five of planes and [4]'s in [7] define a sixth associated plane, forming with them a symmetrical system, which is such that any five planes of it belong to a double-five.*

THEOREM 2.3. *The six planes which meet five of the planes of an associated set of six, also meet the sixth.*

To find points of the sixth associated plane we have to use the last syzygy, and this we want expressed in terms of points $A_{\gamma\delta}^{\alpha\beta}$ rather than $X_{\alpha\beta}$. Moreover, we want to work with as few points as possible. The smallest possible symmetrical set is $A_{\beta\delta}^{\alpha\beta}$ ($\alpha, \beta = 1, 2, 3, 4$). These are connected by the relations

$$A_{\gamma\delta}^{\beta\beta} + A_{\alpha\delta}^{\gamma\delta} + A_{\beta\delta}^{\alpha\alpha} = A_{\beta\delta}^{\gamma\delta} + A_{\gamma\delta}^{\alpha\alpha} + A_{\alpha\delta}^{\beta\beta} \quad (2.1)$$

[four relations, of which three are independent, directly deducible from equations (1.1)],

$$\sum_{\alpha, \beta} q_{\alpha\delta}^{\beta\delta} A_{\alpha\delta}^{\beta\beta} = 0 \quad (2.2)$$

[see Appendix (C 3), p. 470].

When we project from h_{56} , i.e. from the points $A_{\alpha\beta}^{56}$, we get a figure in [4] which depends on the twelve points $*A_{5\beta}^{6\alpha}$, which are coincident by pairs— $A_{\beta\delta}^{\alpha\alpha}$ and $A_{\alpha\delta}^{\beta\beta}$ being collinear with $A_{\delta\delta}^{\alpha\beta}$ project into the same point $*A_{\beta\delta}^{\alpha\alpha} = *A_{\alpha\delta}^{\beta\beta}$ —and are further connected by the relation (3.2) which reduces to

$$\sum_{\alpha\beta} p_{\alpha\beta} (*A_{\beta\delta}^{\alpha\alpha} + *A_{\alpha\delta}^{\beta\beta}) = 0. \quad (2.3)$$

The projection $*h_{16}$ of h_{16} is the plane through the points $*A_{25}^{16}, *A_{35}^{16}, *A_{45}^{16}$, so that the associated plane of $*h_{16}, *h_{26}, *h_{36}, *h_{46}$ passes through the four points like

$$p_{12} *A_{25}^{16} + p_{13} *A_{35}^{16} + p_{14} *A_{45}^{16}.$$

Thus the [5] which joins it to h_{56} contains the four points like

$$1_5^6 = p_{12} A_{25}^{16} + p_{13} A_{35}^{16} + p_{14} A_{45}^{16}.$$

We have now to find the points common to the five [5]'s defined in this way; this is a matter of straightforward algebra [see Appendix (D 1)], and we get the result: The associated plane h_6 , of $h_{16}, \dots, h_{56}$, contains the ten points $B_{\alpha\beta}^6$ ($\alpha, \beta = 1, \dots, 5$), where

$$p_{ca} p_{cb} B_{ab}^c = p_{cb} \sum_{\gamma} p_{c\gamma} A_{b\gamma}^{ca} - p_{ca} \sum_{\gamma} p_{c\gamma} A_{a\gamma}^{cb}$$

(γ takes in the summation the three values 1, 2, ..., 6 other than abc). From the relations between these points we deduce the following geometrical theorems.

THEOREM 2.4. (1) *The line which meets h_{16}, h_{26}, h_6 , does so in the points $6_2^1, 1_2^6, B_{12}^6$. [Appendix (D 1.2).]*

(2) *The ten points $B_{\alpha\beta}^6$ which lie in h_6 , lie by sets of three on ten lines—i.e. they form a Desargues figure—[Appendix (D 2.2)].*

(3) The plane h_{16} contains the six points $A_{\alpha\beta}^{16}$, the eight points 6_a^1 , and 1_a^6 , and the point $C_{16} = C_{61} = 6_a^1 + 1_a^6$, these fifteen points lying by sets of three on twenty lines, with four lines through every point [Appendix (D 2.4)].

(4) The line $[B_{1\alpha}^6, B_{6\alpha}^1]$ meets h_{16} ; thus the three planes h_1, h_6, h_{16} are met by four lines in sets of four points which are not collinear, and they, like the sets of three planes $h_{\beta\gamma}, h_{\gamma\alpha}, h_{\alpha\beta}$, lie in a [5] [Appendix (D 2.5)].

It is now not difficult to show that the planes $h_1, \dots, h_6$ are themselves associated. The first step towards proving this is to find the line which meets any three of the planes. Here, again, it is only a matter of elementary algebra to show that the point D_{23}^1 of h_1 , defined by

$$\Delta D_{23}^1 = \sum_{\alpha} p_{1\alpha} r_{23}^{\alpha\alpha} B_{2\alpha}^1 \quad (\alpha \neq 1, 2, a; a \text{ arbitrary}),$$

is collinear with the similarly defined points D_{31}^2 and D_{12}^3 of h_2 and h_3 , and that these are the only such set of three points [Appendix (D 3.2)]. Then it appears [Appendix (D 3.3)] that the three points $D_{34}^1, D_{42}^1, D_{23}^1$ in h_1 are themselves collinear, so that the four transversals to sets of three of the planes $h_1, \dots, h_4$ lie in a [4] which meets those planes in lines.

THEOREM 2.5. *The planes $h_1, \dots, h_6$ are associated.*

Denote by $K_{\alpha\beta}$ the [4] which meets in lines the set of four planes obtained by omitting h_α and h_β . $K_{16}, \dots, K_{56}$ form with $h_1, \dots, h_5$ a double-five, and there is therefore a sixth [4] associated with them. K_{16} contains the double-four of lines defined by the point scheme :

$$\begin{array}{ccc} D_{45}^2 & D_{53}^2 & D_{34}^2 \\ D_{54}^3 & D_{25}^3 & D_{42}^3 \\ D_{35}^4 & D_{52}^4 & D_{23}^4 \\ D_{43}^5 & D_{24}^5 & D_{32}^5 \end{array},$$

whose fifth associated line, in consequence of the relation

$$D_{\gamma\delta}^\alpha - D_{\gamma\delta}^\beta = A_{\gamma\delta}^{\alpha\beta} \quad [\text{Appendix (D 3.4)}],$$

contains the three points $A_{25}^{34}, A_{35}^{42}, A_{45}^{23}$. Thus the sixth associated [4] of the five [4]'s $K_{16}, \dots, K_{56}$ contains all the points $A_{\gamma\delta}^{\alpha\beta}$ with $\alpha, \beta, \gamma, \delta = 1, \dots, 5$. It is therefore K_6 .

THEOREM 2.6. *The sixth associated [4] of the [4]'s $K_{16}, \dots, K_{56}$ which form, with the planes $h_1, \dots, h_6$, a double-five is K_6 .*

The final result is:—

THEOREM 2.7. *There is a quadric with respect to which K_a is the polar of h_a , and $K_{a\beta}$ of $h_{a\beta}$.*

By Appendix, (E), this quadric is represented by the equation

$$\sum_{a, \beta} p_{a\beta}^{-1} x_{a\beta}^2 = 0$$

between the fifteen coordinates $x_{a\beta}$, since the points A_{cd}^{ab} and $D_{a\beta}^e$ are conjugate with respect to it for any a, β , provided that $e \neq a, b, c, d$. This gives all that we want, since the points $D_{a\beta}^e$ lie in h_e , and the points A_{cd}^{ab} with $e \neq a, b, c, d$ lie in K_e . Incidentally, it may be noticed that $A_{cd}^{ab}, A_{ef}^{cd}, A_{ab}^{ef}$ form a self-polar triangle with respect to the quadric.

The general trend of the argument may be summarized best by the following table.

Take five [4]'s $K_1, \dots, K_5$ in [7] which are such that any four of them are met in lines by a plane. These define a sixth associated [4] K_6 .

Six associated [4]'s

$K_1, \dots, K_6$.



Fifteen planes

$h_{12}, \dots, h_{56}$.



Six sixth associated planes, $h_1, \dots, h_6$, which are themselves associated.



Fifteen [4]'s

$K_{12}, \dots, K_{56}$.



Six sixth associated [4]'s, which are themselves associated, being, in fact, $K_1, \dots, K_6$.

$K_1, \dots, K_4$ are met by h_{56} in lines, etc., giving

$h_{16}, \dots, h_{56}$ are one row of a double-five of planes and [4]'s in [7], and so define a sixth associated plane, h_6 . We obtain thus

$h_1, \dots, h_4$ are met by a [4], K_{56} in lines, etc., giving

$K_{16}, \dots, K_{56}$ are one row of a double-five of [4]'s and planes in [7], and so define a sixth associated [4]. This [4] is K_6 .

In this form the chain of theorems exhibits a striking analogy with Grace's extension of the double-six†, where the chain is :

Six arbitrary lines,
 $a_1, \dots, a_6$, having a
common transversal
 f (in [3]).

↓
Fifteen lines
 $b_{12}, \dots, b_{56}$.

↓
Six lines $c_1, \dots, c_6$
(which have a com-
mon transversal e).

↓
Fifteen lines
 $d_{12}, \dots, d_{56}$.

↓
Six lines $a_1, \dots, a_6$.

$a_1, \dots, a_4$ have one other transversal b_{56} , giving

f and $b_{16}, \dots, b_{56}$ are one row of a double-six, of which $a_1, \dots, a_5$ are five lines of the other row, so that the double-six will be completed by a unique line c_6 , transversal to $b_{16}, \dots, b_{56}$. We obtain thus

$c_1, \dots, c_4$ have one other transversal d_{56} , etc., giving

$d_{16}, \dots, d_{56}$ are five lines of one row of a double-six, and therefore have a common transversal. This is actually, of course, a_6 .

For both figures there is a quadric with respect to which one-half of the spaces involved are polars of the other.

APPENDIX TO § 2. *The algebra leading to the results of § 2.*

A. *Properties of a certain determinant.*

The determinant is

$$\begin{vmatrix} 0 & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ p_{21} & 0 & p_{23} & p_{24} & p_{25} & p_{26} \\ p_{31} & p_{32} & 0 & p_{34} & p_{35} & p_{36} \\ p_{41} & p_{42} & p_{43} & 0 & p_{45} & p_{46} \\ p_{51} & p_{52} & p_{53} & p_{54} & 0 & p_{56} \\ p_{61} & p_{62} & p_{63} & p_{64} & p_{65} & 0 \end{vmatrix},$$

† Grace, *Trans. Camb. Phil. Soc.*, 16 (1898), 164-5. Wakeford, *Proc. London Math. Soc.*, 21 (1922), 127. Baker, *Principles of geometry*, 4 (1925), 59.

where

$$p_{\alpha\beta} = p_{\beta\alpha},$$

$$\sum_{\alpha} p_{\alpha i} = 0.$$

All first minors of the determinant (taken with suitable sign) have the same value, say Δ . [Proved by adding the terms in each column of the minor.] (A 1)

If $r_{\gamma\delta}^{\alpha\beta}$ is the four-row minor obtained by omitting the α -th and β -th columns, and the γ -th and δ -th rows, and arranging the remaining rows and columns so that the terms in the leading diagonal are $p_{\lambda\lambda'}$, $p_{\mu\mu'}$, $p_{\nu\nu'}$, $p_{\rho\rho'}$, where $(\alpha\beta\lambda\mu\nu\rho)$ and $(\gamma\delta\lambda'\mu'\nu'\rho')$ are both even permutations of (123456), then

$$\left. \begin{aligned} r_{\gamma\delta}^{\alpha\beta} &= r_{\alpha\beta}^{\gamma\delta} = -r_{\gamma\delta}^{\beta\alpha} \\ r_{\delta\epsilon}^{\alpha\beta} + r_{\epsilon\gamma}^{\alpha\beta} + r_{\gamma\delta}^{\alpha\beta} &= 0 \end{aligned} \right\}, \quad (\text{A } 2)$$

and

$$\Delta = \sum_{\alpha} p_{\alpha\alpha} r_{\alpha c}^{ab} \quad (a, b, c \text{ fixed, } b \text{ may be equal to } a \text{ or } c), \quad (\text{A } 3)$$

$$0 = \sum_{\alpha} p_{\alpha d} r_{\alpha c}^{ab} \quad (a, b, c, d \text{ fixed, } d \neq c, a). \quad (\text{A } 4)$$

B. *Points in the [14]† in which the figure is contained.*

We start with fifteen points $X_{\alpha\beta} = X_{\beta\alpha}$ ($\alpha, \beta = 1, \dots, 6$; $\alpha \neq \beta$) in [14], and suppose that a general point of the space is represented by $\sum_{\alpha, \beta} x_{\alpha\beta} X_{\alpha\beta}$. The figure to be considered lies in the [8] whose equations are

$$\sum_{\alpha} x_{\alpha\alpha} = 0 \quad (a = 1 \dots 6),$$

and is a projection from the point

$$W = \sum_{\alpha, \beta} p_{\alpha\beta} X_{\alpha\beta}$$

of the [8], where

$$\sum_{\alpha} p_{\alpha\alpha} = 0.$$

We shall need the following points in the [14].

$$X_{\alpha} \text{ defined by } X_{\alpha} = \sum_{\alpha} p_{\alpha\alpha} X_{\alpha\alpha} \quad (\text{six points}). \quad (\text{B } 1)$$

These six points lie in a [5] which contains W .

† First introduced at equations (1.2).

$$X_a^b \text{ defined by } X_a^b = \sum_a p_{ab} X_{aa} \quad (\text{thirty points}). \quad (\text{B } 2.1)$$

These are connected with the points X_a by the relations

$$\sum_{\beta} X_a^{\beta} + X_a = 0. \quad (\text{B } 2.2)$$

$$\begin{aligned} Z_{bc}^a \text{ defined by } Z_{bc}^a &= -Z_{cb}^a = \frac{1}{\Delta} \sum_{a, \beta} p_{a\beta} (r_{aa}^{bc} + r_{a\beta}^{b*}) X_{a\beta} \\ &= \frac{1}{\Delta} \sum_{\gamma} r_{a\gamma}^{bc} X_{\gamma}, \end{aligned} \quad (\text{B } 3.1)$$

$$Z_{bc}^a - Z_{bc}^d = \frac{1}{\Delta} r_{bc}^{ad} W, \quad (\text{B } 3.2)$$

so that, when the projection is made from W , the top suffix may be omitted. These are connected by the relations

$$Z_{cd}^a + Z_{db}^a + Z_{bc}^a = 0. \quad (\text{B } 3.3)$$

The relations between the X_a and the Z_{bc}^d are

$$\left. \begin{aligned} \sum_{\beta} p_{\beta c} Z_{\beta c}^a &= X_c \\ \sum_{\beta} p_{\beta d} Z_{\beta c}^a &= X_c \end{aligned} \right\}. \quad (\text{B } 3.4)$$

The six points Z_{ab}^c having the same suffixes a, b all lie on a line through W , and the whole family of points $Z_{a\beta}^{\gamma}$ lies in the [5] containing the points X_{δ} .

C. The points in the [8].

The figure in the [8], *i.e.* the figure of § 1, is completely defined by the points

$$A_{\gamma\delta}^{a\beta} = X_{a\gamma} + X_{\beta\delta} - X_{a\delta} - X_{\beta\gamma}, \quad (\text{C } 1) \text{ [also (1.4)]}$$

which initially are points of [14], but which, since they all satisfy the six equations

$$\sum_a x_{aa} = 0 \quad (1.2)$$

all lie in an [8] in the [14].

An expression for the point W is required in terms of the points $A_{\beta\delta}^{a\beta}$. These are connected by the four relations (of which three are

independent)

$$A_{\gamma 5}^{\beta 6} + A_{a 5}^{\gamma 6} + A_{\beta 5}^{a 6} = A_{\beta 5}^{\gamma 6} + A_{\gamma 5}^{a 6} + A_{a 5}^{\beta 6} \quad (a, \beta, \gamma = 1, 2, 3, 4),$$

(C 2) [also (2.1)]

so that the coefficients of W cannot be fixed uniquely. Suppose that

$$\mu W = \mu \sum_{a, \beta=1}^6 p_{a\beta} X_{a\beta} = \sum_{a, \beta=1}^4 q_{\beta 5}^{a 6} A_{\beta 5}^{a 6},$$

then

$$q_{\beta 5}^{a 6} + q_{\beta 6}^{a 5} = \mu p_{a\beta} \quad (\text{six equations}),$$

$$q_{25}^{16} + q_{35}^{16} + q_{45}^{16} = -\mu p_{15}, \text{ etc.}$$

(two sets of four equations),

$$\sum_{a, \beta} q_{\beta 5}^{a 6} = \mu p_{56}.$$

Any symmetrical function of the $p_{a\beta}$ which satisfies these equations may be used. One such is

$$\left. \begin{aligned} q_{\beta 5}^{a 6} &= p_{a\beta} p_{56} + p_{a5} p_{\beta 6} - p_{a6} p_{\beta 5}, \\ 2p_{56} W &= \sum_{a, \beta} q_{\beta 5}^{a 6} A_{\beta 5}^{a 6}. \end{aligned} \right\} \quad (\text{C } 3)$$

Then

D. *The points in the* [7].

D1. *The points of the transversals to* $h_{a\beta}$, $h_{a\gamma}$, h_a .

Define points a_c^b and B_{bc}^a by the relations

$$\begin{aligned} a_c^b &= \frac{1}{p_{ab}} \sum_a p_{aa} A_{ca}^{ab} \quad (\text{three terms in the summation}). \\ &= \frac{1}{p_{ab}} (X_a^b - X_b) - X_{bc} + X_{ba} + X_{ca}. \end{aligned} \quad (\text{D } 1.1)$$

a_c^b is a point of h_{ab} .

$$\left. \begin{aligned} B_{bc}^a &= -B_{cb}^a = a_c^b - a_b^c \\ &= \frac{1}{p_{ab}} (X_a^b - X_b) - \frac{1}{p_{ac}} (X_a^c - X_c) \end{aligned} \right\}. \quad (\text{D } 1.2)$$

We saw that the associated plane of h_{16} , ..., h_{56} was that common to the five [5]'s like $[h_{56}, 1_5^6, 2_5^6, 3_5^6, 4_5^6]$. It is a simple matter to express all these points in terms of $A_{\beta 5}^{a 6}$, and then, using the known syzygies

between these points—C 2 and C 3—to find points of the plane common to the [5]'s. It is found that a general point of the plane expressed in terms of the points of the typical [5] above is

$$\lambda_1 1_5^6 + \lambda_2 2_5^6 + \lambda_3 3_5^6 + \mu_1 A_{56}^{41} + \mu_2 A_{56}^{42} + \mu_3 A_{56}^{43},$$

where

$$\mu_i = \lambda p_{i5} + \lambda_i,$$

and

$$\lambda p_{56} = \lambda_1 + \lambda_2 + \lambda_3.$$

With $\lambda_1 = -\lambda_2$, $\lambda_3 = 0$, $\lambda = 0$, we get, as a point of h_6 , the point

$$6_2^1 - 6_1^2 = B_{12}^6.$$

Thus the transversal to h_{16} , h_{26} , h_6 meets those planes in 6_2^1 , 6_1^2 , and B_{12}^6 ; h_6 contains the ten points $B_{\alpha\beta}^6$.

D 2. *The figures formed by the points A_{cd}^{ab} , B_{cd}^a , a_c^b .*

We have these relations between the points :

$$A_{cd}^{ab} = a_c^b - a_d^b = c_a^d - c_b^d, \quad (\text{D } 2.1)$$

and thence, by (D 1.2),

$$B_{bc}^a + B_{ca}^b + B_{ab}^c = 0. \quad (\text{D } 2.2)$$

The relations (D 1.2), (D 2.1), (D 2.2) also serve as a check on the result that h_{16} , ..., h_{56} , h_6 are associated, because the twelve points of the scheme below define a double-four of lines which lies in a [4] which meets h_{16} , h_{26} , h_{36} , h_6 in lines.

$$\begin{array}{ccc} 6_3^1 & 6_2^1 & A_{23}^{16} \\ 6_3^2 & 6_1^2 & A_{31}^{26} \\ 6_2^3 & 6_1^3 & A_{12}^{36} \\ B_{23}^6 & B_{31}^6 & B_{12}^6. \end{array}$$

The relations (D 2.2) are not sufficient of themselves to make all the points $B_{\beta\gamma}^a$ lie in h_a , but there are additional relations like

$$\sum_a p_a B_{ab}^a = 0 \quad (\text{four terms in the summation}). \quad (\text{D } 2.3)$$

The relations which give rise to the other theorems which are quoted in

Theorem 2.4 are

$$\left. \begin{aligned} a_c^b + b_c^a &= a_d^b + b_d^a = \dots = C_{ab} = C_{ba} \\ C_{ab} &= \frac{1}{p_{ab}} \sum_a (p_{aa} - p_{ab})(X_{aa} - X_{ab}) \end{aligned} \right\}. \quad (\text{D } 2.4)$$

$$p_{ac} B_{bc}^a - p_{bc} B_{ac}^b = \frac{1}{p_{ab}} \sum_a q_{ca}^{ab} A_{ca}^{ab} \quad (\text{D } 2.5)$$

(and the point on the right-hand side lies in h_{ab}).

D3. *The transversals h_a, h_b, h_c .*

Define points D_{bc}^a by the relations

$$D_{bc}^a = X_{ab} - X_{ac} + Z_{bc}. \quad (\text{D } 3.1)$$

We may omit the top suffix of the points Z_{bc}^a of equations (B 3.1), in virtue of equations (B 3.2), since in the [7] we have the syzygy $W = 0$. The ten points $D_{\beta\gamma}^a$ lie in h_a . From (B 3.1) and (D 3.1) we get

$$D_{bc}^a + D_{ca}^b + D_{ab}^c = 0 \quad (\text{D } 3.2)$$

(so that these are three points on the transversal line to h_a, h_b, h_c), and

$$D_{bc}^d + D_{ca}^d + D_{ab}^d = 0 \quad (\text{D } 3.3)$$

(so that the points in which the transversals from h_d to the pairs of h_a, h_b, h_c meet h_d are collinear).

The equations which give K_u as the sixth associated [4] defined by the five [4]'s K_{aa} are

$$D_{bc}^a - D_{bc}^d = D_{ad}^b - D_{ad}^c = X_{ab} - X_{ac} - (X_{db} - X_{dc}) = A_{bc}^{ad}. \quad (\text{D } 3.4)$$

Again, the relations (D 3.3) are not sufficient to make the ten points $D_{\beta\gamma}^a$ coplanar, but there are the additional relations

$$\sum_a p_{aa} D_{ba}^a = 0. \quad (\text{D } 3.5)$$

There are also relations of the type

$$\sum_{\alpha, \beta} q_{\beta\beta}^{aa} D_{b\beta}^a = 0,$$

which are of exactly the same form as (C 3), as well as some of the same form as (C 2), so that it is easily possible to build up, on a set of six associated [4]'s like $K_{16}, \dots, K_{56}, K_6$, a figure precisely equivalent to that defined by $K_1, \dots, K_6$.

E. *The quadric with respect to which the figure in [7] is self-polar.*

A quadratic equation in the fifteen coordinates $x_{a\beta}$ will represent a quadric in the [7] only if, when considered as representing a quadric in the [14], the section of this quadric by the [8]

$$\sum_a x_{aa} = 0$$

is a cone whose vertex is W , the point from which the figure in [8] is to be projected. That is, if the quadric in [14] passes through W and the [8] lies in the tangent prime at this point.

$$Q = \sum_{a, \beta} p_{a\beta}^{-1} x_{a\beta}^2 = 0$$

satisfies these conditions.

The polar—in [14]—of the [8]

$$\sum_a x_{aa} = 0$$

with respect to Q is the [5] through the six points $X_a = \sum_a p_{aa} X_{aa}$. The [8] contains all the points $A_{\gamma\delta}^{a\beta}$ and the [5] contains all the points $Z_{\mu\nu}^\lambda$; thus every point $A_{\gamma\delta}^{a\beta}$ is conjugate to every point $Z_{\mu\nu}^\lambda$, and therefore A_{cd}^{ab} is conjugate to

$$D_{fg}^e = X_{ef} - X_{eg} + Z_{fg},$$

if it is conjugate to $X_{ef} - X_{eg}$, i.e. if $e \neq a, b, c, d$. Thus all points of K_e are conjugate to all points of h_e .

3. *The dual figure in [8] of that considered in § 1.*

The foundation of this figure in [8] is five [3]'s, $k_1, \dots, k_5$, every pair of which has a common point; that is, ten points $A_{a\beta} = (k_a, k_\beta)$, arranged in five sets of four. They may be supposed to be connected by the syzygy

$$\sum_{a\beta} A_{a\beta} = 0 \quad (a, \beta = 1, \dots, 5).$$

The points may also be arranged in five sets of six like A_{ij} ($i, j = 1, \dots, 4$), defining five [5]'s, $H_{16}, \dots, H_{56}$. These [5]'s, with the five [3]'s, form a double-five, which is dual to that considered in § 1; k_1 and H_{16} have a common point

$$-(A_{12} + A_{13} + A_{14} + A_{15}) = A_{34} + A_{42} + A_{23} + A_{25} + A_{35} + A_{45} = A_{16} \quad (\text{say}),$$

and the five points $A_{a6} = (k_a, H_{a6})$ are connected by the syzygy

$$\sum_a A_{a6} = 0.$$

They therefore lie in a [3]. This is the associated [3] k_6 .

THEOREM 3.1. *Five [3]'s in [8], every pair of which has a common point, define a unique sixth associated [3], forming with them a symmetrical system†.*

The complete dual figure consists of fifteen points $A_{a\beta}$ connected by the syzygies

$$\sum_a A_{a\alpha} = 0,$$

of six [3]'s like $k_1 = [A_{12}, A_{13}, A_{14}, A_{15}, A_{16}]$, and of fifteen [5]'s like $H_{56} = [A_{56}, A_{23}, A_{31}, A_{12}, A_{14}, A_{24}, A_{34}]$. We can add ten other [3]'s to make a completely symmetrical system, as in Theorem 1.4, but we are

† From the way in which this result is obtained, it is clear that there is a general theorem:

$n+2$ [n]'s in $[\frac{1}{2}(n+1)(n+2)-2]$, every pair of which has a common point, define a unique $(n+3)$ -th associated [n], forming with them a symmetrical system.

The projection of this figure from one of the [n]'s is the previous member of the family, viz. is the figure of $n+2$ associated $[n-1]$'s in $[\frac{1}{2}n(n+1)-2]$. Dually we have the extension of Theorem 1.1:

$n+2$ $[\frac{1}{2}n(n+1)-2]$'s in $[\frac{1}{2}(n+1)(n+2)-2]$, every pair of which lies in a prime, define a unique $(n+1)$ -th associated $[\frac{1}{2}n(n+1)-2]$ forming with them a symmetrical system.

The figure is built up in this way: any one of the $[\frac{1}{2}n(n+1)-2]$'s is met by the other $n+1$ of the given set in $n+1$ $[\frac{1}{2}(n-1)(n-2)]$'s, every pair of which lies in a prime in the $[\frac{1}{2}n(n+1)-2]$. These $n+1$ $[\frac{1}{2}(n-1)(n-2)]$'s define a $(n+2)$ -th associated $[\frac{1}{2}(n-1)(n-2)]$, and we obtain in this way $n+2$ such spaces, one in each of the given $[\frac{1}{2}n(n+1)-2]$'s, which lie in a $[\frac{1}{2}n(n+1)-2]$, the $(n+3)$ -th associated $[\frac{1}{2}n(n+1)-2]$, and are themselves associated.

Moreover, the nature of the algebra in the appendix to §2 shows that the theorems of that section can be similarly generalized. We take in $[m]$, $m = \frac{1}{2}(n+1)(n+2)-2$, the $\frac{1}{2}n(n+1)(n+2)(n+3)$ points $A_{\gamma\delta}^{\alpha\beta}$ ($\alpha, \beta, \gamma, \delta = 1, \dots, n+3$), which satisfy the relations

$$\left. \begin{aligned} A_{\gamma\delta}^{\alpha\beta} &= A_{\alpha\beta}^{\gamma\delta} = -A_{\gamma\delta}^{\beta\alpha} \\ A_{\gamma\delta}^{\alpha\beta} + A_{\delta\beta}^{\alpha\gamma} + A_{\beta\gamma}^{\alpha\delta} &= 0 \\ A_{\delta\epsilon}^{\alpha\beta} + A_{\delta\epsilon}^{\beta\gamma} + A_{\epsilon\delta}^{\gamma\alpha} &= 0 \end{aligned} \right\}.$$

These can be derived from $\frac{1}{2}(n+2)(n+3)$ points $X_{a\beta}$ ($= X_{\beta a}$), which form a simplex of reference in $[m+n+3]$, by the relations

$$A_{\gamma\delta}^{\alpha\beta} = X_{\alpha\gamma} + X_{\beta\delta} - X_{\alpha\delta} - X_{\beta\gamma}.$$

The coordinates of all the points $A_{\gamma\delta}^{\alpha\beta}$ satisfy the $n+3$ equations

$$\sum_a x_{a\beta} = 0.$$

The points $A_{\gamma\delta}^{\alpha\beta}$ whose suffixes take all possible values from a set of $n+2$ of the numbers $1, \dots, n+3$, lie in a $[m-n-1]$, and define an "associated" figure in that space, while those

mainly concerned in this section with some loci defined by the original five [3]'s.

We may consider the figure as the projection of one in [14] from the six points $U_a = \sum_a A_{aa}$. If $\sum_{a,\beta} x_{a\beta} A_{a\beta}$ is taken to be a general point in the [14], we get for the various spaces systems of equations, of which those for k_6 and H_{56} are typical, viz. :

$$(k_6) \quad x_{a\beta} = 0 \quad (a, \beta = 1, \dots, 5)$$

(ten equations, giving a [4] in [14], which passes through the point U_6 , and consequently projects into a [3] in [8]);

$$(H_{56}) \quad x_{a5} = x_{a6} = 0 \quad (a = 1, \dots, 4)$$

(eight equations, giving a [6], which passes through

$$U_1 + U_2 + U_3 + U_4 - U_5 - U_6,$$

and consequently projects into a [5] in [8]).

which have two given upper suffixes, and the other two suffixes arbitrary, lie in a $[n-1]$, and are the points of intersection of sets on $n-1$ from $n+1$ primes in that space. The figure in $[m]$ consists of $n+3$ "associated" $[m-n-1]$'s and the $\frac{1}{2}(n+2)(n+3)$ $[n-1]$'s which meet sets of $n+1$ of them in $[n-2]$'s.

We now project from the arbitrary point

$$\sum_{a,\beta} p_{a\beta} x_{a\beta},$$

where, since the point is to lie in the $[m]$,

$$\sum_a p_{a\beta} = 0,$$

and define points D_{bc}^a of the $[m-1]$ on to which the projection is made by

$$D_{bc}^a = X_{ab} - X_{ac} + \frac{1}{\Delta} \sum_{a,\beta} p_{a\beta} (r_{aa}^{bc} + r_{a\beta}^{bc}) X_{a\beta} \quad (a, b, c, \alpha, \beta = 1, \dots, n+3),$$

where the $r_{\gamma\delta}^{a\beta}$ are the $(n+1)$ -row minors of the determinant of the $p_{a\beta}$, and Δ is the common value of all the $(n+2)$ -row minors of the determinant.

We get in $[m-1]$:

$n+3$ "associated" $[m-n-1]$'s, K_a .

K_a contains all the points $A_{\gamma\delta}^{a\beta}$ which have $\alpha, \beta, \gamma, \delta \neq a$.

$\frac{1}{2}(n+2)(n+3)$ $[n-1]$'s, $h_{a\beta}$.

h_{ab} contains all the points $A_{\gamma\delta}^{ab}$, and meets in a line each of the spaces K_a except K_a and K_b .

$n+3$ "associated" $[n-1]$'s h_a .

h_a contains all the points $D_{\gamma\delta}^a$, and forms with the $n+2$ $[n-1]$'s $h_{a\beta}$ which have a as one suffix, a set of $n+3$ "associated" spaces.

$\frac{1}{2}(n+2)(n+3)$ $[m-n-1]$'s, $K_{a\beta}$.

K_{ab} contains all the points $D_{\beta\gamma}^a$ and $A_{\beta\gamma}^{ab}$ which have $\alpha, \beta, \gamma, \delta \neq a, b$. The $n+2$ spaces

$K_{a\beta}$, which have a as one suffix, form with K_a an "associated" set of $n+3$ spaces.

Further, the two halves of the system are polars with respect to the quadric

$$\sum_{a,\beta} p_{a\beta}^{-1} x_{a\beta}^2 = 0.$$

Meeting any five [3]'s in [8] there are ∞^3 planes, which form a manifold of order forty†. Neither the dimension nor the total order of this manifold is disturbed by inserting the condition that each pair of the five [3]'s shall have a common point, but the manifold consists of several parts. Consider first the planes which pass through A_{12} and meet k_3, k_4, k_5 . They either lie entirely in all three [7]'s $[A_{12}, k_4, k_5], \dots$, i.e. in the [5] H_{12} common to them, or they pass through one of the lines $[A_{12}, A_{45}], [A_{12}, A_{53}], [A_{12}, A_{34}]$, and meet k_3, k_4, k_5 respectively. In the first case the plane passes through a point A_{12} of a [5], and meets three planes $(k_3, H_{12}), (k_4, H_{12}), (k_5, H_{12})$, and therefore belongs to a family which has exactly one member through each point of the [5]. In each of the other cases the plane passes through a line and meets a [3], and therefore belongs to a family which has one member through a general point of a [5]. Thus the locus of planes meeting $k_1, \dots, k_5$ consists of ten [5]'s $H_{\alpha\beta}$, fifteen [5]'s $[A_{\alpha\beta}, A_{\gamma\delta}, k_\epsilon]$, and some locus of order fifteen.

Suppose that the five points $\Sigma\lambda_a A_{1a}, \Sigma\mu_a A_{2a}, \Sigma\nu_a A_{3a}, \Sigma\rho_a A_{4a}, \Sigma\sigma_a A_{5a}$, which lie one in each of $k_1, \dots, k_5$, are coplanar. The exceptional planes are typified by :

the family in $[A_{23}, A_{45}, k_1]$: λ_a arbitrary, all μ, ν, ρ, σ zero except $\mu_3, \nu_2, \rho_5, \sigma_4$;

the family in H_{45} : all ρ, σ zero except ρ_5, σ_4 ; and

$$\lambda_4 = \mu_4 = \nu_4 = \lambda_5 = \nu_5 = \mu_3 + \nu_2 = \nu_1 + \lambda_3 = \lambda_2 + \mu_1.$$

The non-exceptional family consists of the planes which join the sets of five points‡ :

$$P_1 = \frac{\lambda_2}{\lambda_2 - \lambda_1} A_{21} + \frac{\lambda_3}{\lambda_3 - \lambda_1} A_{31} + \frac{\lambda_4}{\lambda_4 - \lambda_1} A_{41} + A_{51},$$

$$P_2 = \frac{\lambda_1}{\lambda_1 - \lambda_2} A_{12} + \frac{\lambda_3}{\lambda_3 - \lambda_2} A_{32} + \frac{\lambda_4}{\lambda_4 - \lambda_2} A_{42} + A_{52},$$

$$P_3 = \frac{\lambda_1}{\lambda_1 - \lambda_3} A_{13} + \frac{\lambda_2}{\lambda_2 - \lambda_3} A_{23} + \frac{\lambda_4}{\lambda_4 - \lambda_3} A_{43} + A_{53},$$

$$P_4 = \frac{\lambda_1}{\lambda_1 - \lambda_4} A_{14} + \frac{\lambda_2}{\lambda_2 - \lambda_4} A_{24} + \frac{\lambda_3}{\lambda_3 - \lambda_4} A_{34} + A_{54},$$

$$P_5 = \lambda_1 A_{15} + \lambda_2 A_{25} + \lambda_3 A_{35} + \lambda_4 A_{45}.$$

† Segre, *Encyk. der Math. Wiss.*, III C7, 817, foot-note 157; or Todd, *Proc. Camb. Phil. Soc.*, 25 (1929), 46.

‡ This form for the points is precisely analogous to that used for points on lines meeting four planes in [4]. Cf., e.g., Room, *Proc. London Math. Soc.* (2), 28 (1927), 319.

We want now to see which of the planes which meet $k_1, \dots, k_5$ also meet k_6 . Take first those in $[A_{23}, A_{45}, k_1]$. This $[5]$ meets k_6 in the line $[A_{16}, A_{26} + A_{36}, A_{46} + A_{56}]$, and therefore only those planes of the family— ∞^1 members, that is—which meet this line meet k_6 . Take next those in H_{45} . H_{45} meets k_6 in the plane $[A_{16}, A_{26}, A_{36}]$, and therefore the planes required are those which pass through A_{45} and meet the four planes $[A_{\alpha\beta}, A_{\alpha\gamma}, A_{\alpha\delta}]$ ($\alpha, \beta, \gamma, \delta = 1, 2, 3, 6$). There are ∞^2 of these, forming a $[0]$ -cone, whose section is a Segre cubic threefold. Lastly, we find that all the members of the non-exceptional family meet k_6 , because we can write

$$\lambda_1^{-1}P_1 = (\lambda_1 - \lambda_2)^{-1}A_{12} + (\lambda_1 - \lambda_3)^{-1}A_{13} + (\lambda_1 - \lambda_4)^{-1}A_{14} + \lambda_1^{-1}A_{16}, \text{ etc.,}$$

and then

$$P_6 = \lambda_1^{-1}P_1 + \lambda_2^{-1}P_2 + \lambda_3^{-1}P_3 + \lambda_4^{-1}P_4 = \lambda_1^{-1}A_{16} + \lambda_2^{-1}A_{26} + \lambda_3^{-1}A_{36} + \lambda_4^{-1}A_{46},$$

which is a point of k_6 .

THEOREM 3.2. *All planes which meet $k_1, \dots, k_5$, and do not pass through a point $A_{\alpha\beta}$, also meet k_6 †.*

We shall denote the locus of these planes by Γ .

To get the equations of Γ we take a general point $P = \alpha P_1 + \beta P_2 + \gamma P_3$ of it. P is expressed in a form which does not contain $A_{\alpha\beta}$ or A_{45} . Thus, if we take a general point of the $[8]$ to be $\sum_{\alpha, \beta=1}^5 y_{\alpha\beta} A_{\alpha\beta}$ ($y_{\alpha\beta} = x_{\alpha\beta} - x_{\alpha 6} - x_{\beta 6}$), the parametric equations of Γ are obtained by equating $y_{\alpha\beta} - y_{45}$ to the coefficient of $A_{\alpha\beta}$ in P . Then, eliminating the parameters, we get a set of equations of the type

$$(y_{\alpha\beta} - y_{\beta\delta})(y_{\beta\gamma} - y_{\gamma\delta})(y_{\gamma\alpha} - y_{\alpha\delta}) = (y_{\alpha\beta} - y_{\alpha\delta})(y_{\beta\gamma} - y_{\beta\delta})(y_{\gamma\alpha} - y_{\gamma\delta}),$$

five in number, but only three independent. Then, writing $z_{\alpha\beta} = y_{\alpha\beta} - y_{\alpha 5}$, we can express the result as

THEOREM 3.3. *The planes which meet the six $[3]$'s $k_1, \dots, k_6$ form a locus of order fifteen, whose equations can be put in the form*

$$z_{\alpha\beta}z_{\beta\gamma}z_{\gamma\alpha} = z_{\alpha\gamma}z_{\gamma\beta}z_{\beta\alpha} \quad (\alpha, \beta, \gamma = 1, 2, 3, 4),$$

where

$$z_{\alpha\beta} + z_{\beta\gamma} + z_{\gamma\alpha} \equiv z_{\alpha\gamma} + z_{\gamma\beta} + z_{\beta\alpha}.$$

We can check from these equations the result that Γ is of order fifteen. Three of the equations imply the fourth, and the locus repre-

† This theorem is also generalizable to "All $[n-1]$'s which meet $n+2$ of a family of $n+3$ associated $[n]$'s in $[\frac{1}{2}n(n+3)-1]$, and do not pass through a point of intersection of two of the spaces, meet the last $[n]$ ".

sented by them is the intersection of three cubic [3]-cones (whose [4]-sections are Segre cubics) which have common (i) three [5]'s belonging to the family $H_{a\beta}$, (ii) three cubic primals in spaces of six dimensions, the intersections of a [6], $z_{a\beta} = z_{\beta a} = 0$ which satisfies two of the equations, with the cubic represented by the third, and (iii) residually Γ , which is thus of order $27-3-9=15$.

The six points $P_1, \dots, P_6$, in which a plane of Γ meets $k_1, \dots, k_6$, all lie on the conic γ whose general point is

$$P' = (\lambda_1 - \theta)^{-1} P_1 + (\lambda_2 - \theta)^{-1} P_2 + (\lambda_3 - \theta)^{-1} P_3 + (\lambda_4 - \theta)^{-1} P_4,$$

where θ is a parameter, being given respectively by

$$\theta = \lambda_1, \lambda_2, \lambda_3, \lambda_4, \infty, \text{ and } 0.$$

If we write

$$\theta \mu_5 (\lambda_a - \theta)^{-1} = \mu_a - \mu_5 \quad \text{for } a = 1, 2, 3, 4,$$

we have

$$\begin{aligned} \theta \mu_5 P' &= \mu_1 P_1 + \mu_2 P_2 + \mu_3 P_3 + \mu_4 P_4 \\ &= \frac{1}{\mu_5} \left(\sum_{\alpha, \beta=1}^5 \mu_\alpha \mu_\beta A_{\alpha\beta} \right). \end{aligned}$$

Thus the locus Φ of conics γ is the projection from six of its points of the manifold in [14] whose points correspond to those of [4], when the quadrics of the [4] are correlated with primes in [14]; *i.e.* it is the manifold which represents points of [4] when the quadrics of that space which pass through six fixed points are correlated with primes in [8]. The [3]'s, $k_1, \dots, k_6$, which lie on Φ , correspond to the six points, and the conics γ to the rational quartic curves through those six points. Thus the result that there are six planes which contain conics γ lying in any prime in the [8]—*i.e.* six planes meeting six associated planes in [7]—corresponds to the theorem that six quartic curves in [4] can be drawn through six points to lie on an arbitrary quadric through the points.

In passing, it may be noticed that there are an ∞^6 family of conics, and six ∞^3 families of lines on Φ corresponding to the general lines of [4] and to the lines which pass through any of the six fixed points†.

† When we deal with the corresponding problem in [4], namely that of finding the lines which meet four planes each pair of which has a common point—as must always happen—we get Γ and Φ , with equations of the form of Theorem 3.3 and points of the form $\sum_{\alpha, \beta=1}^4 A_{\alpha\beta}$, to be the same—a Segre cubic threefold. In the general case of $[n-1]$'s which meet $n+2$ $[n]$'s in $\frac{1}{2}n(n+3)-1$, every pair of which has a common point, and which therefore also meet an $(n+3)$ -th associated $[n]$, we find the $n+3$ points of intersection lie on a normal $(n-1)$ -ic curve in the $[n-1]$, and the loci of the spaces and of the curves in them correspond exactly to Γ and Φ , having in fact precisely similar forms for their equations and for the symbol of a general point.

4. *The dual figure in [7].*

The double-five of planes and [4]'s in [7] is self-dual, so that the figure to be considered here is not different from that discussed in § 2; there are, however, certain algebraic advantages in treating it as the section of a double-five of [3]'s and [5]'s in [8], rather than as the projection of the double-five of [2]'s and [4]'s.

We take an arbitrary prime section G of the figure in [8], or, what is the same thing analytically, a prime section of the figure in [14] through the points $U_a^\dagger$, say by

$$\sum_{\alpha, \beta} a_{\alpha\beta} x_{\alpha\beta} = 0 \quad (\alpha, \beta = 1, \dots, 6, \quad a_{\alpha\beta} = a_{\beta\alpha}),$$

where

$$\sum_a a_{a\alpha} = 0.$$

To find the associated [4] of the five [4]'s $H_{16}, \dots, H_{56}$ (using the same notation for the sections of the spaces in [8] by G as for the spaces themselves) it is most convenient to work first in terms of points

$\sum_{\alpha, \beta=1}^5 y_{\alpha\beta} A_{\alpha\beta}$ only, the two sets of coordinates being connected by the relations

$$y_{\alpha\beta} = x_{\alpha\beta} - x_{\alpha 6} - x_{\beta 6}^\dagger.$$

G is

$$\sum_{\alpha\beta} a_{\alpha\beta} y_{\alpha\beta} = 0 \quad (\alpha, \beta = 1, \dots, 5),$$

where

$$\sum_{\alpha\beta} a_{\alpha\beta} = 0.$$

In [8] the plane (H_{16}, H_{26}) was $[A_{34}, A_{45}, A_{53}]$, so that in [7] the line (H_{16}, H_{26}) contains the points which are the differences of $A_{34}/a_{34}, A_{45}/a_{45}, A_{53}/a_{53}$. Thus the lines (H_{16}, H_{26}) give rise to the scheme of points :

$$\begin{array}{ccc} 34-35 & 45-43 & 53-54 \\ 25-24 & 42-45 & 54-52 \\ 23-25 & 32-34 & 52-53 \\ 24-23 & 35-32 & 43-42 \end{array}$$

where 23 is written for A_{23}/a_{23} , etc. Points in the same column lie on

† P. 475, line 4.

‡ I.e., we project first from the five points $U_1, \dots, U_5$ on to the [9] in which the coordinate-simplex is $A_{12} \dots A_{45}$, then from the point U_6 , which is $A_{12} + \dots + A_{45}$, and lastly take a section by G .

a line (H_{16}, k_a) , and points in the same row on a line (H_{16}, H_{a6}) . The associated line of either of these sets of four therefore contains the points like $34+25-35-24$. The points of this line, and of the corresponding lines in the other [4]'s H_{a6} , all satisfy the equations

$$\sum_{a=1}^5 a_{ai} y_{ai} = 0 \quad (i = 1, \dots, 5)$$

(five equations, giving therefore a [4] in [9], which does not pass through the point $\Sigma A_{a\beta}$, but does lie in G , and giving therefore also a [4] in the [7]). These are the equations of the associated [4] H_6 . Substituting for the $y_{a\beta}$ in terms of the $x_{a\beta}$, we get to the symmetrical form for the equations of H_6 , viz. :

$$\sum_{a=1}^6 a_{ai} x_{ai} = \sum_{a=1}^5 a_{ai} x_{a6} \quad (i = 1, \dots, 5)$$

(five equations, giving a [9] in [14] through the five points $U_1, \dots, U_5$, but not through U_6 , and lying in the prime G , giving thus a [4] in the [7]).

The main interest in this aspect of the configuration, however, is not the new algebraic expression of the results already obtained, but the proof of the corollary to Theorem 2.2, and the investigation of the properties of the planes of that theorem. From Theorem 3.2 we have, since G does not pass through any of the points $A_{a\beta}$,

THEOREM 4.1. *All planes which meet the five planes $k_1, \dots, k_5$ also meet k_6 , and their points of intersection lie on a conic.*

There are actually six of these planes, as can be seen by making the points $P_1, \dots, P_6$ of § 3† satisfy the equation of G . Take, e.g., P_1, P_2, P_5 , and treat the λ_a as coordinates in [3]. The points of this [3] which correspond to the planes which lie in G are the common intersections of a plane with two four-nodal cubic surfaces with three common nodes, omitting those on the triangle common to the surfaces. It is to be shown that these six planes are associated.

The algebra that we have to hand enables this to be done more easily for the corresponding [4]'s of the figure than for the planes. Suppose that $F_1, \dots, F_6$ are the six [4]'s which meet in planes $H_{16}, \dots, H_{56}$, and therefore also H_6 . F_1 and F_2 have a common line, and this line must meet all the planes (F_1, H_{a6}) [because the planes (F_1, H_{16}) and

† P. 476.

(F_2, H_{16}) necessarily have a common point]. Thus in F_1 we have five lines (F_1, F_a) which meet the six planes (F_1, H_{a6}) and (F_1, H_6), and which therefore are associated, unless there are lines other than they which meet the six planes. This can only happen if all the lines which meet five of the planes also meet the sixth, *i.e.* if the six planes belong to the quintic threefold— ∞^1 planes—defined by any five of them. Also the number of lines which meet six general [4]'s in [7] is fifteen, exactly the number of the lines (F_a, F_β); thus either the five lines (F_a, F_β) in every F_a are associated, or there is an ∞ of lines meeting the six [4]'s, forming six quintic surfaces, one in each of the [4]'s F_a . To show that the five lines in any F_a are associated, we have therefore only to show that the number of lines is finite.

Use as points of reference the ten $A_{\alpha\beta}$ ($\alpha, \beta = 1, \dots, 5$), connected by the syzygy $\sum_{\alpha\beta} A_{\alpha\beta} = 0$, so that they lie in [8]. Then

(i) Any [4] F , which meets in planes the [5]'s $H_{16}, \dots, H_{56}$, has equations

$$\sum_{\alpha=1}^5 b_{\alpha i} y_{\alpha i} = 0 \quad (i = 1, \dots, 5),$$

where

$$\sum_{\alpha} b_{\alpha i} = 0$$

and

$$b_{\alpha\beta} + b_{\beta\alpha} = 0.$$

(H_{16}, F) contains the four points of the form $A_{45}/b_{45} + A_{53}/b_{53} + A_{34}/b_{34}$.

(ii) The transversal lines of $H_{16}, \dots, H_{56}$ meet H_{16} in points of the locus

$$(y_{34} - y_{45})(y_{42} - y_{25})(y_{23} - y_{35}) = (y_{34} - y_{35})(y_{42} - y_{45})(y_{23} - y_{25}).$$

i.e. a [0]-cone formed by joining a Segre cubic threefold to a point, here A_{16} , outside its [4]. To each plane of this cone there corresponds a plane in each of the other spaces H_{a6} , and the five planes so obtained are associated. The planes of intersection of a general [4] of the family (i) with $H_{16}, \dots, H_{56}$ meet these cones in cubic curves, which are the loci of the points of intersection with the planes of the lines transversal to them all.

(iii) Any [7] G through F is

$$\sum_{\alpha\beta} (c_{\alpha} b_{\alpha\beta} + c_{\beta} b_{\beta\alpha}) y_{\alpha\beta} = 0,$$

and the equations of the H_6 associated with the sections of $H_{16}, \dots, H_{56}$ by this [7] are

$$\sum_a c_a a_{ai} y_{ai} = 0 \quad (i = 1, \dots, 5).$$

It is now easy to see that H_6 does not meet (H_{16}, F) in a point of the locus whose equation is given in (ii), and therefore that (H_6, F) cannot belong to the system of planes meeting all the lines which meet the five planes (H_{16}, F) .

Thus the five lines (F_a, F_β) in any of the spaces F_a are associated, and this theorem, in conjunction with the work of the Note at the end of this paper (p. 485), is sufficient to prove that the six [4]'s $F_1, \dots, F_6$ are associated in the same sense as are $K_1, \dots, K_6$.

THEOREM 4.2. *The six planes which meet six associated planes in [7] are themselves associated.*

A direct proof of this theorem is given in the next section.

5. The planes which meet six associated planes in [7].

To prove that the six planes which meet six associated planes in [7] are themselves associated, it will clearly be sufficient to prove: "If $h_1, \dots, h_5$ are five planes in [7] which meet four planes $f_1, \dots, f_4$, and are such that each set of four of them is met by a [4] in lines, then there is a [4] which meets $f_1, \dots, f_4$ in lines".

Denote by $A_{a\beta}$ ($\neq A_{\beta a}$, $a = 1, \dots, 4$, $\beta = 1, \dots, 5$) the point of intersection of f_a with h_β , and suppose that these points are connected by the relations

$$\sum_{a=1}^4 A_{aj} = 0 \quad (j = 1, \dots, 5),$$

$$\sum_{\beta=1}^5 a_{i\beta} A_{i\beta} = 0 \quad (i = 1, \dots, 4),$$

$$\sum_{\beta=1}^5 b_{i\beta} A_{i\beta} = 0 \quad (i = 1, \dots, 4).$$

These thirteen relations connect twenty points in [7], so that only twelve of them can be linearly independent. There must therefore be constants λ, μ, ν , for which

$$\sum_{\beta} \lambda_{\beta} \sum_a A_{a\beta} + \sum_a \mu_a \sum_{\beta} a_{a\beta} A_{a\beta} + \sum_a \nu_a \sum_{\beta} b_{a\beta} A_{a\beta} \equiv 0,$$

i.e.

$$\lambda_{\beta} + \mu_a a_{a\beta} + \nu_a b_{a\beta} = 0.$$

Now take

$$\lambda_{\beta} A_{\alpha\beta} = A'_{\alpha\beta}, \quad \lambda_{\beta}^{-1} \mu_{\alpha} a_{\alpha\beta} = a'_{\alpha\beta}, \quad \lambda_{\beta}^{-1} \nu_{\alpha} b_{\alpha\beta} = b',$$

and drop the dashes. The relations may then be reduced to

$$\left. \begin{aligned} \sum_{\alpha} A_{\alpha j} &= 0 \\ \sum_{\beta} A_{i\beta} &= 0 \\ \sum_{\beta} a_{i\beta} A_{i\beta} &= 0 \end{aligned} \right\}.$$

To find the points of intersection of the transversal line to h_1, h_2, h_3 with the planes, the syzygy has to be expressed in terms of points that lie in those planes only. Such a syzygy is

$$\sum_{\beta=1}^3 \sum_{\alpha=1}^4 \frac{a_{\alpha\beta} - a_{\alpha 5}}{a_{\alpha 4} - a_{\alpha 5}} A_{\alpha\beta} = 0.$$

Thus the point of intersection with h_1 is $\sum_{\alpha=1}^4 \frac{a_{\alpha 1} - a_{\alpha 5}}{a_{\alpha 4} - a_{\alpha 5}} A_{\alpha 1}$. Similarly, the points in h_1 , from which transversals can be drawn to h_2, h_4 , and h_3, h_4 respectively, are $\sum \frac{a_{\alpha 1} - a_{\alpha 5}}{a_{\alpha 3} - a_{\alpha 5}} A_{\alpha 1}$ and $\sum \frac{a_{\alpha 1} - a_{\alpha 5}}{a_{\alpha 2} - a_{\alpha 5}} A_{\alpha 1}$. Since

$$\sum_{\alpha} A_{\alpha 1} = 0,$$

the condition for these three points to be collinear is

$$\Delta_5 = \begin{vmatrix} (a_{11} - a_{15})^{-1} & (a_{21} - a_{25})^{-1} & (a_{31} - a_{35})^{-1} & (a_{41} - a_{45})^{-1} \\ (a_{12} - a_{15})^{-1} & (a_{22} - a_{25})^{-1} & (a_{32} - a_{35})^{-1} & (a_{42} - a_{45})^{-1} \\ (a_{13} - a_{15})^{-1} & (a_{23} - a_{25})^{-1} & (a_{33} - a_{35})^{-1} & (a_{43} - a_{45})^{-1} \\ (a_{14} - a_{15})^{-1} & (a_{24} - a_{25})^{-1} & (a_{34} - a_{35})^{-1} & (a_{44} - a_{45})^{-1} \end{vmatrix} = 0.$$

This is the condition that there shall be a [4] which meets h_1, h_2, h_3, h_4 in lines. Thus, for every set of four from the five planes $h_1, \dots, h_5$ to be met by a [4] in lines, the five equations

$$\Delta_i = |(a_{\alpha\beta} - a_{\alpha i})^{-1}| = 0$$

$$(\beta = 1, \dots, 5, \text{ but } \beta \neq i; \alpha = 1, \dots, 4)$$

must be true. By Theorem 2.1, all five are true if four of them are.

In the same way the condition that $f_1, \dots, f_4$ should be met by a

[4] in lines is

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{vmatrix} = 0.$$

It has to be shown that, if

$$\Delta_i = 0 \quad \text{for } i = 1, \dots, 5,$$

then

$$\Delta = 0.$$

Effect the transformation

$$c_{a\beta} - c_{a5} = (a_{a\beta} - a_{a5})^{-1}.$$

We get, for $i = 1, \dots, 4$,

$$\Delta_i = k | (c_{a\beta} - c_{a5})^{-1} |,$$

where k is a product of factors of the type $c_{a\beta} - c_{a\gamma}$ and therefore does not vanish, while

$$\Delta_5 = \begin{vmatrix} 1 \\ c_{a\beta} \end{vmatrix}$$

and

$$\Delta = | (c_{a\beta} - c_{a5})^{-1} |.$$

Thus $\Delta_1, \dots, \Delta_4, \Delta = 0$ form exactly the same system of equations when expressed in terms of the $c_{a\beta}$ as do $\Delta_1, \dots, \Delta_4, \Delta_5 = 0$ when expressed in terms of $a_{a\beta}$. By Theorem 2.1 we know that, if

$$\Delta_i = 0 \quad (i = 1, \dots, 4).$$

then

$$\Delta_5 = 0,$$

and therefore, by this transformation, also

$$\Delta = 0.$$

Thus Theorem 4.2 is true†.

† The algebraic theorem, " $\Delta_1, \dots, \Delta_4 = 0$ imply $\Delta_5 = 0$ and $\Delta = 0$ ", can be expressed geometrically thus: "If in [3] four points are such that cubic surfaces can be drawn through them to have nodes at four of the sets of four points from a given five points, then one can be drawn to have nodes at the fifth set of four points, and the original four points are coplanar"; or "If $A_{a\beta}$ ($a, \beta = 1, \dots, 5$) are ten points forming a plane Desargues figure of perspective triangles, and ϖ_i denotes the set of six of these points which do not have suffix i , and cubic curves are drawn through $\varpi_1, \varpi_2, \varpi_3$ to have four other points common, then cubic curves can be drawn through ϖ_4 and ϖ_5 to pass through these four points".

NOTE.—*The planes which meet five general planes in [7].*

Let $h_1, \dots, h_5$ be any five planes in [7]. Every three of these will have a transversal line, so that we can assume the following scheme of points, points in the same column being collinear, and those in the same row being coplanar.

$$\begin{array}{cccccc} (h_1) & B_1 & C_1 & D_1 & B'_1 & C'_1, \\ (h_2) & A_2 & & C_2 & D_2 & A'_2 & C'_2, \\ (h_3) & A_3 & B_3 & & D_3 & A'_3 & B'_3 & , \\ (h_4) & A_4 & B_4 & C_4 & & & & , \\ (h_5) & & & & A'_5 & B'_5 & C'_5. \end{array}$$

Suppose that the syzygies are expressed as :—

The sum of points in the same column is zero. (N 1)

$$\left. \begin{array}{l} B'_1 = bB_1 + cC_1 + dD_1 \\ C'_1 = b'B_1 + c'C_1 + d'D_1 \end{array} \right\}, \quad \text{etc.} \quad \text{(N 2)}$$

The planes transversal to h_1, h_2, h_3, h_4 meet h_1, h_2, h_3 in points

$$\left. \begin{array}{l} h_1, \quad \beta'B_1 + \gamma'C_1 + \delta'D_1 \\ h_2, \quad \alpha'A_2 + \quad \gamma'C_2 + \delta'D_2 \\ h_3, \quad \alpha'A_3 + \beta'B_3 \quad + \delta'D_3 \end{array} \right\}, \quad \text{(N 3)}$$

and those to h_1, h_2, h_3, h_5 in points

$$\left. \begin{array}{l} h_1, \quad \beta B'_1 + \gamma C'_1 + \delta D'_1 \\ h_2, \quad \alpha A'_2 + \quad \gamma C'_2 + \delta D'_2 \\ h_3, \quad \alpha A'_3 + \beta B'_3 + \quad \delta D'_3 \end{array} \right\}. \quad \text{(N 4)}$$

The planes which meet all five planes $h_1, \dots, h_5$ can be found by identifying members of these two families, rejecting the apparent solution $\alpha, \beta, \gamma = 0, \alpha', \beta', \gamma' = 0$, which gives only the line meeting h_1, h_2, h_3 . We get three identities like

$$\lambda(\beta'B_1 + \gamma'C_1 + \delta'D_1) \equiv \beta(bB_1 + cC_1 + dD_1) + \gamma(b'B_1 + c'C_1 + d'D_1) + \delta D_1.$$

Now write

$$X_2 = b\beta + b'\gamma,$$

$$X_3 = c\beta + c'\gamma,$$

$$X = d\beta + d'\gamma + \delta,$$

Y_3, Y_1 for similar functions of α and γ , and Y of α, γ , and δ , and Z_1, Z_2 for similar functions of α and β , and Z of α, β , and δ . Eliminating all the parameters but $\alpha, \beta, \gamma, \delta$, we get

$$\left. \begin{aligned} YZ_1 &= ZY_1 \\ ZX_2 &= XZ_2 \\ XY_3 &= YX_3 \end{aligned} \right\}. \quad (\text{N } 5)$$

Thus the planes which meet $h_1, \dots, h_5$ are those of the family (N 4), whose parameters satisfy the relations (N 5). Treating the $\alpha, \beta, \gamma, \delta$ as coordinates in [3], the planes are represented by the intersections of three quadrics, the points $X, Y, Z = 0$ and $\alpha, \beta, \gamma = 0$ being irrelevant. There are thus six of them. It is now not difficult to see that these six planes can have no transversal planes other than $h_1, \dots, h_5$, as they would if $h_1, \dots, h_5$ belonged to an associated set of six. For using such a plane, supposing it to exist, instead of h_4 , we should get three new quadrics like $YZ_1 = ZY_1$, etc. The one corresponding to $YZ_1 = ZY_1$ must pass through the six points representing the six transversal planes, and through $(0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0)$. Since these do not lie on a quartic curve, it must be the same as $YZ_1 = ZY_1$. Thus the only chance that there should be another transversal plane is that the same three quadrics should be expressible in the same form with two different sets of coefficients, and this is easily seen not to be possible. Moreover, a set of planes $h_1, \dots, h_5$ which has the required property cannot be found by just specializing the coefficients in the relations (N 2); the whole scheme must be inapplicable. The simplest such system is that in which the sets of three points such as B_1, C_1, D_1 are collinear, and consequently do not define the plane h_1 ; namely, the system of § 2.

THE SIMULTANEOUS SYSTEM OF A QUADRIC SURFACE AND TWO LINEAR COMPLEXES

By P. N. DAS GUPTA.

[Received 26 September, 1929.—Read 14 November, 1929.]

Introduction.

The invariants and covariants of a quaternary system of a linear complex and a quadric have been discussed by Weitzenböck*, but he excludes the mixed concomitants. In the present paper, the concomitants (which will include mixed forms) of a quadric and two linear complexes are discussed. The system for n linear complexes and a quadric can be obtained by the same methods.

Notation.

The symbols† x, p, u are the variables in homogeneous coordinates such that

$$x = x_1, x_2, x_3, x_4 \quad (\text{point coordinates}),$$

$$u = u_1, u_2, u_3, u_4 \quad (\text{plane coordinates}),$$

$$p = p_{12}, p_{23}, p_{13}, p_{14}, p_{24}, p_{34} \quad (\text{line coordinates}),$$

$$= \left\| \begin{array}{c} x_1, x_2, x_3, x_4 \\ y_1, y_2, y_3, y_4 \end{array} \right\|,$$

where y is cogredient to x , and

$$p_{ik} = x_i y_k - x_k y_i.$$

* "Zum System eines linearen Komplexes und einer Fläche zweiter Ordnung", *Journal für Math.*, 137 (1910), 65–82.

† This notation and its applications are more fully explained in a paper by H. W. Turnbull, *Proc. London Math. Soc.*, 2, 25 (1926), 303–327.

In the above $u_1 = (xyz)_{234}$, etc. Thus, if $v_1 = (xy\zeta)_{234}$, etc., where ζ is cogredient to x , we have

$$p_{ik} = (xy)_{ik} \quad \text{or} \quad (uv)_{lm},$$

where $i, k, l, m = 1, 2, 3, 4$ in some order. Also

$$a_x = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4.$$

The quadric $\Sigma a_{ij}x_ix_j$ is symbolically represented by

$$f = a_x^2 = b_x^2 = c_x^2 = \dots,$$

where $a, b, c, \dots$ are equivalent symbols.

The plane and the line equations of the quadric are $u_x^2 = 0$ and $(Ap)^2 = 0$, where

$$A = (ab), \quad \text{a second degree element,}$$

$$a = (abc), \quad \text{a third degree element,}$$

such that $(abcd) = \Sigma \pm a_1b_2c_3d_4$, a determinant of the fourth order,

$$a_1 = \text{cofactor of } a_1 \text{ in } (abcd),$$

$$a_2 = \quad , \quad a_2 \quad , \quad .$$

The capitals $K, L, M, N, Q, \dots$ are used in connection with complexes $(Kp), (Lp), \dots$, where

$$(Kp) = \Sigma K_{ij}p_{ij} = K_{12}p_{34} + \dots$$

to six terms $[i, j, k, l = 1, 2, 3, 4]$. The capitals B, C are symbols equivalent to A ; β, γ are equivalent to a ; $K', L', \dots$ are equivalent to $K, L, \dots$. The coefficient K is symbolically represented by two symbols k_1, k_2 , cogredient with a , so that

$$K_{ij} = k_{1i}k_{2j} - k_{2i}k_{1j}.$$

Similarly, for the quadric symbol A ; we mean by $A = a_1a_2$ the six equations

$$A_{ij} = a_{1i}a_{2j} - a_{2i}a_{1j},$$

where a_1, a_2 are symbols equivalent to a, b .

Thus, elements a_1, a_2, k_1, k_2, u, v , which are of degree 1, are defined to be of currency 1, and are cogredient. Contragredient to these are a, β, x, y , which are of currency 3; while capitals together with p are compounded of two cogredient elements a, b , etc., or of $a, \beta, \dots$.

Unless otherwise mentioned, a quadric symbol of currency 2 is taken to be resolved into a product of two elements in corresponding small letters with suffixes 1 and 2,

$$e.g. \quad M = m_1 m_2; \quad K = k_1 k_2; \quad p = p_1 p_2.$$

We put

$$(AB) = A_{12}B_{34} + A_{23}B_{14} + A_{31}B_{24} + A_{14}B_{23} + A_{24}B_{31} + A_{34}B_{12}.$$

Besides the usual bracket factors, certain compound factors are useful. They resolve into the usual factors by determinantal permutation,

$$e.g. \quad (uKAx) = (uKa_1)(a_2x) - (uKa_2)(a_1x).$$

An expression with two or more *dotted* symbols means a determinantal permutation of all dotted symbols; thus

$$(\dot{a}KLM\dot{b}) = (aKLMb) - (bKLMa).$$

*Definition** of Ω .

$$\begin{aligned} (uKAx) &= (uK\dot{a}_1)(\dot{a}_2x), \\ &= \Omega_a(uKa_1)(a_2x). \end{aligned}$$

$$\begin{aligned} \text{Also,} \quad (uKALMx) &= \Omega_a(uKa_1)(a_2LMx) \\ &= \Omega_a \Omega_m(uKa_1)(a_2Lm_1)(m_2x) \\ &= \Omega_{a,m}(uKa_1)(a_2Lm_1)(m_2x). \end{aligned}$$

Here $\Omega_{a,m}$ denotes a summation of $4 = (2 \times 2)$ terms obtained by permuting a_1, a_2 and m_1, m_2 independently with proper signs. Where the *end* elements are of currency 2, the bracket factors have a special meaning :

$$(AKLMNQ) = (\dot{a}_2KLMNQ\dot{a}_1) = \Omega_a(a_2KLMNQa_1).$$

The number of symbols of currency 2 in a bracket factor is called its *grade*.

$$e.g. \quad (aKLMa) \quad \text{is of grade 3;}$$

$$(AKLMQp) \quad \text{is of grade 6,}$$

because p as well as the capitals is of currency 2.

“*Total grade*” is the sum of the grades of all the factors in a concomitant.

The symbol ϖK stands for $(AK)(Ap)$.

* Turnbull, *Proc. Royal Soc. Edinburgh*, 46 (1926), 210–222.

If two concomitants X , Y are connected by the relation $X \equiv Y$, i.e. X equivalent to Y , it means that X is expressible in terms of concomitants of the type Y , along with others which have already been taken as belonging to the irreducible set of concomitants. This implies that, for the purpose of constructing a set, or a typical set, of irreducible concomitants, if we retain X we need not retain Y .

$X \equiv 0$ means that X is *reducible*.

2. For the purpose of discussing properties of the bracket factors the following identities are fundamental* :

$$(uKAx) + (uAKx) = -(AK)u_x,$$

or, more generally,

$$(uKALMx) + (uAKLMx) = -(AK)(uLMx). \quad (1)$$

$$(uKLx) = -(xLKu), \quad (2)$$

or, generally,

$$(uKLMNx) = -(xNMLKu). \quad (2')$$

$$(uKLLNx) = -\frac{1}{2}(LL)(uKNx). \quad (3)$$

$$(xPQRx) = -(xQPRx) = (xQRPx) = \text{etc.} \quad (4)$$

$$(\dot{a}KLN\dot{b}) = -(KL)(NA) - (LN)(KA) + (KN)(LA),$$

$$\text{where } A = ab. \quad (5)$$

Identities similar to (1) to (4) hold for any bracket factors of higher grades, but (5) holds only for grade 3. They have been proved elsewhere†

The proofs are reproduced here.

Proof of identity (1).

$$\begin{aligned} (uKALMx) &= \Omega_a(uKa_1)(a_2LMx), \\ &= (a_2Ka_1)(uLMx) + \Omega_k(uk_1a_2a_1)(k_2LMx) \\ &\quad \text{(where } A = a_1a_2, K = k_1k_2), \\ &= -(AK)(uLMx) - (uAKLMx). \end{aligned}$$

* Turnbull, *Theory of determinants* (1928), 210-212.

† Turnbull, *Proc. London Math. Soc.*, 2, 25 (1926), 303-327.

Proof of identity (2').

$$\begin{aligned}(uKLMNx) &= \Omega_{l,n}(uKl_1)(l_2Mu_1)(n_2x), \\ &= +\Omega_{l,n}(xn_2)(n_1ml_1)l_2Ku, \\ &= -(xNMLKu).\end{aligned}$$

Identities (3) and (4) follow immediately from (1) for a factor of the type $(xRx) = -(xRx) = 0$.

Identity (5) is proved in a similar way.

Prepared forms.

3. By the fundamental theorem a concomitant of the complexes (Kp) , (Lp) , and the quadric f may consist of factors of the simple bracketed type, e.g. $(ubcd)$, $(abcu)$, (abp) , a_x , (AK) , (aKu) , etc., or may have compound bracket factors of the type $(uKAx)$, $(aKpa)$, etc. Every symbol a , A , or a , which involves the coefficients of the quadric, must occur twice in a prepared form. We call them *quadric symbols*. If two of these are convolved in one bracket they need not be paired in another bracket, but may occur separately in other bracket factors.

A product such as $(abuv)a_\delta b_\epsilon$ is admitted by the fundamental theorem. Here a , b are equivalent symbols. In prepared forms it is replaced by

$$\frac{1}{2}(abuv)(ab|\delta\epsilon).$$

A feature of the prepared system* is that *all* quadric symbols a , b , c can be replaced by a , a , a , and a , β , γ by a , a , a , e.g. there is no need to consider the type $(apKa)(bpKa)$, since the second a can be substituted for this b . The formation of a prepared system takes this possibility into account.

A list of all possible factors which build up the prepared forms is called the *Complete prepared system*, which is given below.

If a , x are the end elements of a bracket factor, we denote such factors as being of type $\{a, x\}$.

The following Table A, for the prepared system, is constructed by convolving the letters a , A , a , u , p , x , K , L , or their equivalent symbols. No irreducible factor contains two equivalent symbols such as a , b ; nor can a , A occur in an irreducible factor because an identity, viz. § 2 (1), replaces $(a \dots A \dots)$ by $(aA \dots)$, in which the three letters a , a_1 , a_2 are

* Turnbull, *Proc. London Math. Soc.* (2), 21 (1922), 381-388; *Theory of Determinants* (1928), 210-212.

convolved together, which then becomes of the type $(a \dots)$. Similarly, a factor of the type $(\dots A \dots B \dots)$ is replaced by $(\dots AB \dots)$, which contains three equivalent symbols, say a_1, a_2, b_1 , convolved, which is equivalent to a . The cases $(a \dots)$, $(a \dots a)$, or $(a \dots a)$, in the prepared forms are irreducible, because we cannot shift the end elements by such an identity.

We suppose that

- (ϕ) denotes a factor of the type $\{a, u\}$ or $\{a, x\}$,
 (ψ) „ „ „ „ $\{a, x\}$ „ $\{a, u\}$,
 $(\phi\psi)$ „ „ „ „ $\{a, a\}$,
 (θ) denotes a factor which involves the symbol A .

TABLE A.

The prepared system for two linear complexes and a quadric.

10 Concomitants. Type $\{a, a\}$, $\{u, x\}$, $\{a, a\}$, etc.	a_a^2 , u_x , (KL) , (KK') , (LL') , (Kp) , (Lp) , $(uKLx)$, $(aKLpa)$, $(aKLpa)$.
15. (θ) , (ϕ) , (ψ) . Type $\{a, u\}$, $\{a, x\}$, etc.	a_x , u_a , (Ap) , (auK) , (auL) , (aKx) , (aLx) , (AK) , (AL) , $(uKAx)$, $(uLAX)$, $(aKLx)$, $(aKLu)$, $(xKALx)$, $(uKALu)$.
3. $(\phi\psi)$. Type $\{a, a\}$	$(apKa)$, $(apLa)$, $(aKLa)$.

From this Table (A) a complete system, including all irreducible forms and some redundancies, is at once written down. It consists of ten concomitants, together with the types

$$(\theta)(\theta), (\phi)(\phi), (\psi)(\psi), (\phi\psi)(\phi\psi), \text{ and } (\phi)(\psi)(\phi\psi).$$

Products which are more extended are obviously reducible.

Duality in prepared forms.

4. A principle of duality is evident from the prepared system. The symbols a, u , each of currency 1, are reciprocal to a and x , each of currency 3, while the symbols K, L, A , each of currency 2, are self-reciprocal. A number of members of the prepared system are self-dual, e.g. (Ap) , u_x .

Reducibility.

5. *Concomitants obtained by squaring any one, or pairing any two, of the following forms are reducible:—*

$$(uKAx), (uL Ax), (xKALx), (uKALu). \quad (6)$$

This can be shown by resolving A ; for example,

$$\begin{aligned} (uKAx)(xKALx) &= -(uKAx)(xAKLx) \\ &= -\{uKa_1(a_2x) - (uKa_2)(a_1x)\} \{xa_1(a_2KLx) - (xa_2)(a_1KLx)\}, \\ &\quad \text{where } A = a_1a_2, \\ &= -(uKa_1)(xa_1)(a_2x)(u_2KLx) - (a_1x)^2(uKa_2)(a_2KLx) \\ &\quad - (uKa_1)(a_1KLx)(a_2x)^2 + (a_1x)(a_1KLx)(uKa_2)(xa_2). \end{aligned}$$

Each term in the above is palpably a product of two reducible forms.

Again,

$$\begin{aligned} (xKALx)^2 &= (xa_1)^2(a_2KLx)^2 + (xa_2)^2(a_1KLx)^2 - 2(xa_1)(a_1KLx)(xa_2)(a_2KLx). \quad (7) \end{aligned}$$

Evidently this is reducible.

6. $(aKLa)(aMNa)$ is reducible.

Call this ψ .

Let $\alpha = bcd = Ad$, so that $A = bc$.

Let $A' = da$; $K = k_1k_2$; $M = m_1m_2$.

Bracketing K in the first factor of ψ , we have

$$\begin{aligned} \psi &= \Omega_{k,m}(bcdk_1)(k_2La)(bcdm_1)(m_2Na) \\ &= -3(AK)(A'L)[- (bcM)(dNa) + (bdM)(cNa) - (cdM)(bNa)]. \\ &= 3(AK)(A'L)(AM)(A'N) - 3(AK)(A'L)(bMd)(cNa) \\ &= 3\theta - 3\phi - 3(AK)(A'L)(cbaNMd), \end{aligned}$$

where

$$\theta = (AK)(AM)(A'L)(A'N),$$

$$\phi = (AK)(AN)(A'L)(A'M).$$

Hence

$$\begin{aligned}\psi &= 3\theta - 3\phi + 3(AK)(A'L)(aANM\bar{d}). \\ &= 3\theta - 3\phi + (a'KL\bar{d})(a'NM\bar{d}), \quad \text{where } a' = abc, \\ &= 3\theta - 3\phi - (a'KL\bar{d})(a'MN\bar{d}) + \frac{1}{2}(KL)\Delta(MN), \\ &\quad \text{where } \Delta = (abcd)^2.\end{aligned}$$

Therefore

$$2(aKLa)(aMNa) = 3\theta - 3\phi + \frac{1}{2}(KL)\Delta(MN),$$

and

$$\begin{aligned}\frac{2}{3}(aKLa)(aMNa) \\ = (AK)(AM)(A'L)(A'N) - (AK)(AN)(A'L)(A'M) + \frac{1}{6}(KL)\Delta(MN). \quad (8)\end{aligned}$$

Putting $M = K$, $N = L$ in (8), we have

$$\frac{2}{3}(aKLa)^2 = (AK)^2(A'L)^2 - [(AK)(AL)]^2 + \frac{1}{6}(KL)^2\Delta. \quad (9)$$

Putting $L = p$, we see that $(aKpa)^2$ is reducible. By substitution, the reduction of $(aKLa)(apLa)$ can be readily obtained.

7. *Proof that $(apKa)(a\sigma)(aN\rho) \equiv (apKa)(ap)(aN\sigma)$, where ρ and σ have currency 3 (mod 4).* (10)

We have

$$\begin{aligned}(apKa)(a\sigma)(aN\rho) &= -3(apKaa')(AN)(a'\rho)(a\sigma) \quad (a = Aa'), \\ &= 3(AK)(apa')(AN)(a'\rho)(a\sigma) \\ &\quad - 3(Ap)(aKa')(AN)(a'\rho)(a\sigma) \\ &\quad - 3(aApKa')(AN)(a'\rho)(a\sigma).\end{aligned}$$

The first two terms are reducible; when A and a are convolved, the last term is

$$(apKa')(aN\sigma)(a'\rho) = (apKa)(aN\sigma)(a\rho) + \text{reducible terms, which proves (10).}$$

Since L is cogredient to p we may write L for p in (10). If μ and ν are the grades of $(a\sigma)$ and $(aN\rho)$ respectively, we shall describe $(apKa)(a\sigma)(aN\rho)$ as being of characteristics 2, μ , ν , and we shall write this symbolically as $(2; \mu, \nu)$.

From (10) we deduce

$$(2; \mu, \nu) \equiv (2; \nu-1, \mu+1). \quad (11)$$

This relation is reversible since

$$(2; \nu-1, \mu+1) \equiv (2; \mu, \nu) \quad (12)$$

We next prove that $(2; 2, 0) \equiv (2; 0, 2)$.

We have

$$\begin{aligned}(aKLx)u_a &= \Omega_L(aKl_1)(l_2x)u_a \\ &= (aKu)(xLa) + (uKLx)u_a - \Omega_L(al_1uKa)(l_2x) \\ &= (aKu)(xLa) + (uKLx)a_a - (auL)(xKa) \\ &\quad + a_x(uLKa) - u_x(aLKa) \quad (14)\end{aligned}$$

Hence

$$\begin{aligned}(apKa)(aKLx)u_a &\equiv (apKu)(aKu)(xLa) + (apKa)a_x(uLKa) \\ &\quad - (apKa)(aLu)(xKa),\end{aligned}$$

i.e. $(2; 2, 0) \equiv (2; 1, 1) + (2; 0, 2).$

But $(2; 1, 1) \equiv (2; 0, 2).$

Hence $(2; 2, 0) \equiv (2; 0, 2).$

8. *The irreducible set of concomitants of two linear complexes and a quadric, of the type $(\phi\psi)(\phi)(\psi)$, where*

$$(\phi\psi) = \{a, a\}; \quad \phi = \{a, u\} \text{ or } \{a, x\}; \quad \psi = \{a, u\} \text{ or } \{a, x\},$$

contains the following types:—

$$(2; 2, 0); \quad (2; 1, 0), (2; 0, 2); \quad (2; 0, 1), \text{ and } (2; 0, 0).$$

By applying § 7 to all possible forms we get the irreducible forms, thus:—

$$(2; 2, 2) \equiv (2; 1, 3)^\dagger \equiv 0.$$

$$(2; 2, 1) \equiv (2; 0, 3)^\dagger \equiv 0.$$

$$(2; 2, 0) \equiv (2; 0, 2), \text{ by (13), § 7.}$$

$$(2; 1, 2)^*$$

$$(2; 1, 1) \equiv (2; 0, 2).$$

$$(2; 1, 0)^*$$

$$(2; 0, 2)^*$$

$$(2; 0, 1)^*$$

$$(2; 0, 0)^*$$

† A factor of grade 3 is of the type $(aKLLx)$ or $(aLLKx)$ and is therefore reducible because (LL) is a factor by (3), § 2.

* The irreducible forms are those marked with an asterisk.

TABLE B.

The complete system for two linear complexes and a quadric.

Invariants :— (7)	$a_a^2, (KK'), (LL'), (AK)^2, (AL)^2, (AK)(AL), (KL).$
Covariants :— (10)	$a_x^2, a_x(uKLx), (aKx)^2, (aLx)^2, (aKLx)^2, (AK)(xKALx), (AL)(xKALx),$ $(aKx)(aLx), (aKL\alpha) a_x(aKx), (aKL\alpha) a_x(aLx).$
Contravariants :— (10)	$u_a^2, u_a(aKLu), (aKu)^2, (aLu)^2, (aKLu)^2, (AK)(uKALu),$ $(AL)(uKALu), (aKu)(aLu), (aKL\alpha) u_a(aKu), (aKL\alpha) u_a(aLu).$
Complexes :— (7)	$(Kp), (Lp), (Ap)^2, (aKLpa), (aKLpx), (AK)(Ap), (AL)(Ap).$
Mixed Concomitants :— (32)	$u_x, (uKLx), a_x(auK), a_x(auL), u_a(aKx), u_a(aLx), (aKLx)(auK),$ $(aKLx)(auL), (aKx)(aKLu), (aLx)(aKLu), (uKAx)(Ap),$ $(uLx)(Ap), (Ap)(uKALu), (Ap)(xKALx), (AK)(uKAx),$ $(AL)(uLx), (AK)(uLx), (AL)(uKAx),$ $(ApKa) a_x(aKx), (ApKa) a_x u_a, (ApKa) a_x(aLx),$ $(ApKa) a_x(aKLu), (ApKa)(auK) u_a, (ApKa)(auL) u_a.$
Six others, similar to the last six, in which $(apL\alpha)$ is substituted for $(apKa).$	$(aKL\alpha) a_x u_a, (aKL\alpha) a_x(aKLu).$

From the above table it follows that

The covariant and the contravariant forms of two linear complexes and a quadric are quadrics, while, except for the complex equation of the quadric itself, the concomitant complexes are all linear. (15)

9. To find combined invariants of the quadric and the covariant quadrics.

If g_x^2 is a covariant quadric, we define

$$I_{31} = (abcg_1)^2,$$

$$I_{22} = (abg_1g_2)^2,$$

$$I_{13} = (ag_1g_2g_3)^2,$$

where a, b, c and g_1, g_2, g_3 are two sets of equivalent symbols.

One typical case is worked out.

Let $(a'KLx)^2 = g_x^2$, where a' is equivalent to a . Then

$$I_{31} = (abcLK a')^2 = (aLK a')^2 = \frac{3}{2}\phi \quad (\text{say}),$$

where $\phi = (AK)^2(AL)^2 - \overline{(AK)}(\overline{AL})^2 + \frac{1}{6}(KL)^2\Delta$ from (9),

$$\text{and } I_{22} = (abg_1g_2)^2 = (g_1abg_2)^2 = (aKLcdLKb)^2 \\ = [- (cdL)(aKLKb) + (cdK)(aLLKb) + (acdKLLKb)]^2.$$

Hence

$$I_{22} = (AK)^2(AL)^2(KL)^2 + \frac{1}{4}(KK)^2(AL)^2(A'L)^2 + \frac{1}{4}(LL)^2(AK)^2(A'K)^2 \\ - (KK)(KL)(AL)^2[(AK)(AL)] - (LL)(KL)[(AK)(AL)](AK)^2 \\ - \frac{1}{48}(LL)^2(KK)^2\Delta + \frac{1}{12}(LL)(KK)(KL)^2\Delta \\ + \frac{1}{2}(LL)(KK)[(AK)(AL)]^2, \quad (16)$$

$$I_{13} = (aKLcdLKKLf)^2$$

from the expression for I_{22} . Thus

$$I_{13} = \frac{3}{32}(KK)^2(LL)^2\phi \quad (17)$$

TABLE C.

The combined invariants of α_x^2 and the covariant quadrics.

Quadrics	I_{31}	I_{22}	I_{13}
$(aKLx)^2$	$\frac{3}{2}[(AK)^2(AL)^2 - \overline{(AK)}(\overline{AL})^2 + \frac{1}{6}(KL)^2\Delta]$	$(AK)^2(AL)^2(KL)^2 + \frac{1}{4}(KK)^2(AL)^2(A'L)^2 + \frac{1}{4}(LL)^2(AK)^2(A'K)^2 - (KK)(KL)(AL)^2 \times [(AK)(AL)] - (LL)(KL)(AK)^2 [(AK)(AL)] - \frac{1}{48}(KK)^2(LL)^2\Delta + \frac{1}{12}(LL)(KK)(KL)^2\Delta + \frac{1}{2}(LL)(KK)[(AK)(AL)]^2$	$\frac{3}{32}(KK)^2(LL)^2 \times [(AK)^2(AL)^2 - \overline{(AK)}(\overline{AL})^2 + \frac{1}{6}(KL)^2\Delta]$
$\alpha_x(aKLx)$	$-\frac{1}{2}(KL)\Delta$	$\frac{5}{24}\Delta(KL)^2 + \frac{1}{24}(LL)(KK)\Delta - \frac{1}{4}(AK)^2(AL)^2 + \frac{1}{4}(AK)(\overline{AL})^2$	$\frac{3}{8}\{(AK)^2(AL)^2 - \overline{(AK)}(\overline{AL})^2\} - \frac{1}{16}(KK)(LL)(KL)\Delta - \frac{1}{16}\Delta(KL)^3$
$(\alpha xK)^2$	$\frac{3}{2}\Delta(AK)^2$	$\frac{3}{2}\Delta(AK)^4 + \frac{1}{8}(KK)^2\Delta^2$	$\frac{9}{16}(AK)^2\Delta^2(KK)^2$
$(\alpha Kx)(\alpha Lx)$	$\frac{3}{2}\Delta(AK)(AL)$	$\frac{1}{8}(\overline{AK})(\overline{AL})^2\Delta - \frac{3}{8}(AK)^2(AL)^2\Delta + \frac{1}{16}\Delta^2(KK)(LL) + \frac{1}{16}\Delta^2(KL)^2$	$\frac{3}{2}\Delta^2(KK)(LL)(AK)(AL) + \frac{9}{32}(KL)^2(AK)(AL)\Delta^2 + \frac{27}{16}(AK)(\overline{AL})^3\Delta - \frac{27}{16}(\overline{AK})(\overline{AL})\Delta(AK)^2(AL)^2$
$(AK)(xKALx)$	0	$\frac{1}{6}\Delta(KK)(KL)(AK)(AL) - \frac{1}{12}\Delta(AK)^3(KL)^2 - \frac{1}{12}\Delta(KK)^2(AL)^2 + \frac{1}{2}(AK)^2(A'K)^2(AL)^2 - \frac{1}{2}(AK)^2(\overline{AK})(\overline{AL})^2$	0
$(aKL\alpha)$ $\alpha_x(aKx)$	0	Same as in the case of $(AK)(xKALx)$.	0

10. *Geometrical verification of the vanishing of the invariants I_{31} and I_{13} for the quadric $(AK)(xKALx)$.*

The quadric $U \equiv (AK)(xKALx) = 0$ is such that the line of the congruence $(Kp) = (Lp) = 0$ through a point on it is also a line of the complex $(AK)(Ap) = 0$. Such lines are generators of U and intersect the directrices of the three congruences defined by pairs of the complexes $(Kp) = 0$, $(Lp) = 0$, $(AK)(Ap) = 0$. These directrices are therefore generators of U of the opposite system. Let Γ_1 , Γ_2 be the directrices of $(Kp) = (AK)(Ap) = 0$; as is remarked by Weitzenböck*, it is easy to see that Γ_1 , Γ_2 are polar reciprocals with respect to the quadric $S \equiv a_x^2 = 0$.

By taking on Γ_1 any pair of points conjugate with respect to S , and a similar pair of points on Γ_2 , we get a tetrahedron self-polar with respect to S and also both inscribed in and circumscribed to U . Hence U is both in-polar and out-polar to S , and the invariants I_{31} and I_{13} both vanish, as has been proved symbolically in the construction of Table C.

Hence the quadrics U and S are in-polar and out-polar to each other. (24)

The property is symmetrical.

This theorem admits further geometrical consequences which will be considered in § 13.

11. *Line equations of the concomitant quadrics.*

Quadric $(auK)^2$.

If u and v are two tangent planes, $\lambda u + v$ touches the above quadric if the quadratic

$$\lambda^2(auK)(buK') + \lambda[(auK)(bvK') + (avK)(buK')] + (avK)(bvK')$$

in λ is a perfect square, i.e. if

$$[\Omega_{u,v}(aKu)(vK'b)]^2 = 0,$$

or
$$(aKpK'b)^2 = 0,$$

i.e.
$$(Kp)^2(AK)^2 + \frac{1}{4}(KK')^2(Ap)^2 - (Kp)(KK')(AK)(Ap). \quad (25)$$

* *Journal für Math.*, loc. cit., 69.

Quadric $(axK)^2$.

Putting $\lambda x + y$ for x we get, in the same way as above,

$$(aKpK'\beta)^2 = 0, \quad (25')$$

which gives the same line form as in (25), multiplied by $\frac{3}{2}\Delta$.

Hence the quadrics $(auK)^2 = 0$ and $(axK)^2 = 0$ are identical. (26)

Weitzenböck has obtained these results by the use of complex symbols*.

Geometrical explanation of (26).

If x, y, z are three points on S , and ζ is the point where the null planes at x, y , and z intersect, then, when x, y, z are coincident, the locus of ζ is the envelope of the null plane at x . This is $(aKu)^2 = 0$. Again the null plane of ζ passes through x, y, z . So when x, y, z are coincident, this null plane touches S . The locus of ζ from this consideration is $(aKx)^2 = 0$.

The line equations of the remaining quadrics can be found as in (25) and (25'), or more readily as follows.

$$\text{If} \quad a_x^2 = b_x^2 = l_x m_x = l'_x m'_x,$$

$$\text{the line form is} \quad (apb)^2 = (apl)(apm), \quad (27)$$

$$\text{i.e.} \quad (apb)^2 = \frac{1}{2} [(l'pl)(m'pm) + (l'pm)(m'pl)]. \quad (28)$$

$$\text{Similarly, if} \quad u_a^2 = u_\beta^2 = (u\theta)(u\phi) = (u\theta')(u\phi'),$$

$$\begin{aligned} (ap\beta)^2 &= (ap\theta)(ap\phi) \\ &= \frac{1}{2} [(\theta'p\theta)(\phi'p\phi) + (\theta'p\phi)(\phi'p\theta)]. \end{aligned}$$

$$\text{Since} \quad (ap\beta)^2 = \frac{3}{2}\Delta(apb)^2,$$

$$\text{we have} \quad (apb)^2 = \frac{1}{3}\Delta[(\theta'p\theta)(\phi'p\phi) + (\theta'p\phi)(\phi'p\theta)]. \quad (29)$$

From the above results it follows that the quadrics $(aKLx)^2$ and $(aKLu)^2$, the quadrics $(AK)(xKALx)$ and $(AK)(uKALu)$, as also $(aKLa)a_x(aKx)$ and $(aKLa)U_a(aKu)$, are identical. (30)

If there are three linear complexes (Kp) , (Lp) , (Mp) , the quadric of their common lines† is $(xKLx)$ or, alternatively, $(uKLx) = 0$. If for (Mp) we substitute the polar complex $(AK)(Ap)$, the identity of the

* *Journal für Math.*, loc. cit., 70.

† Turnbull, "On the invariant theory of mixed quaternary forms", *Proc. London Math. Soc.* (2), 25 (1925), 303-321 (319).

TABLE D.

The line equations of the remaining quadrics.

Quadrics	Line equations
$(aKLx)^2 = 0$	$(aKLpLKb)^2 = 0,$ <i>i.e.</i> $[(Lp)(KL)(AK) - \frac{1}{2}(KK)(AL)(Lp) - \frac{1}{2}(LL)(Kp)(AK) + \frac{1}{4}(LL)(KK)(Ap)]^2 = 0$
$(aKLu)^2 = 0$	$(aKLpLK\beta)^2 = 0,$ <i>i.e.</i> $\frac{3}{2}\Delta[(Lp)(KL)(AK) - \frac{1}{2}(KK)(AL)(Lp) - \frac{1}{2}(LL)(Kp)(AK) + \frac{1}{4}(LL)(KK)(Ap)]^2 = 0$
$a_x(aKLx) = 0$	$(\alpha p\beta)(aKLpLKb) + (\alpha pLKb)(aKLp\beta) = 0,$ <i>i.e.</i> $(aKLp\alpha)^2 - 2(Lp)(AK)(Ap)(LK) + (KK)(Lp)(AL)(Ap) + (LL)(Ap)(AK)(Kp) - \frac{1}{2}(LL)(KK)(Ap)^2 = 0$
$u_a(aKLu) = 0$	$(\alpha p\beta)(aKLpLK\beta) + (\alpha pLK\beta)(aKLp\beta) = 0,$ <i>i.e.</i> $(aKLp\alpha)^2 - 3(Lp)(AK)(Ap)(LK)\Delta + \frac{3}{2}(KK)(AL)(Ap)(Lp) + \frac{3}{2}\Delta(LL)(AK)(Ap)(Kp) - \frac{3}{4}(LL)(KK)(Ap)^2\Delta = 0$
$(aKu)(aLu) = 0$	$(aKpKb)(aLpLb) + (aKpLb)(aLpKb) = 0,$ <i>i.e.</i> $(aLpKa)^2 - 2(AL)(AK)(Kp)(Lp) + (KK)(AL)(Ap)(Lp) + (LL)(AK)(Ap)(Kp) - \frac{1}{2}(LL)(KK)(Ap)^2 = 0$
$(aKx)(aLx) = 0$	$(\alpha KpK\beta)(\alpha LpL\beta) + (\alpha KpL\beta)(\alpha LpK\beta) = 0,$ <i>i.e.</i> $(aLpKa)^2 - 3(AL)(AK)(Kp)(Lp)\Delta + \frac{3}{2}\Delta(KK)(AL)(Ap)(Lp) + \frac{3}{2}(LL)(AK)(Ap)(Kp)\Delta - \frac{3}{4}(LL)(KK)(Ap)^2\Delta = 0$
$(AK)(xKALx) = 0$	$(AK)(BK)[(b_1KpKALpLb_2) + (b_2LpKALpKb_1)] = 0,$ where $B = b_1b_2$ and is equivalent to A , <i>i.e.</i> $(Lp)^2[(AK)^2(A'K)^2 - \frac{1}{8}(KK)^2\Delta] + (Kp)^2[(AK)(AL)^2 - (KK)(LL)\frac{1}{8}\Delta] + 2(AK)(Ap)Lp\{(KK)(AK)(AL) - (KL)(A'K)^2\} + 2(AK)(Ap)(Kp)\{(LL)(A'K)^2 - (LK)(AK)(AL)\} + (AK)(Ap)^2\{(KL)^2 - (KK)(LL)\} + (Lp)(Kp)[\frac{1}{8}(KK)(KL)\Delta - 2(AK)^2 \cdot (AK)(AL)] = 0$
$(AK)(uKALu) = 0$	Same line equation as $(AK)(xKALx)$
$(aKL\alpha)a_x(aKx)$	$(aKL\alpha)(bKL\beta)(b\beta Ka)(\alpha pK\beta) + (aKL\alpha)(bKL\beta)(\beta KpKa)(\alpha p\beta) = 0,$ <i>i.e.</i> $\frac{9}{4}(\alpha K)^2(\overline{\alpha L})(AK)^2 + \frac{9}{4}(AK)^2(A'K)^2(\overline{\alpha L})^2 + \frac{1}{16}(LK)^2(Kp)^2\Delta^2 - \frac{9}{8}(\alpha K)(\overline{\alpha L})(AK)^2 \cdot (A'L)(A'K) - \frac{3}{4}(AK)^2(LK)(Kp)(\overline{\alpha L})\Delta + \frac{3}{8}(LL)(\alpha K)(Kp)(AK)^2\Delta + \frac{3}{8}(KK)(Kp)(AL)(AK)(\overline{\alpha L})\Delta - \frac{1}{16}(KK)(LL)(Kp)^2\Delta^2 + \frac{3}{4}(\alpha K)(\overline{\alpha L})(KL)(KK)\Delta - \frac{9}{8}(LL)(KK)\Delta(\alpha K)^2 - \frac{3}{8}(AK)^2(\overline{\alpha L})^2\Delta - \frac{3}{4}(KL)(Kp)(AL)(AK)(\alpha K)\Delta = 0,$ where $(\alpha K) = (AK)(Ap), (\overline{\alpha L}) = (AL)(Ap)$
$(aKL\alpha)u_a(aKu)$	$(aKL\alpha)(bKL\beta)(\alpha p\beta)(aKpKb) + (aKL\alpha)(bKL\beta)(\alpha Kp\beta)(\alpha pKb) = 0,$ which reduces to the same line form as in the case of $(aKL\alpha)a_x(aKx)$

quadrics $(AK)(xKALx)$ and $(AK)(uKALu)$ is apparent. Similar considerations apply to the identity of $(aKLa)a_x(aKx)$ and $(aKLa)U_a(aKu)$, for they denote, as is explained later in § 13, the quadric of the three complexes

$$(Kp), (AK)(Ap), (AL)(Ap).$$

Further Geometrical Properties.

12. Reference has been made in § 10 to the "complex generators" of the quadric S . These are generators also of $(axK)^{2*}$. We proceed to discuss the relationship of the directrices of the congruences given by pairs of the complexes

$$(Kp) = 0, \quad (Lp) = 0, \quad (AK)(Ap) = 0, \quad (AL)(Ap) = 0$$

with the covariant quadrics. For brevity the complexes will hereafter be called K , L , ϖK , ϖL respectively.

$a_x(aKLx) = 0$ is the locus of x such that x is conjugate to the null-point with reference to K of the null plane of x with reference to L . Each of the four points of intersection of the directrices of $K = L = 0$ with S satisfy the above condition. Thus

the points of intersection of the directrices of $K = L = 0$ with S are on $a_x(aKLx) = 0$. (31)

Dually,

the tangent planes to S drawn through the directrices of the congruence $K = L = 0$ touch $u_a(aKLu) = 0$. (32)

$(aKx)(aLx) = 0$ is the locus of x such that the null plane of x with respect to K is conjugate to the null plane with respect to L . Let the tangent planes to S , through the directrix a_1 of the congruence $K = L = 0$, intersect the directrix a_2 in the points θ_1, θ_2 . Similarly, let θ_3, θ_4 be obtained on the other directrix. These points are on the quadric. Thus

the four tangent planes to S , through the directrices of the congruence $K = L = 0$, determine a tetrahedron inscribed in $(aKx)(aLx) = 0$. (33)

Dually,

the four points, where the directrices of $K = L = 0$ intersect S ,

* Weitzenböck, *Journal für Math.*, loc. cit., 70.

determine a tetrahedron whose faces are tangent planes to
 $(aKu)(aLu) = 0$. (34)

13. If we denote the quadric which has, as generators, the common lines of the complexes $K = 0$, $L = 0$, $M = 0$, viz. $(xKLMx) = 0$, by $[K, L, M] = 0$, we get

$$\begin{aligned} [K, \varpi K, \varpi L] &= (AK)(BL)(xAKBx), \\ &A, B \text{ being equivalent quadric symbols,} \\ &= 4(abK)(cdL)(xa)(bKc)(dx), \text{ where } A = ab, B = cd, \\ &= 2\{ (cbK)(adL) - (abcKLd) \} (xa)(bKc)(dx) \\ &= \frac{2}{3}(abcKLd)(xKabc)(dx). \end{aligned}$$

Except for a constant factor, this is

$$(aKL\alpha)(aKx)\alpha_x, \quad (35)$$

Call this quadric Q . Similarly,

$$(aKL\alpha)(\alpha Lx)\alpha_x \text{ is } [L, \varpi K, \varpi L].$$

Call this quadric R . Also

$$(AL)(xKALx) \text{ is } [K, L, \varpi L].$$

Call this quadric V .

Since a generator of a quadric reciprocates into a generator on the reciprocal quadric, U and V are the reciprocals of Q and R respectively with regard to S . (36)

Also, any two of the quadrics U, V, Q, R have, as common generators, a pair of directrices from one of the congruences obtained from $K = 0$, $\varpi K = 0$, $\varpi L = 0$, $L = 0$. (37)

Since each directrix of $K = \varpi K = 0$ is a generator of Q , the tetrahedron of § 10 can be viewed as both inscribed and circumscribed to Q and self-conjugate to S .

Hence, an infinite number of tetrahedra self-conjugate to S can be inscribed and also circumscribed to each of the quadrics U, Q, V, R , their vertices being points in involution on the directrices of $K = \varpi K = 0$ in the case of U, Q , and of $L = \varpi L = 0$ in the case of V, R , harmonically separated by the quadric S . (38)

This is expressed by saying that the quadric S is apolar either way to the quadrics U, V, Q, R . Again consider the point and

plane equations of the quadrics Q and U ; namely, let

$$\begin{aligned} g_x^2 &= (aKLa) a_x (aKx), & \text{correlative to } u_7^2; \\ u_\phi^2 &= (AK)(uKALu), & \text{correlative to } f_x^2. \end{aligned}$$

We find that

$$(abgu)^2 = (dKLa)(uabKa)(uabd),$$

where a, b, d are equivalent symbols,

$$\begin{aligned} &= -(abK) u_a (dKLa)(uabd) - 2(dKLa)(uKa) b_a (uabd) \\ &= \frac{1}{4}(AK)(uAKLu) \Delta, \quad \text{after some transformation.} \end{aligned} \quad (39)$$

From (39) we get $(abgu)^2 = 0$, if $u_\phi^2 = 0$. Thus

a tangent plane to U intersects Q in a conic apolar to the conic in which this tangent plane intersects S . (40)

Similarly, it can be shown that

$$(abfu)^2 = \text{const} \times u_7^2. \quad (41)$$

Hence, a tangent plane to Q intersects U in a conic apolar to the conic in which this tangent plane intersects S . (42)

This result can also be deduced correlatively from (40).

Again, the vanishing of the I_{22} invariant of two quadrics means that the edges of a self-conjugate tetrahedron, with respect to either quadric, all touch the other. Since an edge reciprocates into the opposite edge of the tetrahedron, it follows that, when the I_{22} invariant of U and S vanishes, so also does the I_{22} invariant of Q and S . Hence it follows geometrically that the two invariants are identical, as has been indicated in the table of invariants § 9.

14. *Invariants of concomitant complexes.*

- (1) (Kp) .
- (2) (Lp) .
- (3) $(AK)(Ap)$.
- (4) $(AL)(Ap)$.
- (5) $(aKLpa)$.
- (6) $(aKLpa)$.

Suppose that the linear complexes are numbered as above, and that

R_{23} denotes the invariant of complexes (2) and (3) above, R_{22} denotes the invariant of the complex (2), and so on.

The following table gives the invariants of the system.

TABLE E.

$R_{11} = (KK')$	$R_{22} = (LL)$	$R_{33} = \frac{1}{6}\Delta(KK')$
$R_{12} = (KL)$	$R_{23} = (AK)(AL)$	$R_{34} = \frac{1}{6}\Delta(KL)$
$R_{13} = (AK)^2$	$R_{24} = (AL)^2$	$R_{44} = \frac{1}{6}\Delta(LL)$
$R_{14} = (AK)(AL)$		
$R_{55} = (AL)(AK)(KL) - \frac{1}{2}(KK)(AL)^2 - \frac{1}{2}(LL)(AK)^2$		
$R_{56} = \frac{1}{6}\Delta\{(KK)(LL) - (KL)^2\} + (AL)^2(AK)^2 - \{(AL)(AK)\}^2$		
$R_{66} = -\frac{1}{3}\Delta\{(AL)(AK)(KL) - \frac{1}{2}(KK)(AL)^2 - \frac{1}{2}(LL)(AK)^2\}$		
$R_{15} = R_{16} = R_{25} = R_{26} = R_{35} = R_{36} = R_{45} = R_{46} = 0$		

These invariants are obtained by removing p from the two complexes and convolving the remaining parts together, separate quadric symbols being used for each factor.

$$\text{Thus} \quad R_{55} = (aKLabLKb). \quad (43)$$

$$\text{Similarly} \quad R_{56} = (aAKLAabKLb). \quad (44)$$

$(aKLpa)$ is apolar to (Mp) when $(aKLMa) = 0$, i.e. if a_x^2 is apolar to the quadric of three complexes (Kp) , (Lp) , (Mp) , i.e. $(xKLMx) = 0$.

Thus $(aKLpa) = 0$ is apolar to any complex (Mp) which is such that the quadric of the three complexes (Kp) , (Lp) , (Mp) is apolar to a_x^2 . (45)

But a_x^2 is apolar to the quadrics of the three complexes (Kp) , (Lp) , ϖK , and (Kp) , (Lp) , ϖL . It follows that $(aKLpa)$ is apolar to ϖK and ϖL . This explains $R_{35} = 0$, $R_{45} = 0$. Similarly, we can explain $R_{36} = 0$, $R_{46} = 0$.

We may show directly by substitution that $(aKLpa) = 0$ is the complex of lines conjugate to those of $(aKLpa)$ with respect to the quadric a_x^2 . Such a polar complex is

$$\begin{aligned} (Ap)(AaKLa) &= \Omega_k(Ap)(Aak_1)(k_2La) \\ &= \frac{1}{3}(apLKa). \end{aligned}$$

Two complexes can be simultaneously apolar to the four complexes (Kp) , (Lp) , ϖK , ϖL , and, if (Mp) is one of these two, it can easily be proved analytically that $(AM)(Ap)$ is also apolar to (Kp) , (Lp) , ϖK , ϖL . This explains $R_{16} = 0$, $R_{26} = 0$, $R_{36} = 0$, and $R_{46} = 0$.

I have to express my thanks to Professor Turnbull for guiding this work and to Dr. Saddler for helpful suggestions.

AN EXTENSION OF WHIPPLE'S THEOREM ON WELL-POISED HYPERGEOMETRIC SERIES

By W. N. BAILEY.

[Received 21 November, 1929.—Read 12 December, 1929.]

1. The formula

$$\begin{aligned}
 (1.1) \quad {}_7F_6 \left[\begin{matrix} a, 1+\frac{1}{2}a, c, d, e, f, g; \\ \frac{1}{2}a, 1+a-c, 1+a-d, 1+a-e, 1+a-f, 1+a-g \end{matrix} \right] \\
 = \frac{\Gamma(1+a-e)\Gamma(1+a-f)\Gamma(1+a-g)\Gamma(1+a-e-f-g)}{\Gamma(1+a)\Gamma(1+a-f-g)\Gamma(1+a-e-f)\Gamma(1+a-e-g)} \\
 \times {}_4F_3 \left[\begin{matrix} 1+a-c-d, e, f, g; \\ e+f+g-a, 1+a-c, 1+a-d \end{matrix} \right]
 \end{aligned}$$

is the most general formula obtained by Whipple* giving a well-poised series in terms of a Saalschützian series. It is true when the series on the right is terminating, *i.e.* when one of the four parameters e, f, g , or $1+a-c-d$ is a negative integer.

In a recent paper† I have given the formula

$$\begin{aligned}
 (1.2) \quad {}_9F_8 \left[\begin{matrix} a, 1+\frac{1}{2}a, b, c, d, e, f, g, -m; \\ \frac{1}{2}a, 1+a-b, 1+a-c, 1+a-d, 1+a-e, \\ 1+a-f, 1+a-g, 1+a+m \end{matrix} \right] \\
 = \frac{(1+a)_m(1+k-e)_m(1+k-f)_m(1+k-g)_m}{(1+k)_m(1+a-e)_m(1+a-f)_m(1+a-g)_m} \\
 \times {}_9F_8 \left[\begin{matrix} k, 1+\frac{1}{2}k, k+b-a, k+c-a, \\ k+d-a, e, f, g, -m; \\ \frac{1}{2}k, 1+a-b, 1+a-c, 1+a-d, 1+k-e, \\ 1+k-f, 1+k-g, 1+k+m \end{matrix} \right],
 \end{aligned}$$

* Whipple, 1 (7.7), and Whipple, 2.

† Bailey, 5 (8.6).

where $k = 1 + 2a - b - c - d$, and the parameters are subject to the restriction

$$b + c + d + e + f + g - m = 2 + 3a,$$

m being a positive integer.

If we substitute for k and g , and let $b \rightarrow \infty$, we derive (1.1) in the case when e , f , or g is a negative integer. The question naturally arises whether we can let m tend to infinity through positive integral values instead of letting $g \rightarrow \infty$, and so prove that Whipple's formula is true even when the series on the right does not terminate. As a matter of fact this result is not true. It will be seen that, in proceeding to the limit, some care is necessary, a merely formal process leading to an incorrect result.

The result of the limiting process will give the relation corresponding to (1.1) when the series on the right does not terminate. It will be seen that in this case the ${}_7F_6$ can be expressed in terms of two series of the type ${}_4F_3$, both these series being of Saalschützian type, except that they do not terminate.

2. We begin with the formula (1.2), replacing k and b by their values in terms of the other constants. We thus have

$$\begin{aligned} (2.1) \quad {}_9F_8 & \left[\begin{matrix} a, 1 + \frac{1}{2}a, \beta + m, c, d, e, f, g, -m; \\ \frac{1}{2}a, 1 + a - \beta - m, 1 + a - c, 1 + a - d, 1 + a - e, \\ 1 + a - f, 1 + a - g, 1 + a + m \end{matrix} \right] \\ &= \frac{(1+a)_m (e-a)_m (f-a)_m (g-a)_m}{(-a)_m (1+a-e)_m (1+a-f)_m (1+a-g)_m} \\ & \quad \times {}_9F_8 \left[\begin{matrix} a-m, 1 + \frac{1}{2}a - \frac{1}{2}m, a + \beta - a, a - m + c - a, \\ a - m + d - a, e, f, g, -m; \\ \frac{1}{2}a - \frac{1}{2}m, 1 + a - \beta - m, 1 + a - c, 1 + a - d, \\ 1 + a - m - e, 1 + a - m - f, 1 + a - m - g, 1 + a \end{matrix} \right], \end{aligned}$$

where

$$a = e + f + g - 1 - a,$$

$$\beta = 2 + 3a - c - d - e - f - g,$$

and we shall let $m \rightarrow \infty$ through positive integral values.

The series on the left of (2.1) does not present any great difficulty. The series of limits is

$$(2.2) \quad {}_7F_6 \left[\begin{matrix} a, 1 + \frac{1}{2}a, c, d, e, f, g; \\ \frac{1}{2}a, 1 + a - c, 1 + a - d, 1 + a - e, 1 + a - f, 1 + a - g \end{matrix} \right],$$

which converges absolutely if

$$R(2+2a-c-d-e-f-g) > 0.$$

To get the $(r+1)$ -th term of the original series from the corresponding term of (2.2) we have to multiply by

$$(2.3) \quad \frac{(\beta+m)_r(-m)_r}{(1+a-\beta-m)_r(1+a+m)_r}.$$

The factor for the next term is obtained from this by multiplying by

$$\frac{(\beta+m+r)(-m+r)}{(1+a-\beta-m+r)(1+a+m+r)} = \frac{1+\{(\beta-a-1)/(r+1+a+m)\}}{1+\{(\beta-a-1)/(m-r)\}},$$

and, supposing at present that a and β are real and that $\beta \geq a+1$, we see that this factor is less than or equal to 1 for $r \geq -\frac{1}{2}(1+a)$. Under these conditions, if N is any positive integer greater than $-\frac{1}{2}(1+a)$, the value of the factor (2.3) for any r greater than N cannot exceed its value for $r=N$, which evidently has a finite upper limit independent of m . Hence the remainder after r terms in the series on the left of (2.1) cannot have a modulus exceeding that of a certain finite multiple of the corresponding remainder in the series (2.2). It thus follows that (2.2) is the limit when $m \rightarrow \infty$ of the series on the left of (2.1), provided that a and β are real and that $\beta \geq a+1$.

3. We now turn to the series ${}_9F_8$ on the right of (2.1), and here the limiting process is not so simple. For large values of m the terms in the middle of the series become negligible, but those near both ends are important*. We therefore suppose, for convenience, that m is an odd positive integer, and we divide the series into two parts of $\frac{1}{2}(m+1)$ terms each. The second half of the series is then reversed, and we can proceed to the limit in each case.

The first half of the series ${}_9F_8$ on the right of (2.1) is

$$(3.1) \quad \sum_{r=0}^{\frac{1}{2}(m-1)} \frac{(\alpha-m)_r(1+\frac{1}{2}\alpha-\frac{1}{2}m)_r(\alpha+\beta-a)_r(\alpha-m+c-a)_r}{r! (\frac{1}{2}\alpha-\frac{1}{2}m)_r(1+a-\beta-m)_r(1+a-c)_r(1+a-d)_r} \\ \times \frac{(\alpha-m+d-a)_r(e)_r(f)_r(g)_r(-m)_r}{(1+a-m-e)_r(1+a-m-f)_r(1+a-m-g)_r(1+a)_r},$$

* Cf. Dougall, 7, §12. Dougall notices a similar peculiarity in obtaining from his theorem the extension of Saalschütz's formula for the sum of a ${}_3F_2$. My discussion of the present problem is similar to that given by Dougall.

and the series of term by term limits is

$$(3.2) \quad {}_4F_3 \left[\begin{matrix} a+\beta-a, e, f, g; \\ 1+a-c, 1+a-d, 1+a \end{matrix} \right],$$

which converges absolutely provided that no terms are infinite.

Now the greatest value of r is $\frac{1}{2}(m-1)$, and so $m-r$ is large whenever m is.

The $(r+1)$ -th term of (3.1) is obtained from the corresponding term of (3.2) by multiplying by the factor

$$(3.3) \quad \frac{(2r+a-m)(a-m)_r(a-m+c-a)_r(a-m+d-a)_r(-m)_r}{(a-m)(1+a-\beta-m)_r(1+a-m-e)_r(1+a-m-f)_r(1+a-m-g)_r},$$

and, using the approximate value of the gamma function for large values of the argument, we see that the factor (3.3) is asymptotically equivalent to

$$\left(\frac{m-a-2r}{m-a} \right) \left(\frac{m-r}{m} \right)^2,$$

and this clearly lies between finite limits independent of m as r goes from 0 to $\frac{1}{2}(m-1)$.

It follows by an argument similar to that used in § 2 that the limit of (3.1) is the series (3.2).

Now the second half of the series on the right of (2.1) can be written, when reversed, as the first half of the series

$$\begin{aligned} & \frac{(\alpha-m)_m(1+\frac{1}{2}\alpha-\frac{1}{2}m)_m(a+\beta-a)_m(a-m+c-a)_m}{m!(\frac{1}{2}\alpha-\frac{1}{2}m)_m(1+a-\beta-m)_m(1+a-c)_m(1+a-d)_m} \\ & \times \frac{(\alpha-m+d-a)_m(e)_m(f)_m(g)_m(-m)_m}{(1+a-m-e)_m(1+a-m-f)_m(1+a-m-g)_m(1+a)_m} \\ & \times {}_9F_8 \left[\begin{matrix} -\alpha-m, 1-\frac{1}{2}\alpha-\frac{1}{2}m, \beta-a, c-a-m, d-a-m, \\ e-a, f-a, g-a, -m; \\ -\frac{1}{2}\alpha-\frac{1}{2}m, 1-a-\beta+a-m, 1-a+a-c, \\ 1-a+a-d, 1-e-m, 1-f-m, 1-g-m, 1-a \end{matrix} \right], \end{aligned}$$

and we find, as above, that the limit of this half series as $m \rightarrow \infty$ through odd positive integral values is

$$\begin{aligned} & - \frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(\beta-a)\Gamma(e-a)\Gamma(f-a)\Gamma(g-a)\Gamma(1+a)}{\Gamma(1-a)\Gamma(1-a+a-c)\Gamma(1-a+a-d)\Gamma(a+\beta-a)\Gamma(e)\Gamma(f)\Gamma(g)} \\ & \times {}_4F_3 \left[\begin{matrix} \beta-a, e-a, f-a, g-a; \\ 1-a+a-c, 1-a+a-d, 1-a \end{matrix} \right]. \end{aligned}$$

Combining these results we have finally the relation

$$\begin{aligned}
 (3.4) \quad {}_7F_6 & \left[\begin{matrix} a, 1+\frac{1}{2}a, c, d, e, f, g; \\ \frac{1}{2}a, 1+a-c, 1+a-d, 1+a-e, 1+a-f, 1+a-g \end{matrix} \right] \\
 &= \frac{\Gamma(1+a-e)\Gamma(1+a-f)\Gamma(1+a-g)\Gamma(1+a-e-f-g)}{\Gamma(1+a)\Gamma(1+a-f-g)\Gamma(1+a-g-e)\Gamma(1+a-e-f)} \\
 &\quad \times {}_4F_3 \left[\begin{matrix} 1+a-c-d, e, f, g; \\ 1+a-c, 1+a-d, e+f+g-a \end{matrix} \right] \\
 &\quad + \frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-f)\Gamma(1+a-g)}{\Gamma(1+a)\Gamma(1+a-c-d)\Gamma(e)\Gamma(f)\Gamma(g)} \\
 &\quad \times \frac{\Gamma(e+f+g-1-a)\Gamma(2+2a-c-d-e-f-g)}{\Gamma(2+2a-c-e-f-g)\Gamma(2+2a-d-e-f-g)} \\
 &\quad \times {}_4F_3 \left[\begin{matrix} 2+2a-c-d-e-f-g, 1+a-f-g, \\ 1+a-g-e, 1+a-e-f; \\ 2+a-e-f-g, 2+2a-c-e-f-g, \\ 2+2a-d-e-f-g \end{matrix} \right].
 \end{aligned}$$

This relation is proved under certain restrictions, but an appeal to the theory of analytic continuation shows that it is true if

$$R(2+2a-c-d-e-f-g) > 0.$$

4. As we should expect, the formula (3.4) contains as particular cases a number of known results. Thus, if one of the quantities $1+a-c-d$, e , f , or g is a negative integer, the second term on the right vanishes owing to an infinity in the denominator, and we have Whipple's theorem (1.1). If $f+g=1+a$, the first term on the right vanishes, and the second series reduces to unity, and we obtain the known formula*

$$\begin{aligned}
 (4.1) \quad {}_5F_4 & \left[\begin{matrix} a, 1+\frac{1}{2}a, c, d, e; \\ \frac{1}{2}a, 1+a-c, 1+a-d, 1+a-e \end{matrix} \right] \\
 &= \frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-c-d-e)}{\Gamma(1+a)\Gamma(1+a-d-e)\Gamma(1+a-e-c)\Gamma(1+a-c-d)}.
 \end{aligned}$$

If $c+d=1+a$, the first series reduces to unity and the second term vanishes, and we are again led to (4.1).

* Whipple, 1 (5.2), and Dougall, 7 (9).

Now put $c = 0$ in (3.4), and we obtain the formula

$$\begin{aligned}
 (4.2) \quad {}_3F_2 \left[\begin{matrix} e, f, g; \\ 1+a, e+f+g-a \end{matrix} \right] \\
 = \frac{\Gamma(1+a) \Gamma(1+a-f-g) \Gamma(1+a-g-e) \Gamma(1+a-e-f)}{\Gamma(1+a-e) \Gamma(1+a-f) \Gamma(1+a-g) \Gamma(1+a-e-f-g)} \\
 - \frac{\Gamma(1+a) \Gamma(1+a-f-g) \Gamma(1+a-g-e) \Gamma(1+a-e-f) \Gamma(e+f+g-1-a)}{\Gamma(e) \Gamma(f) \Gamma(g) \Gamma(2+2a-e-f-g) \Gamma(1+a-e-f-g)} \\
 \times {}_3F_2 \left[\begin{matrix} 1+a-f-g, 1+a-g-e, 1+a-e-f; \\ 2+a-e-f-g, 2+2a-e-f-g \end{matrix} \right].
 \end{aligned}$$

If one of e , f , or g is a negative integer, the second term on the right vanishes, and we have Saalschütz's formula.

5. The formula (3.4) has an interesting connection with a recent paper* by Whipple. If we put $a = -m + \frac{1}{2}A$, $c = \frac{1}{2}B$, $d = \frac{1}{2}(1+B)$, $e = -\frac{1}{2}m$, $f = \frac{1}{2}(1-m)$, $g = \frac{1}{2}A$, and define C by the relation $m = \frac{1}{2}(A-B-C+1)$, we get, after changing capital letters to small letters,

$$\begin{aligned}
 (5.1) \quad & \frac{\Gamma(1-m+\frac{1}{2}a) \Gamma(\frac{1}{2}-\frac{1}{2}m) \Gamma(1-\frac{1}{2}m) \Gamma(\frac{1}{2}+\frac{1}{2}a)}{\Gamma(1-\frac{1}{2}m+\frac{1}{2}a) \Gamma(\frac{1}{2}-\frac{1}{2}m+\frac{1}{2}a) \Gamma(1-m) \Gamma(\frac{1}{2})} \\
 & \times {}_7F_6 \left[\begin{matrix} -m+\frac{1}{2}a, 1-\frac{1}{2}m+\frac{1}{4}a, \frac{1}{2}b, \frac{1}{2}+\frac{1}{2}b, \\ -\frac{1}{2}m, \frac{1}{2}-\frac{1}{2}m, \frac{1}{2}a; \\ -\frac{1}{2}m+\frac{1}{4}a, 1-m+\frac{1}{2}a-\frac{1}{2}b, \frac{1}{2}-m+\frac{1}{2}a-\frac{1}{2}b, \\ 1-\frac{1}{2}m+\frac{1}{2}a, \frac{1}{2}-\frac{1}{2}m+\frac{1}{2}a, 1-m \end{matrix} \right] \\
 & = {}_4F_3 \left[\begin{matrix} -\frac{1}{2}m, \frac{1}{2}-\frac{1}{2}m, \frac{1}{2}a, \frac{1}{2}(c-b); \\ \frac{1}{2}, \frac{1}{2}c, \frac{1}{2}(c+1) \end{matrix} \right] + \frac{2m}{c} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}c-\frac{1}{2}b) \Gamma(\frac{1}{2}+\frac{1}{2}a)}{\Gamma(\frac{1}{2}c-\frac{1}{2}b) \Gamma(\frac{1}{2}a)} \\
 & \times {}_4F_3 \left[\begin{matrix} \frac{1}{2}(1-m), 1-\frac{1}{2}m, \frac{1}{2}(a+1), \frac{1}{2}(c-b+1); \\ \frac{3}{2}, \frac{1}{2}(c+1), \frac{1}{2}(c+2) \end{matrix} \right],
 \end{aligned}$$

where $m = \frac{1}{2}(a-b-c+1)$ and m is not necessarily an integer.

Using the formula (10.3) of Whipple's paper, we find that

$$\begin{aligned}
 (5.2) \quad {}_2F_1 \left[\begin{matrix} a, b; \\ c \end{matrix} -1 \right] &= \frac{\Gamma(\frac{1}{2}+\frac{1}{2}b+\frac{1}{2}c) \Gamma(\frac{1}{2}+\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}c)}{\Gamma(\frac{1}{2}-\frac{1}{2}b+\frac{1}{2}c) \Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}c)} \\
 & \times {}_7F_6 \left[\begin{matrix} -m+\frac{1}{2}a, 1-\frac{1}{2}m+\frac{1}{4}a, \frac{1}{2}b, \frac{1}{2}+\frac{1}{2}b, \\ -\frac{1}{2}m, \frac{1}{2}-\frac{1}{2}m, \frac{1}{2}a; \\ -\frac{1}{2}m+\frac{1}{4}a, 1-m+\frac{1}{2}a-\frac{1}{2}b, \frac{1}{2}-m+\frac{1}{2}a-\frac{1}{2}b, \\ 1-\frac{1}{2}m+\frac{1}{2}a, \frac{1}{2}-\frac{1}{2}m+\frac{1}{2}a, 1-m \end{matrix} \right].
 \end{aligned}$$

* Whipple, 4.

We have thus obtained a relation giving the general ${}_2F_1(-1)$ as a well-poised series of the type ${}_7F_6$, neither series necessarily terminating. The only relation of this type given by Whipple* involved terminating series.

If, in (5.2), we put $c = 1 + a - b$, so that $m = 0$, we obtain Kummer's formula for the sum of a ${}_2F_1(-1)$.

6. In my paper referred to above I have shown that the formula (1.2) gives as limiting cases Whipple's relations† between terminating well-poised series of the type ${}_7F_6$. I take this opportunity to note that it also includes as limiting cases Whipple's relations involving well-poised series which do not terminate. The process of deriving these formulae is simply that of making $m \rightarrow \infty$ through integral values instead of making one of the other parameters tend to infinity. The justification is similar to that of § 2 above, and presents no great theoretical difficulty.

In conclusion, I note that the formula (1.2) can be proved in a very simple manner by a method similar to that used in proving the analogous formula involving Heinean series‡.

References.

1. F. J. W. Whipple, "On well-poised series, generalized hypergeometric series having parameters in pairs, each pair with the same sum", *Proc. London Math. Soc.* (2), 24 (1925), 247-263.
2. ——— "A fundamental relation between generalized hypergeometric series", *Journal London Math. Soc.*, 1 (1926), 138-145.
3. ——— "Well-poised and other generalized hypergeometric series", *Proc. London Math. Soc.* (2), 25 (1926), 525-544.
4. ——— "On series allied to the hypergeometric series with argument -1 ", *Proc. London Math. Soc.* (2), 30 (1930), 81-94.
5. W. N. Bailey, "Some identities involving generalized hypergeometric series", *Proc. London Math. Soc.* (2), 29 (1929), 503-516.
6. ——— "An identity involving Heine's basic hypergeometric series", *Journal London Math. Soc.*, 4 (1929), 254-257.
7. J. Dougall, "On Vandermonde's theorem and some more general expansions", *Proc. Edinburgh Math. Soc.*, 25 (1907), 114-132.

* Whipple, 4 (11.41).

† Whipple, 3.

‡ Bailey, 6.

THE REPRESENTATION OF A NUMBER AS THE SUM OF ONE SQUARE AND A NUMBER OF k -th POWERS

By G. K. STANLEY.

[Received 9 November, 1929.—Read 14 November, 1929.]

1.1. The problem considered here is that of finding the smallest value of s for which the equation

$$x_0^2 + x_1^k + x_2^k + \dots + x_s^k = n$$

is soluble for all values of n , k being any positive integer greater than two. Clearly this is a problem closely related to Waring's Problem, and in dealing with it I have made considerable use of the work done by Hardy and Littlewood on that subject. My method is that of their later papers, but I use results from the whole set of papers called by them "Partitio Numerorum"* , which bear on the question. I have also used Landau's† work where his form of the result, or of the argument, was more convenient for my purposes.

The method is to approximate to the functions

$$f(x) = 1 + 2x + 2x^{2k} + 2x^{3k} + \dots,$$

$$\mathfrak{Z}(x) = 1 + 2x + 2x^{2^2} + 2x^{3^2} + \dots$$

by means of functions which we call ψ and θ respectively, and then to consider the integral

$$\int_{\Gamma} (f^s \mathfrak{Z} - \psi^s \theta)^2 d\alpha,$$

* G. H. Hardy and J. E. Littlewood, "Some problems of 'Partitio Numerorum' (I): A new solution of Waring's problem", *Göttinger Nachrichten* (1920), 33–54; "(II): Proof that every large number is the sum of at most 21 biquadrates", *Math. Zeitschrift*, 9 (1921), 14–27; "(IV): The singular series in Waring's problem and the value of the number $G(k)$ ", *Math. Zeitschrift*, 12 (1922), 161–188; "(VI): Further researches in Waring's problem", *Math. Zeitschrift*, 23 (1925), 1–37; "(VIII): The number $\Gamma(k)$ in Waring's problem", *Proc. London Math. Soc.*, 28 (1928), 519–542. These memoirs will be referred to as P.N. 1, P.N. 2, P.N. 4, P.N. 6, and P.N. 8 respectively.

† E. Landau, *Vorlesungen über Zahlentheorie*, 1 (1927), 235–360.

where Γ is a circle with centre at the origin and radius $e^{-1/n}$. The circle of integration is divided into "Farey arcs", as in all the P.N. papers, and these arcs are classified as "major" or "minor" arcs in a manner explained later.

My paper is divided into two parts. In the first part I consider the "singular series" which arises from the consideration of our approximation $\psi^s \theta$. I show that, for sufficiently large values of s , this series is absolutely convergent and does not vanish. The values of s required to ensure the absolute convergence of the series, and to ensure that the series does not vanish, are, of course, functions of k . I have given general results for these values of s , but have also considered certain particular cases more carefully where the general result is not sufficient for the results required in the second part of the paper.

In Part 2 I find an inequality satisfied by the function $G_1(k)$, which denotes the smallest value of s such that almost all numbers are sums of one square and s k -th powers. This result is embodied in Theorems 7, 8, 9, which state that, if $k > 4$, $G_1(k) \leq 2^{k-2}(\frac{1}{2}k-1)+2$, while $G_1(4) \leq 6$, $G_1(3) \leq 3$. I then consider lower bounds for the function $G(k)$, which denotes the smallest value of s for which *all* large numbers are the sums of one square and s k -th powers. I give these results in Theorems 10, 11, 12, where I find that, if $k > 4$,

$$G(k) \leq 2^{k-2}(\frac{1}{2}k-1)+k+4+[\xi_k],$$

where

$$\xi_k = \frac{(k-2) \log 2 + \log(k-2) - \log k}{\log k - \log(k-1)},$$

and $G(4) \leq 14$, $G(3) \leq 7$.

1.2. The following notation is used throughout: k, s, n are integers, positive or zero, and, unless otherwise stated, $k > 2$; p and q are also positive integers, $p < q$, $(p, q) = 1$, and p is zero when $q = 1$, otherwise $p > 0$; ϖ denotes a prime. In addition we have

$$K = 2^{k-1}, \quad \kappa = 1 - \frac{1}{K}, \quad \alpha = \frac{1}{k}, \quad \mathbf{C} = \frac{\sqrt{\pi} \{2\Gamma^*(1+a)\}^s}{\Gamma(\frac{1}{2} + as)},$$

$$e(x) = e^{2\pi i x}, \quad e_q(x) = e\left(\frac{x}{q}\right),$$

$$S_{p,q,k} = \sum_{h=0}^{q-1} e_q(h^k p), \quad S_{p,q,2} = \sum_{h=0}^{q-1} e_q(h^2 p),$$

$$f(x) = 1 + 2x^{1^k} + 2x^{2^k} + 2x^{3^k} + \dots,$$

$$\mathfrak{Z}(x) = 1 + 2x^{1^2} + 2x^{2^2} + 2x^{3^2} + \dots,$$

$$\rho(n) = \rho_s(n) = \rho_{s,k}(n) = \mathbf{C} n^{s\alpha - \frac{1}{2}} \sum_{q=1}^{\infty} A_q(n) = n^{s\alpha - \frac{1}{2}} \mathbf{S}(n),$$

$$\text{where} \quad A_q(n) = \sum_p \left(\frac{S_{p,q,k}}{q} \right)^s \left(\frac{S_{p,q,2}}{q} \right)^s e_q(-np).$$

$$\text{Also} \quad \sigma(n) = \sigma_s(n) = \sigma_{k,s}(n) = r_{k,s}(n) - \rho_{k,s}(n),$$

where $r_{k,s}(n)$ denotes the coefficient of x^n in the expansion of $\{f(x)\}^s \mathfrak{S}(x)$. A is an absolute constant, B a constant depending upon k , and C a constant depending upon k and s . When A , B , or C depend also upon some other quantity this is explicitly stated. They are all essentially positive and vary from one occurrence to another.

1.3. It follows from the above definitions that

$$\mathfrak{S}(x) \{f(x)\}^s = 1 + \sum_{n=1}^{\infty} r_{k,s}(n) x^n,$$

and clearly

$$r_{k,s}(n) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\{f(x)\}^s \mathfrak{S}(x)}{x^{n+1}} dx.$$

We divide the circle of integration Γ into Farey arcs $\xi_{p,q}$ of order $N = [n^{1-a}]$ and define major and minor arcs by $q \leq n^a$ and $q > n^a$ respectively. These are denoted by M and m . We write

$$(1.311) \quad x = |x| e^{i\omega} = e^{-(1/n)+i\omega},$$

and on the arc $\xi_{p,q}$

$$(1.312) \quad x = e_q(p) e^{-y}, \quad y = \frac{1}{n} + i\alpha, \quad \alpha = \frac{2p\pi}{q} - \omega.$$

Also

$$(1.32) \quad \psi = \psi_{p,q} = 2\Gamma(1+a) \frac{S_{p,q,k}}{q} \cdot \frac{1}{y^a}, \quad \theta = \theta_{p,q} = \sqrt{\pi} \frac{S_{p,q,2}}{q} \cdot \frac{1}{y^{\frac{1}{2}}},$$

$$(1.33) \quad g(z) = \sum_{m=1}^{\infty} m^{sa-1} z^m \quad (|z| < 1),$$

$$(1.34) \quad F(x) = \sum_{m=1}^{\infty} \rho_s(m) x^m = \sum_{m=1}^{\infty} m^{sa-\frac{1}{2}} \mathbf{S}(m) x^m,$$

$$\begin{aligned} (1.35) \quad F_{p,q} &= \mathbf{C} \left(\frac{S_{p,q,k}}{q} \right)^s \left(\frac{S_{p,q,2}}{q} \right)^s \sum_{m=1}^{\infty} m^{sa-1} \{x e_q(-p)\}^m \\ &= \mathbf{C} \left(\frac{S_{p,q,k}}{q} \right)^s \left(\frac{S_{p,q,2}}{q} \right)^s g \{x e_q(-p)\}. \end{aligned}$$

It is necessary to consider separately values of q which are greater, and those which are less than

$$(1.36) \quad \mu = n^{\gamma},$$

where γ will be chosen later. We write

$$(1.37) \quad \mathbf{S}_1(n) = \mathbf{C} \sum_{q \leq \mu} A_q(n), \quad \mathbf{S}_2(n) = \mathbf{C} \sum_{q > \mu} A_q(n),$$

so that

$$(1.371) \quad \mathbf{S}(n) = \mathbf{S}_1(n) + \mathbf{S}_2(n);$$

and

$$(1.38) \quad F_1(x) = \sum_{m=1}^{\infty} m^{s_1-\frac{1}{2}} \mathbf{S}_1(m) x^m, \quad F_2(x) = \sum_{m=1}^{\infty} m^{s_2-\frac{1}{2}} \mathbf{S}_2(m) x^m,$$

so that

$$(1.381) \quad F(x) = F_1(x) + F_2(x).$$

Then it follows that

$$(1.391) \quad F_1 = \mathbf{C} \sum_{p, q \leq \mu} \sum_{m=1}^{\infty} \left(\frac{S_{p, q, k}}{q} \right)^s \left(\frac{S_{p, q, 2}}{q} \right) m^{s_1-\frac{1}{2}} \{x e_q(-p)\}^m = \sum_{p, q \leq \mu} F_{p, q},$$

where the summation implies that we sum for all values of $q \leq \mu$ and all associated values of p . Similarly, we have

$$(1.392) \quad F_2 = \sum_{p, q > \mu} F_{p, q}.$$

I use the notation $\varpi^\lambda \| n$ to denote that λ is the highest power of ϖ which is a divisor of n . Similarly $(\varpi^k)^\lambda \| n$ is used to denote that λ is the highest power of ϖ^k which divides n . Obviously this must be distinguished from $\varpi^{k\lambda} \| n$.

PART 1. *The singular series.*

2.1. We prove first that, if the singular series is absolutely convergent, it is expressible as an infinite product.

LEMMA 1. *If $(q, q') = 1$, then*

$$A_{qq'} = A_q A_{q'}.$$

We have

$$S_{p, qq', k} = S_{pq^{k-1}, q, k} S_{pq^{k-1}, q', k},$$

where $k \geq 2$. The proof is well known when $k = 2$. When $k > 2$, see P.N. 2, Lemma 1. We now notice that, if p describes a complete system of residues prime to modulus q , and p' a system prime to modulus q' , then $\mathbf{p} = pq' + p'q$ describes a similar system for modulus qq' . Also

$$S_{\mathbf{p}q'^{k-1}, q, k} = \sum_h e\left(\frac{(pq' + p'q)q'^{k-1}h^k}{q}\right) = \sum_h e\left(\frac{(p'qh)^k}{q}\right) = \sum_h e\left(\frac{p'h^k}{q}\right) = S_{p', q, k}.$$

Thus we have

$$\begin{aligned}(qq')^{s+1}A_qA_{q'} &= \sum_{p,p'}(S_{p,q,k})^s(S_{p',q',k})^s(S_{p,q,2})(S_{p',q',2})e\left\{-n\left(\frac{p}{q}+\frac{p'}{q'}\right)\right\}\\&= \sum_{\mathbf{p}}(S_{\mathbf{p}q^{k-1},q',k})^s(S_{\mathbf{p}q^{k-1},q,k})^s(S_{\mathbf{p}q,q',2})(S_{\mathbf{p}q',q,2})e\left(-\frac{n\mathbf{p}}{qq'}\right)\\&= \sum_{\mathbf{p}}(S_{\mathbf{p},qq',k})^s(S_{\mathbf{p},qq',2})e_{qq'}(-n\mathbf{p})=(qq')^{s+1}A_{qq'};\end{aligned}$$

which proves the result.

It follows that, when the singular series is absolutely convergent, it may be written in the form

$$\mathbf{S}=\mathbf{C}\chi_2\chi_3\chi_5\dots=\mathbf{C}\Pi\chi_{\varpi},$$

where

$$\chi_{\varpi}=1+A_{\varpi}+A_{\varpi^2}+\dots=\Sigma A_{\varpi^{\lambda}}.$$

2.2. We shall show later that $\mathbf{S}$ is absolutely convergent for all sufficiently large values of s . At the moment we determine the form of the factors χ_{ϖ} . Suppose that $q=\varpi^{\lambda}$ ($\lambda\geqslant 1$), and let $\nu(\zeta,\xi,q,n)$ denote the number of solutions of the congruence

$$(2.21)\qquad x_0^2+\varpi^{\xi}\sum_{r=1}^sx_r^k\equiv n\pmod{q},$$

for which

$$(2.211)\qquad 0\leqslant x_t<\xi\quad(t=0,1,2,\dots,s).$$

Both here and later we use the suffix t to denote one of the suffixes 0, 1, 2, ..., s , while the suffix r denotes one of the suffixes 1, 2, 3, ..., s . We write

$$(2.22)\quad M(q)=M(q,n)=\nu(0,q,q,n),\quad J(\xi,q,n)=\nu(\xi,q,q,n)$$

and denote by

$$(2.23)\qquad L(\xi,q,n)=L(\xi,q)$$

the number of solutions $\nu(\xi,q,q,n)$ for which not every x is divisible by ϖ . Such solutions are called primitive solutions. We also write

$$(2.24)\qquad L(0,q,n)=N(q,n)=N(q).$$

We now prove a series of preliminary lemmas.

2.3. LEMMA 2. *If*

$$\begin{aligned}\varpi^{\theta}|k\quad(\theta\geqslant 0),\quad \lambda>\theta+l\quad(\varpi>2),\quad \lambda>\theta+l+1\quad(\varpi=2),\\1\leqslant l\leqslant \lambda-\theta,\\(2.31)\qquad x=\xi+\alpha\varpi^{\lambda-\theta-l},\end{aligned}$$

then

$$(2.32) \quad x^k \equiv \xi^k + \frac{k}{\varpi^\theta} \alpha \xi^{k-1} \varpi^{\lambda-l} \pmod{\varpi^{\lambda-l+1}}, \quad k \geq 2.$$

We have

$$x^k = \sum_{r=0}^k \binom{k}{r} \alpha^r \xi^{k-r} \varpi^{r(\lambda-\theta-l)}.$$

The first two terms of this expression are those of (2.32). We shall, therefore, suppose that $r \geq 2$. The index of the highest power of ϖ which divides $r!$ is*

$$\left[\frac{r}{\varpi} \right] + \left[\frac{r}{\varpi^2} \right] + \dots < \frac{r}{\varpi-1}.$$

Hence the r -th term is divisible by ϖ^c , where

$$c > \theta - \frac{r}{\varpi-1} + r(\lambda-\theta-l).$$

Now this gives

$$c - \lambda + l - 1 > (r-1)(\lambda-\theta-l-1) + r \frac{\varpi-2}{\varpi-1} - 2.$$

Thus, when $\varpi > 2$, $c - \lambda + l - 1 > \frac{1}{2}r - 2 \geq -1$, while, when $\varpi = 2$, $\lambda \geq \theta + l + 2$, and so $c - \lambda + l - 1 > r - 3 \geq -1$. In either case $c - \lambda + l - 1 \geq 0$, which proves the lemma.

2.4. LEMMA 3.

$$\sum_{\lambda=0}^{\mu} A_{\varpi^{\lambda}}(n) = \varpi^{-s\mu} M(\varpi^{\mu}, n).$$

For, writing $q = \varpi^{\lambda}$, we have

$$\begin{aligned} A_q(n) &= \sum_p \left(\frac{S_{p,q,k}}{q} \right)^s \left(\frac{S_{p,q,2}}{q} \right) e_q(-np) \\ &= q^{-s-1} \sum_{p, x_0, x_1, \dots, x_s=0}^{q-1} e_q \{ p(x_0^2 + x_1^k + x_2^k + \dots + x_s^k - n) \} \\ &= q^{-s-1} \sum_{x_0, x_1, \dots, x_s} e_q(X), \end{aligned}$$

where $X = x_0^2 + x_1^k + x_2^k + \dots + x_s^k - n$ and $c_q(X) = \sum_p e_q(pX)$.

If $\lambda = 0$, $q = 1$ and $A_1 = 1$.

* See P.N. 4, Lemma 1; the proof there given is a proof of our lemma when $l = 1$.

If $\lambda = 1$,

$$q = \varpi, \quad c_{\varpi}(X) = -1 \quad (\varpi \nmid X), \quad c_{\varpi}(X) = \varpi - 1 \quad (\varpi \mid X),$$

and

$$(2.41) \quad A_{\varpi} = \varpi^{-s-1} \left\{ \sum_X (-1) + \sum_{\varpi \mid X} \varpi \right\} = \varpi^{-s-1} \{ -\varpi^{s+1} + \varpi M(\varpi) \} \\ = \varpi^{-s} M(\varpi) - 1.$$

If $\lambda > 1$,

$$c_{\varpi^\lambda}(X) = 0 \quad (\varpi^{\lambda-1} \nmid X), \quad c_{\varpi^\lambda}(X) = -\varpi^{\lambda-1} \quad (\varpi^{\lambda-1} \parallel X),$$

$$c_{\varpi^\lambda}(X) = \varpi^{\lambda-1}(\varpi - 1) \quad (\varpi^\lambda \mid X),$$

$$A_{\varpi^\lambda} = \varpi^{-\lambda(s+1)} \left\{ \sum_{\varpi^{\lambda-1} \mid X} (-\varpi^{\lambda-1}) + \sum_{\varpi^\lambda \mid X} \varpi^\lambda \right\} \\ = \varpi^{-\lambda s} M(\varpi^\lambda) - \varpi^{-\lambda s-1} \nu(0, \varpi^\lambda, \varpi^{\lambda-1}, n).$$

If now we put

$$x_i = \xi_i + a_i \varpi^{\lambda-1} \quad (0 \leq \xi_i < \varpi^{\lambda-1}, \quad 0 \leq a_i < \varpi),$$

we have

$$x_0^2 + \sum x_r^k \equiv \xi_0^2 + \sum \xi_r^k \pmod{\varpi^{\lambda-1}},$$

and hence

$$\nu(0, \varpi^\lambda, \varpi^{\lambda-1}, n) = \sum_{a_0, a_1, a_2, \dots, a_s} \nu(0, \varpi^{\lambda-1}, \varpi^{\lambda-1}, n) = \varpi^{s+1} M(\varpi^{\lambda-1}),$$

$$(2.42) \quad A_{\varpi^\lambda} = \varpi^{-\lambda s} M(\varpi^\lambda) - \varpi^{-s(\lambda-1)} M(\varpi^{\lambda-1}).$$

From the results (2.41) and (2.42) the lemma follows immediately. As corollaries we have

LEMMA 4.

$$\chi_{\varpi} \geq 0.$$

LEMMA 5. If n is representable in any manner as a sum of one square and s k -th powers, then $\chi_{\varpi} > 0$.

To see this we have only to observe that, from a certain point onward, $A_{\varpi^\lambda} = 0$ (as will be shown later), and under the hypothesis of the lemma $M(\varpi^\lambda) > 0$ for every λ .

2.5. LEMMA 6. If $\varpi^{\theta_1} \parallel k$ ($\theta_1 \geq 0$), $\varpi^{\theta_2} \parallel 2$ ($\theta_2 = 0$ or 1), and if $\phi_1 = \theta_1 + 1$, $\phi_2 = 1$ ($\varpi > 2$), $\phi_1 = \theta_1 + 2$, $\phi_2 = 3$ ($\varpi = 2$), then

$$(2.51) \quad L(\xi, \varpi^\lambda, m) = \varpi^{s(\lambda-\phi-\xi)} L(\xi, \varpi^{\phi+\xi}, m),$$

where $\phi = \max(\phi_1, \phi_2)$, $\lambda \geq \phi + \xi$, $0 \leq \xi < k$, and m is arbitrary.

Obviously we can suppose that $\lambda > \phi + \zeta$, for when $\lambda = \phi + \zeta$ there is nothing to prove. We write

$$(2.521) \quad x_r = \xi_r + a_r \varpi^{\lambda - \theta_1 - \zeta - 1} \quad (0 \leq \xi_r < \varpi^{\lambda - \theta_1 - \zeta - 1}, \quad 0 \leq a_r < \varpi^{\theta_1 + \zeta + 1}),$$

$$(2.522) \quad x_0 = \xi_0 + a_0 \varpi^{\lambda - \theta_2 - 1} \quad (0 \leq \xi_0 < \varpi^{\lambda - \theta_2 - 1}, \quad 0 \leq a_0 < \varpi^{\theta_2 + 1}).$$

Then it follows from Lemma 2 that

$$(2.531) \quad x_r^k \equiv \xi_r^k + \frac{k}{\varpi^{\theta_1}} a_r \xi_r^{k-1} \varpi^{\lambda - \zeta - 1} \pmod{\varpi^{\lambda - \zeta}},$$

$$(2.532) \quad x_0^2 \equiv \xi_0^2 + \frac{2}{\varpi^{\theta_2}} a_0 \xi_0 \varpi^{\lambda - 1} \pmod{\varpi^{\lambda}}.$$

And thus, if

$$(2.54) \quad m = m_1 + m_2 \varpi^{\lambda - 1},$$

then the congruence

$$(2.55) \quad x_0^2 + \varpi^{\zeta} \sum x_r^k \equiv m \pmod{\varpi^{\lambda}}, \quad 0 \leq x_t < \varpi^{\lambda}$$

is equivalent to the pair of congruences

$$(2.551) \quad \xi_0^2 + \varpi^{\zeta} \sum \xi_r^k \equiv m_1 \equiv m \pmod{\varpi^{\lambda - 1}},$$

$$(0 \leq \xi_r < \varpi^{\lambda - \theta_1 - \zeta - 1}, \quad 0 \leq \xi_0 < \varpi^{\lambda - \theta_2 - 1})$$

and

$$(2.552) \quad \frac{2}{\varpi^{\theta_2}} \xi_0 a_0 + \sum \frac{k}{\varpi^{\theta_1}} a_r \xi_r^{k-1} \equiv m_2 - \frac{\xi_0^2 + \varpi^{\zeta} \sum \xi_r^k - m_1}{\varpi^{\lambda - 1}} \pmod{\varpi},$$

where

$$0 \leq a_r < \varpi^{\theta_1 + \zeta + 1}, \quad 0 \leq a_0 < \varpi^{\theta_2 + 1}.$$

We shall consider only primitive solutions of (2.551) and (2.552). In such a solution some ξ is not divisible by ϖ ; suppose that ξ_r is such a ξ . Then the value of every a in (2.552) except a_r can be assigned arbitrarily, and then, since $(\xi_r^{k-1} k / \varpi^{\theta_1}, \varpi) = 1$, the value of a_r will be determined uniquely to modulus ϖ . Hence there will be $\varpi^{\theta_1 + \zeta}$ possible values of a_r , and so $\varpi^{\theta_1 + \zeta} (\varpi^{\theta_1 + \zeta + 1})^{s-1} \varpi^{\theta_2 + 1} = \varpi^{(\theta_1 + \zeta + 1)s + \theta_2}$ sets of a 's associated with every solution of (2.551). But now suppose that ξ_0 is the ξ which is not divisible by ϖ . Then the values of $a_1, a_2, \dots, a_s$ may be assigned arbitrarily, and, since $(2\xi_0 / \varpi^{\theta_2}, \varpi) = 1$, the value of a_0 will be determined uniquely to modulus ϖ . Hence there will be ϖ^{θ_2} possible values of a_0 and

$$\varpi^{\theta_2} (\varpi^{\theta_1 + \zeta + 1})^s = \varpi^{(\theta_1 + \zeta + 1)s + \theta_2}$$

sets of a 's associated with every solution of (2.551). Thus in either case

$$(2.56) \quad L(\zeta, \varpi^{\lambda}, m) = \varpi^{s(\theta_1 + \zeta + 1) + \theta_2} L_1,$$

where L_1 denotes the number of primitive solutions of (2.551).

But now $L(\xi, \varpi^\lambda, m)$ is the number of primitive solutions of the congruence (2.55). If in $L(\xi, \varpi^{\lambda-1}, m)$ we write

$$x_r = z_r + a_r \varpi^{\lambda-\theta_1-\zeta-1} \quad (0 \leq z_r < \varpi^{\lambda-\theta_1-\zeta-1}, \quad 0 \leq a_r < \varpi^{\theta_1+\zeta}),$$

$$x_0 = z_0 + a_0 \varpi^{\lambda-\theta_2-1} \quad (0 \leq z_0 < \varpi^{\lambda-\theta_2-1}, \quad 0 \leq a_0 < \varpi^{\theta_2}),$$

we have, by Lemma 2,

$$x_0^2 + \varpi^\zeta \sum x_r^k \equiv z_0^2 + \varpi^\zeta \sum z_r^k \pmod{\varpi^{\lambda-1}}.$$

Hence

$$(2.57) \quad L(\xi, \varpi^{\lambda-1}, m) = \sum_{\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_s} L_1 = \varpi^{\theta_2 + (\theta_1 + \zeta)s} L_1,$$

and from (2.56) and (2.57) it follows that

$$L(\xi, \varpi^\lambda, m) = \varpi^s L(\xi, \varpi^{\lambda-1}, m) \quad (\lambda > \phi + \xi),$$

and the lemma follows immediately.

LEMMA 7. Unless $\varpi = 2$ and $k \leq 4$, we have $\phi \leq k-1$. In the exceptional cases $\phi = k$.

For, if $\varpi > 2$, $\phi = \max(\theta_1+1, 1) = \theta_1+1 \leq 2^{\theta_1} < \varpi^{\theta_1} \leq k$. If $\varpi = 2$ and $\theta_1 = 1$, $\phi = 3 < k$, since obviously in this case k is oddly even. If $\varpi = 2$ and $\theta_1 \geq 3$, $\phi = \theta_1+2 < 2^{\theta_1} \leq k$. There remain only the two cases $\varpi = 2$, $\theta_1 = 0$, and $\varpi = 2$, $\theta_1 = 2$. In the first of these $\phi = 3 < k$, except when $k = 3$, in which case $\phi = k$. In the second $\phi = 4 < k$, except when $k = 4$, when we have $\phi = k$.

2.6. THEOREM I. If

$$\varpi^{\theta_1} \parallel k \quad (\theta_1 \geq 0), \quad \varpi^{\theta_2} \parallel 2 \quad (\theta_2 = 0 \text{ or } 1), \quad (\varpi^k)^\beta \parallel n \quad (\beta \geq 0), \quad \varpi^{\beta k + \nu} \parallel n,$$

and ϕ is defined as in Lemma 6, while $p = [\frac{1}{2}k]$, $\mathbf{B} = 0$ ($u = 0$), $\mathbf{B} = 1$ ($u > 0$), $\eta = 1$ when k and β are both odd and is otherwise zero, $u = [\frac{1}{2}(\nu + \eta)]$, and $\epsilon = 1$ when k and α are both odd and is otherwise zero, then

$$(2.61) \quad \chi_\varpi = \varpi^{-\phi s} \sum_{\alpha=0}^{\beta-1} \varpi^{\alpha(\frac{1}{2}k-s) - \epsilon(\frac{1}{2}+s)} L(\epsilon, \varpi^{\phi+\epsilon}, 0) \\ + \varpi^{\beta(\frac{1}{2}k-s) - \eta(\frac{1}{2}+s) - \phi s} L\left(\eta, \varpi^{\phi+\eta}, \frac{n}{\varpi^{\beta k - \eta}}\right) \\ + \mathbf{B} \varpi^{\beta(\frac{1}{2}k-s) + u - \frac{1}{2}\eta - s(1+\phi+k+\eta-2u)} \\ \times L\left(k+\eta-2u, \varpi^{\phi+k+\eta-2u}, \frac{n}{\varpi^{\beta k - \eta + 2u}}\right).$$

Suppose that $\lambda > \lambda_0$, where

(2.621)

$$\lambda_0 = \beta k + k + \phi \quad (\nu + \eta \geq 2), \quad \lambda_0 = \text{Max}(\beta k + \nu + 1, \beta k + \phi) \quad (\nu + \eta < 2).$$

We can divide the solutions of the congruence

$$(2.622) \quad x_0^2 + \sum_{r=1}^s x_r^k \equiv n \pmod{\varpi^\lambda}, \quad 0 \leq x_t < \varpi^\lambda$$

into two classes, the primitive and the imprimitive solutions. There will be $N(\varpi^\lambda, n)$ solutions of the first kind. Those of the second kind will be correlated with those of the congruence

$$y_0^2 + \varpi^{k-2} \sum y_r^k \equiv \frac{n}{\varpi^2} \pmod{\varpi^{\lambda-2}}, \quad 0 \leq y_t < \varpi^{\lambda-1}.$$

This will have no solutions unless (i) both y_0^2 and n/ϖ^2 are divisible by ϖ , (ii) neither is divisible by ϖ . If case (ii) holds the solutions will necessarily be primitive. If case (i) holds the number of solutions will be equal to the number of solutions of

$$y_0'^2 + \varpi^{k-4} \sum y_r^k \equiv \frac{n}{\varpi^4} \pmod{\varpi^{\lambda-4}}, \quad 0 \leq y_0' < \varpi^{\lambda-2}, \quad 0 \leq y_r < \varpi^{\lambda-1},$$

provided that $k \geq 4$. Proceeding in this way, we see that the number of solutions of our second class is the same as the number of solutions of

$$(2.623) \quad z_0^2 + \varpi^\zeta \sum z_r^k \equiv \frac{n}{\varpi^{2p}} \pmod{\varpi^{\lambda-2p}}, \quad 0 \leq z_0 < \varpi^{\lambda-p}, \quad 0 \leq z_r < \varpi^{\lambda-1},$$

where $\zeta = 1$ if k is odd, and is otherwise zero. If, in this, we write

$$z_r = \xi_r + \alpha_r \varpi^{\lambda-2p} \quad (0 \leq \xi_r < \varpi^{\lambda-2p}, \quad 0 \leq \alpha_r < \varpi^{2p-1}),$$

$$z_0 = \xi_0 + \alpha_0 \varpi^{\lambda-2p} \quad (0 \leq \xi_0 < \varpi^{\lambda-2p}, \quad 0 \leq \alpha_0 < \varpi^p),$$

then we have, by Lemma 2,

$$z_0^2 + \varpi^\zeta \sum z_r^k \equiv \xi_0^2 + \varpi^\zeta \sum \xi_r^k \pmod{\varpi^{\lambda-2p}},$$

so that the number of solutions of (2.623) is obviously

$$(2.624) \quad \varpi^{s(2p-1)+p} J\left(\xi, \varpi^{\lambda-2p}, \frac{n}{\varpi^{2p}}\right).$$

This congruence has primitive and imprimitive solutions, and by repeating the above argument we see that, if $\beta \geq 2$, the number of imprimitive

solutions included in $J(\xi, \varpi^{\lambda-2p}, n/\varpi^{2p})$ is

$$(2.6251) \quad \varpi^{s(2k-2p-1)+k-p} M\left(\varpi^{\lambda-2k}, \frac{n}{\varpi^{2k}}\right).$$

The number of imprimitive solutions included in $M(\varpi^{\lambda-2k}, n\varpi^{-2k})$ can be shown to be

$$(2.6252) \quad \varpi^{2ps+p-s} J\left(\xi, \varpi^{\lambda-2k-2p}, \frac{n}{\varpi^{2k-2p}}\right),$$

and the process can be repeated until the number of imprimitive solutions is

$$(2.626) \quad J\left(\eta, \varpi^{\lambda-\beta k+\eta}, \frac{n}{\varpi^{\beta k-\eta}}\right)$$

multiplied by the appropriate power of ϖ . Now, repeating the process as far as possible, we find that

$$(2.627) \quad J\left(\eta, \varpi^{\lambda-\beta k+\eta}, \frac{n}{\varpi^{\beta k-\eta}}\right) \\ = \mathbf{B}\varpi^{(2s+1)u-s} L\left(k+\eta-2u, \varpi^{\lambda-\beta k+\eta-2u}, \frac{n}{\varpi^{\beta k-\eta+2u}}\right) \\ + L\left(\eta, \varpi^{\lambda-\beta k+\eta}, \frac{n}{\varpi^{\beta k-\eta}}\right).$$

Hence we have, on collecting these results,

$$(2.628) \quad M(\varpi^{\lambda}, n) = \sum_{a=0}^{\beta} \varpi^{\alpha(k s+\frac{1}{2}k-s)-\frac{1}{2}\epsilon(2s+1)} L\left(\epsilon, \varpi^{\lambda-\alpha k+\epsilon}, \frac{n}{\varpi^{\alpha k-\epsilon}}\right) \\ + \mathbf{B}\varpi^{\mu+(2s+1)u-s} L\left(k+\eta-2u, \varpi^{\lambda-\beta k+\eta-2u}, \frac{n}{\varpi^{\beta k-\eta+2u}}\right),$$

where μ denotes the highest power of ϖ which has previously occurred.

2.63. Now if $\lambda \geq \lambda_0$, Lemma 6 is applicable to every term of this series. Multiplying by $\varpi^{-\lambda s}$ and making use of this lemma, our result takes the form

$$(2.631) \quad \varpi^{-\lambda s} M(\varpi^{\lambda}, n) = \varpi^{-s\phi} \sum_{a=0}^{\beta} \varpi^{\alpha(\frac{1}{2}k-s)-\frac{1}{2}\epsilon(2s+1)} L\left(\epsilon, \varpi^{\phi+\epsilon}, \frac{n}{\varpi^{\alpha k-\epsilon}}\right) \\ + \mathbf{B}\varpi^{\mu+u-s(1+\beta k+k-2u+\phi)} L\left(k+\eta-2u, \varpi^{\phi+k+\eta-2u}, \frac{n}{\varpi^{\beta k-\eta+2u}}\right).$$

If $\alpha < \beta$ and $\phi \leq k$, clearly $n/\varpi^{\alpha k-\epsilon}$ is divisible by $\varpi^{\phi+\epsilon}$, and so may be replaced in any L term by zero. But, by Lemma 7, we always have

$\phi \leq k$, hence

$$(2.632) \quad \varpi^{-\lambda s} M(\varpi^\lambda, n) = \varpi^{-\phi s} \sum_{a=0}^{\beta-1} \varpi^{a(\frac{1}{2}k-s)-\epsilon(s+\frac{1}{2})} L(\epsilon, \varpi^{\phi+\epsilon}, 0) \\ + \varpi^{\beta(\frac{1}{2}k-s)-\frac{1}{2}\eta-\phi s-\eta s} L\left(\eta, \varpi^{\phi+\eta}, \frac{n}{\varpi^{\beta k-\eta}}\right) \\ + B \varpi^{\beta(\frac{1}{2}k-s)+u-s-\frac{1}{2}\eta-s(\phi+k+\eta-2u)} L\left(k+\eta-2u, \varpi^{\phi+k+\eta-2u}, \frac{n}{\varpi^{\beta k-\eta+2u}}\right),$$

it being understood that, when $\beta = 0$ or 1 , the series is simply omitted. Since the right-hand side of (2.632) is independent of λ , it is, by Lemma 3, equal to

$$(2.633) \quad \lim_{\lambda \rightarrow \infty} \varpi^{-\lambda s} M(\varpi^\lambda, n) = \chi_\varpi,$$

whence we have the result (2.611).

We may notice that we have shown incidentally that

$$(2.634) \quad A_{\varpi^\lambda} = 0 \quad (\lambda > \lambda_0).$$

3.1. We shall need various properties of the Gaussian sums

$$(3.11) \quad S_{p,q,k} = \sum_{j=0}^{q-1} e_q(j^k p)$$

in the study of the singular series. I quote the relevant lemmas from P.N. 4. In this section we denote by

$$\chi = \chi_\kappa = \chi_\kappa(m) \quad (1 \leq \kappa \leq h)$$

the $*h$ Dirichlet "characters" to modulus q . χ_1 is the principal character, $\bar{\chi}_\kappa$ the character conjugate to χ_κ , and we write

$$(3.12) \quad S'_{p,q,k} = \sum_{j=0}^{q-1} \chi_1(j) e_q(j^k p) = \sum_{(j,q)=1} e_q(j^k p). \\ \delta = (h, k) = \{\varpi^{\lambda-1}(\varpi-1), k\},$$

when $q = \varpi^\lambda$, which is the only case considered here.

$$3.2. \text{ LEMMA } 8^+. \text{ If } q = \varpi^\lambda \quad (1 \leq \lambda \leq k),$$

* Here h denotes the function of q which is usually denoted by $\phi(q)$.

† Lemmas 8, 9 and 10 are Lemmas 11, 12, and 13 of P.N. 4.

then

$$(3.21) \quad S'_{p,q,k} = S'_{p,q,\delta}.$$

LEMMA 9. If $q = \varpi^\lambda$ ($1 < \lambda \leq k$) and $\varpi \nmid k$, then

$$(3.22) \quad S_{p,q,k} = \varpi^{\lambda-1}.$$

LEMMA 10. If $\lambda = 1$, $q = \varpi$, and $\delta = 1$, then

$$(3.23) \quad S_{p,q,k} = 0.$$

But if $\delta > 1$, then

$$(3.24) \quad S_{p,q,k} = \sum_{\kappa'} \tau_{\bar{\kappa}} \chi_{\kappa'}(p),$$

where

$$(3.25) \quad \tau_{\kappa'} = \sum_l e_q(l) \bar{\chi}_{\kappa}(l),$$

and the summation with respect to κ' extends over the special characters χ' , exclusive of the principal character χ_1 . Also

$$(3.26) \quad |S_{p,q,k}| \leq (\delta-1)\sqrt{q}.$$

LEMMA 11. We have

$$(3.27) \quad A_{\varpi} = \varpi^{-s-1} \sum_p e_{\varpi}(-np) \left(\sum_{\kappa'} \tau_{\bar{\kappa}} \chi_{\kappa'}(p) \right)^s \left(\sum_{\kappa''} \tau_{\bar{\kappa}''} \chi_{\kappa''}(p) \right).$$

where $\tau_{\bar{\kappa}}$ is the same function of 2 that τ is of k , and the double dashes denote that the characters are special in the sense that

$$\chi_{\kappa}^2 = \chi_1.$$

The summation for κ' extends over all characters which are special in the sense of P.N. 4, except χ_1 , and the summation over κ'' extends over all characters except χ_1 , which are "special" in the above sense.

This follows at once from (3.24).

LEMMA 12. We have

$$(3.28) \quad \left| \frac{S_{p,q,k}}{q} \right| < Bq^{-a}, \quad \left| \frac{S_{p,q,2}}{q} \right| < Aq^{-\frac{1}{2}},$$

and

$$(3.29) \quad |A_q(m)| < Cq^{\frac{1}{2}-as}.$$

The first two inequalities are proved in Lemma 3 of P.N. 6, and the third is an obvious corollary of the first two.

3.3. LEMMA 13. *If $s > 0$, we have*

$$(3.311) \quad |A_{\varpi}| < C\varpi^{-\frac{1}{2}s} \quad (\varpi \nmid n).$$

$$(3.312) \quad |A_{\varpi}| < C\varpi^{\frac{1}{2}-\frac{1}{2}s} \quad (\varpi | n).$$

$$(3.321) \quad |\chi_{\varpi}-1| < C\varpi^{-\frac{1}{2}s} \quad (\varpi \nmid n).$$

$$(3.322) \quad |\chi_{\varpi}-1| < C(\varpi^{\frac{1}{2}-\frac{1}{2}s} + \varpi^{-s+\frac{1}{2}\nu}) \quad (\varpi | n, \nu \text{ even}, \varpi^k \nmid n).$$

$$(3.323) \quad |\chi_{\varpi}-1| < C\varpi^{\frac{1}{2}-\frac{1}{2}s} \quad (\varpi | n, \nu \text{ odd}, \varpi^k \nmid n).$$

Using (3.27) and noticing that the product of χ 's is a χ , we have, on expanding by the multinomial theorem and reversing the order of summation,

$$A_{\varpi} = \varpi^{-s-1} \Sigma_1 T \Sigma \chi(p) e_{\varpi}(-np),$$

where T is a product of s τ 's and one τ' , χ is a product of $(s+1)$ χ 's, and the number of terms in Σ_1 is $O(1)$. The inner sum is $O(\sqrt{\varpi})$ when $\varpi \nmid n$, and is $O(\varpi)$ when $\varpi | n$, for every χ and all values of n . Hence, if $\varpi \nmid n$, we have

$$|A_{\varpi}| < C\varpi^{-s-1} \varpi^{\frac{1}{2}(s+1)} \varpi^{\frac{1}{2}} = C\varpi^{-\frac{1}{2}s},$$

and (3.312) is obtained similarly.

Now if $\varpi \nmid 2kn$, so that $\beta = 0$, $\nu = 0$, $\phi = 1$, and $\lambda_0 = 1$, then, by (2.634),

$$(3.33) \quad |\chi_{\varpi}-1| = |A_{\varpi}| \leq C\varpi^{-\frac{1}{2}s}.$$

Next suppose that $\varpi | n$, $\varpi^k \nmid n$, so that $0 < \nu < k$, $\beta = 0$. Then, if $\varpi \nmid 2k$ so that $\phi = k+1$,

$$(3.34) \quad \chi_{\varpi} = 1 + A_{\varpi} + \sum_{\lambda=2}^{k+1} A_{\varpi^{\lambda}}.$$

But now

$$S_{p, \varpi^{k+1}, k} = \varpi^{k-1} S_{p, \varpi, k}, \quad S_{p, \varpi^{k+1}, 2} = \varpi^{\frac{1}{2}(k+1)} \text{ or } \varpi^{\frac{1}{2}k} S_{p, \varpi, 2},$$

according as k is odd or even. If, in the expression for $A_{\varpi^{k+1}}$, we write

$$p = \varpi z + p',$$

where $0 \leq z < \varpi^k$, and $p' < \varpi$, we have, when k is even,

$$A_{\varpi^{k+1}} = \varpi^{-2s-\frac{1}{2}k-1} \sum_{p'} \sum_z (S_{p', \varpi, k})^s (S_{p', \varpi, 2}) e\left(-\frac{nz}{\varpi^k} - \frac{np'}{\varpi}\right),$$

and the summation with respect to z vanishes since $\varpi^k \nmid n$. Similarly we can deal with the case when k is odd, so that, whether k is odd or even,

$$A_{\varpi^{k+1}} = 0.$$

Now, by Lemma 9, we have, if $2 \leq \lambda \leq k$,

$$A_{\varpi^\lambda} = \varpi^{-s} A'_{\varpi^\lambda},$$

where

$$A'_{\varpi^\lambda} = \sum_p \left(\frac{S_{p, \varpi, 2}}{\varpi^\lambda} \right) e_{\varpi^\lambda}(-np).$$

So, if ν is even,

$$\begin{aligned} (3.341) \quad \sum_2^k A_{\varpi^\lambda} &= \varpi^{-s} \left[(\varpi-1) + 0 + (\varpi-1)\varpi + 0 \right. \\ &\quad \left. + \dots + (\varpi-1)\varpi^{\frac{1}{2}\nu-1} + 0 + \varpi^{\frac{1}{2}\nu} \left(\frac{n_0}{\varpi} \right) \right] \\ &= -\varpi^{-s} + \varpi^{-s+\frac{1}{2}\nu} \left\{ 1 + \frac{n_0}{\varpi} \right\}^*, \end{aligned}$$

where $n_0 \varpi^\nu = n$. On the other hand, if ν is odd, we have

$$\begin{aligned} (3.342) \quad \sum_2^k A_{\varpi^\lambda} &= \varpi^{-s} [(\varpi-1) + 0 + (\varpi-1)\varpi \\ &\quad + \dots + (\varpi-1)\varpi^{\frac{1}{2}(\nu-3)} + 0 - \varpi^{\frac{1}{2}(\nu-1)}]. \\ &= -\varpi^{-s}. \end{aligned}$$

We still have to consider the case $\varpi \mid 2k$, $\varpi^k \nmid n$. Then $\varpi \leq k$, and, by (3.29), we have

$$|A_q(n)| < Cq^{\frac{1}{2}}.$$

Now $\lambda_0 \leq k + \phi \leq 2k$, since $\phi \leq k$, and so

$$(3.35) \quad |\chi_{\varpi} - 1| \leq C \sum_{\lambda=1}^{2k} (\varpi^\lambda)^{\frac{1}{2}} \leq C \sum_{\lambda=1}^{2k} k^{\frac{1}{2}\lambda} = C \leq C\varpi^{-\frac{1}{2}s}.$$

Thus (3.321) follows from (3.33) and (3.35); (3.322) from (3.34), (3.341), (3.35), and (3.312), since $\frac{1}{2} - \frac{1}{2}s > -s$ if $s \geq 0$; and (3.323) follows from (3.34), (3.342), (3.312), and (3.35).

3.4. LEMMA 14. *We have, when $s \geq \nu-1$,*

$$(3.41) \quad \chi_{\varpi} > 1 - C\varpi^{\frac{1}{2}-\frac{1}{2}s}.$$

* For the value of A'_{ϖ^λ} see G. H. Hardy, "On the representation of a number as the sum of any number of squares, and in particular of five", *Trans. Amer. Math. Soc.*, 17 (1920), 255-284. See, in particular, §§ 4.42, 4.43.

When $\beta = 0$, this is proved in Lemma 13. If $\beta > 0$ and $\varpi > k$ we have $\phi = 1$, and so, from Theorem 1,

$$\chi_{\varpi}(n) > \varpi^{-s} N(\varpi, 0) = \varpi^{-s} N(\varpi, \varpi) = \chi_{\varpi}(\varpi).$$

But, from (3.342),

$$\chi_{\varpi}(\varpi) = 1 + A_{\varpi} - \varpi^{-s} > 1 - C\varpi^{\frac{1}{2} - \frac{1}{2}s}.$$

Also, if $\beta > 0$ and $\varpi < k$, we have

$$\chi_{\varpi}(n) \geq 0 > 1 - C\varpi^{\frac{1}{2} - \frac{1}{2}s}.$$

THEOREM 2. *The singular series $\mathbf{S} = \mathbf{C}\Sigma A_q$ and the product $\mathbf{P} = \mathbf{C}\Pi \chi_{\varpi}$ are absolutely convergent for $s \geq 3$; and $\mathbf{S} = \mathbf{P}$.*

For, if n is fixed, $\varpi \nmid n$ from a certain point onwards, and so the result follows immediately from (3.311) and (3.321).

LEMMA 15. We have, when $s \geq \text{Max}(k, 4)$,

$$(3.42) \quad \chi_{\varpi} > 1 - C\varpi^{-\frac{3}{2}}.$$

This is an immediate corollary of Lemma 14.

4.1. Given k , ϖ , and any positive integer m , then either there exists a number h_{ϖ} such that

$$(4.12) \quad \chi_{\varpi} \geq h_{\varpi}$$

for $s \geq m$, and all values of n , or there is no such number. Let us define

$$\gamma_{\varpi} = \gamma(k, \varpi)$$

as the least value of m for which h_{ϖ} exists, and then define $\Gamma(k)$ by

$$(4.13) \quad \Gamma(k) = \underset{\varpi}{\text{Max}} \gamma_{\varpi}.$$

Further, we define $\gamma'_{\varpi} = \gamma'(k, \varpi)$

as the least value of m such that

$$(4.14) \quad \chi_{\varpi} > 0$$

for $s \geq m$ and all values of n , so that obviously

$$\gamma'_{\varpi} \leq \gamma_{\varpi}.$$

LEMMA 16. *If $\chi_{\varpi} > 0$ for all sufficiently large values of n , then $\chi_{\varpi} > 0$ for all values of n .*

From Lemma 7, $\phi < k$ except when $\varpi = 2$, $k \leq 4$. Neglecting these exceptional cases for the moment, we write

$$n_1 = \varpi^\phi m_1 + n', \quad n_2 = \varpi^\phi m_2 + n' \quad (0 \leq n' < \varpi^\phi).$$

If $n' \neq 0$, then $\beta = 0$, and, by Theorem 1, we have

$$\chi_\varpi(n_1) = \varpi^{-\phi s} N(\varpi^\phi, n_1) + \mathbf{B}\varpi^{\lceil \frac{1}{2}\nu \rceil - s(1+k-2\lceil \frac{1}{2}\nu \rceil + \phi)} L\left(k-2\lceil \frac{1}{2}\nu \rceil, \varpi^{\phi+k-2\lceil \frac{1}{2}\nu \rceil} \frac{n}{\varpi^{2\lceil \frac{1}{2}\nu \rceil}}\right).$$

Now $n_1 \equiv n_2 \pmod{\varpi^\phi}$, and ν is the same for both n_1 and n_2 . Thus it follows that

$$\chi_\varpi(n_1) = \chi_\varpi(n_2),$$

and, since corresponding to every n_2 there is an n_1 which is as large as we please, it follows that, if $\chi_\varpi > 0$ for all large values of n_1 , $\chi_\varpi > 0$ for all values of n which are not divisible by ϖ^ϕ .

Next suppose that $n' = 0$. Then, if $(\varpi^{k-\phi}, m) = 1$, we have $\beta = 0$, and so

$$\chi_\varpi(n_1) = \varpi^{-\phi s} N(\varpi^\phi, 0) + \mathbf{B}\varpi^{u-s(1+k-2u+\phi)} L\left(k-2u, \varpi^{\phi+k-2u} \frac{n}{\varpi^{2u}}\right).$$

Thus at least one of

$$N(\varpi^\phi, 0), \quad L\left(k-2u, \varpi^{\phi+k-2u} \frac{n}{\varpi^{2u}}\right)$$

is positive. On the other hand, if $(\varpi^{k-\phi}, m) \neq 1$, at least one of

$$N(\varpi^\phi, 0), \quad L(\xi, \varpi^{\phi+\xi}, 0), \quad L\left(k+\eta-2u, \varpi^{\phi+k+\eta-2u} \frac{n}{\varpi^{3k-\eta+2u}}\right)$$

is positive. But, given any specified n_2 where $\varpi^\phi | n$, we can always find an n_1 such that (1) n_1 is as large as we please, (2) n_1 and n_2 have the same u and η ,

$$(3) \quad \frac{n_1}{\varpi^{\beta_1 k}} - \frac{n_2}{\varpi^{\beta_2 k}} \equiv 0 \pmod{\varpi^{\phi+k}}.$$

Thus, if $\chi_\varpi(n) > 0$ for sufficiently large values of n_1 , $\chi_\varpi(n) > 0$ for all values of n such that $\varpi^\phi | n$. This completes the lemma, except when $\varpi = 2$, $k \leq 4$.

Suppose first that $k = 4$, so that $\phi = k$. Then, if $\nu = 0$, $\beta = 0$, the only term in χ_ϖ involves $N(16, n)$ where n is odd. But $x^2 \equiv 0, 1, 4$, or $9 \pmod{16}$ and $x^4 \equiv 0, 1 \pmod{16}$, so that $N(16, n) > 0$ if, and only if, $s \geq 6$. Also it is easy to verify that $\chi_\varpi(n)$ always involves $N(16, n')$ where $16 \nmid n'$ and that $N(16, n') > 0$ if $s \geq 6$. Thus $s \geq 6$ is a necessary and sufficient condition that $\chi_\varpi(n) > 0$ for all large values of n . But it is also a sufficient condition that $\chi_\varpi(n) > 0$ for all values of n .

Next suppose that $k = 3$, so that $\varpi^\phi = 8$. Then, if $\varpi^2 \nmid n$, we have

$$\chi_\varpi(n) = \varpi^{-\phi s} N(\varpi^\phi, n),$$

and it is easy to see that, whatever n may be, $\chi_\varpi(n) > 0$ if $s \geq 1$, since $x^2 \equiv 0, 1$, or $4 \pmod{8}$, and $x^3 \equiv 0, 1, 3, 5$, or $7 \pmod{8}$. Hence the necessary and sufficient condition that $\chi_\varpi(n) > 0$ for all large values of n is that $s \geq 1$, in which case $\chi_\varpi(n) > 0$ for all values of n .

4.2. LEMMA 17. *The necessary and sufficient condition that either*

$$(4.21) \quad N(\varpi^\phi, n) > 0 \text{ or } \mathbf{BL}\left(k-2u, \varpi^{\phi+k-2u}, \frac{n}{\varpi^{2u}}\right) > 0$$

for every n is that $s \geq \gamma_\varpi$. Further

$$(4.22) \quad \gamma'_\varpi = \gamma_\varpi,$$

except when $\varpi = 2$, $k = 4$, in which case

$$(4.23) \quad \gamma'_\varpi = 6, \quad \gamma_\varpi = 7.$$

Suppose first that, if $\varpi = 2$, $k > 4$. Then we have $\phi < k$. Now, if $s \geq \gamma'_\varpi$, $\chi_\varpi(l\varpi^\phi) > 0$. But if $(\varpi, l) = 1$, $\beta = 0$, and so

$$\chi_\varpi(l\varpi^\phi) = \varpi^{-\phi s} N(\varpi^\phi, l\varpi^\phi) + \mathbf{B}\varpi^{u+s-\phi} L\left(k-2u, \varpi^{\phi+k-2u}, \frac{l\varpi^\phi}{\varpi^{2u}}\right).$$

Obviously $v = \phi$ and $u = [\frac{1}{2}\phi]$, so that the L term does not occur if $\phi = 1$. Suppose that ϕ is an odd integer greater than 1. Then the L term is the number of solutions of

$$x_0^2 + \varpi^{k-\phi+1} \sum x_r^k \equiv l\varpi \pmod{\varpi^{k+1}}, \quad 0 \leq x_t < \varpi^{k+1},$$

and clearly, since $k-\phi+1 \geq 2$ (except in the excluded cases), this can have no solutions. Next suppose that ϕ is even and that l is not a quadratic residue to modulus ϖ if $\varpi > 2$, and that $l \equiv 3 \pmod{4}$ when $\varpi = 2$. The L term now gives the number of solutions of the congruence

$$x_0^2 + \varpi^{k-\phi} \sum x_r^k \equiv l \pmod{\varpi^k}.$$

Then, since $\varpi \nmid (x_0^2 - l)$ unless $\varpi = 2$, there can be no solutions when $\varpi > 2$. If $\varpi = 2$, we can select x_0 , so that $x_0^2 - l$ is divisible by 2, but it cannot be divisible by 4. Thus there are no solutions except when $k-\phi = 1$, which is impossible. Thus (except possibly when $\varpi = 2$, $k = 3$ or 4), if $s > \gamma_\varpi$ and n is divisible by ϖ^ϕ ,

$$N(\varpi^\phi, 0) > 0.$$

If, on the other hand, $n \not\equiv 0 \pmod{\varpi^\phi}$, then $\beta = 0$, and hence

$$\chi_{\varpi}(n) = \varpi^{-\phi s} N(\varpi^\phi, n) + \mathbf{B} \varpi^{u-s(1+k-2u+\phi)} L\left(k-2u, \varpi^{\phi+k-2u}, \frac{n}{\varpi^{2u}}\right),$$

so that $s \geq \gamma'_{\varpi}$ is a sufficient condition that (4.21) should hold for every n .

Next suppose that (4.21) holds if $s = s_1$ for every n . Then it necessarily holds if $s \geq s_1$ for every n . Also one of the N 's or the final L that occur in $\chi_{\varpi}(n)$ is positive, and $N(\varpi^\phi, 0)$ is always positive and occurs except when $\beta = 0$. In that case there are only two terms, the second of which vanishes if $u = 0$. Hence, from Theorem 1,

$$\chi_{\varpi} \geq \varpi^{-\phi s + s + 1 - ks} \quad (s \geq s_1),$$

it follows that

$$s_1 \geq \gamma_{\varpi} \geq \gamma'_{\varpi}.$$

Thus $s \geq \gamma_{\varpi}$ is both a necessary and a sufficient condition that (4.21) should hold, and also $s \geq \gamma'_{\varpi}$ involves $s \geq \gamma_{\varpi}$. So we get $\gamma'_{\varpi} = \gamma_{\varpi}$.

When $\varpi = 2$, $k = 3$, we have $\varpi^\phi = 8$, $x_0^2 \equiv 0, 1$, or $4 \pmod{8}$, $x^3 \equiv 0, 1, 3, 5$, or $7 \pmod{8}$, and so

$$N(8, n) > 0 \quad (s \geq 1)$$

for all values of n ;

$$L(1, 2^4, n) = 0 \quad (s = 0)$$

except when $n = 1$ or 9 . Also

$$N(2^3, n) = 0 \quad (s = 0),$$

except when $n = 1$, so that

$$\chi_2 > 2^{-3s} \quad (s \geq 1), \quad \chi_2 = 0 \quad (s = 0),$$

and we have

$$\gamma_{\varpi} = \gamma'_{\varpi} = 1.$$

When $\varpi = 2$, $k = 4$, we have $\varpi^\phi = 16$, and $x^4 \equiv 0$ or $1 \pmod{16}$, while $x^2 \equiv 0, 1, 4$, or $9 \pmod{16}$. Then it follows that $N(16, n) > 0$ when $s \geq 7$ for every n ; and that

$$N(16, n) > 0 \quad (16 \nmid n), \quad N(16, 0) = 0 \quad (s = 6);$$

also that $N(16, 15) = N(16, 0) = 0$ when $s < 6$. Since k is even and $\phi = k$ there must be one $N(16, n)$ term when $16 \nmid n$, and not more than one L term, which will occur at the end. Hence we have

$$\chi_2 > h_2 \quad (s \geq 7), \quad \chi_2 > 0 \quad (s = 6),$$

$$\chi_2(16^s, 15) = 2^{-4\beta - 24} N(16, 15) = 2^{-4\beta - 24 + 18 + 2} = 2^{-4\beta - 4} \quad (s = 6),$$

since x_0 may have any of the values 3, 5, 11, or 13, and each x_r any of the eight values 1, 3, 5, ..., 15. Hence, as $\beta \rightarrow \infty$,

$$\chi_2(16^\beta \cdot 15) \rightarrow 0.$$

Thus we have the results (4.23).

4.3. THEOREM 3. $G(k) \geq \Gamma(k)$.

We neglect the case $k = 4$, $\varpi = 2$ for the moment, and suppose that $s \geq G(k)$. Then any sufficiently large value of n is the sum of a square and s k -th powers, so that $\chi_\varpi > 0$ for every ϖ and all sufficiently large values of n . Thus, by Lemma 16, $\chi_\varpi > 0$ for every ϖ and every n , and it follows that $s \geq \gamma_\varpi$. Hence $G(k) \geq \gamma_\varpi$ for every ϖ , which proves the theorem.

When $k = 4$, $\gamma_2 = 7$, and let us take $n = 16^\beta \cdot 47$ where $\beta > 0$. Then

$$x_0^2 + x_1^4 + x_2^4 + \dots + x_s^4 = 16^\beta \cdot 47.$$

If not all the x 's are even, we shall have, as before, $s \geq 7$. If they are all even, but $4 \nmid x_0$, we have, on division by 4,

$$y_0^2 + 4(y_1^4 + y_2^4 + \dots + y_s^4) = 4 \cdot 16^{\beta-1} \cdot 47 \quad (y_0 \text{ odd})$$

which is obviously impossible. If $4 \mid x_0$ we have

$$y_0'^2 + y_1^4 + y_2^4 + \dots + y_s^4 = 16^{\beta-1} \cdot 47,$$

and, again, either $s \geq 7$ or all the y 's are even. Thus we have $s \geq 7$ unless

$$z_0^2 + z_1^4 + z_2^4 + \dots + z_3^4 = 47,$$

where $s < 7$. But the most economical way of representing 47 is $5^2 + 2^4 + 6 \cdot 1 = 47$, so that $G(k) \geq 7 = \gamma_2$, so that the theorem is true even in the exceptional case.

4.4. LEMMA 18. Suppose that θ_1 , θ_2 , ϕ_1 , ϕ_2 , ϕ are defined as in Lemma 6, and further that

$$(4.41) \quad k = \varpi^{\theta_1} \epsilon_1 k_0, \quad 2 = \varpi^{\theta_2} \epsilon_2 a_0,$$

where

$$(4.42) \quad \begin{cases} \epsilon_1 = (\varpi^{-\theta_1} k, \varpi - 1), & \epsilon_2 = (\varpi^{-\theta_2} 2, \varpi - 1), \\ \epsilon = \epsilon_1 \quad (\phi = \phi_1), & \epsilon = \epsilon_2 \quad (\phi \neq \phi_1), \end{cases}$$

and

$$(4.43) \quad d = \frac{\varpi - 1}{\epsilon};$$

then it follows that

$$(4.441) \quad \gamma_{\varpi} \leq c-1 = \frac{\varpi^{\phi}-1}{\varpi-1} \epsilon \quad (k \text{ even or } \varpi = 2),$$

$$(4.442) \quad \gamma_{\varpi} \leq c = \frac{\varpi^{\phi}-1}{\varpi-1} \epsilon + 1 \quad (k \text{ odd, } \varpi \neq 2).$$

We consider first the cases when $\varpi > 2$. We have shown that $\chi_{\varpi} > 0$ if $N(\varpi^{\phi}, n) > 0$ for every n , and $\chi_{\varpi} > 0$ if $s \geq \gamma'_{\varpi} = \gamma_{\varpi}$ (since $\varpi > 2$). Now, when k is even, any x_k is also an x^2 , so that, using P.N.4, Lemma 20, we get (4.441). If k is odd, this argument does not apply, but it is obvious that the congruence

$$x_0^2 + \sum_{r=1}^s x_r^k \equiv n \pmod{\varpi^{\phi}}$$

cannot have fewer primitive solutions than the congruence

$$\sum_{r=1}^s x_r^k \equiv n \pmod{\varpi^{\phi}}.$$

Thus we have (4.442). So far, of course, $\phi = \phi_1$.

If $\varpi = 2$, $\epsilon_1 = \epsilon_2 = 1$, and there is nothing to prove, for any residue to modulus ϖ^{ϕ} is representable by at most $\varpi^{\phi} - 1$'s.

LEMMA 19. *When $k = 3$, we have*

$$\gamma_{\varpi} \leq 3.$$

Except when $\varpi = 2$ or 3, we have $\theta_1 = \theta_2 = 0$, $\phi = 1$, and so, from (4.442), $\gamma_{\varpi} \leq \epsilon + 1$. But now $\epsilon = 1$ unless ϖ is of the form $3m+1$, so that, except in these cases, $\gamma_{\varpi} \leq 2$. Also, when $\varpi = 2$, we showed in Lemma 17 that $\gamma_{\varpi} = 1$.

There remain the cases $\varpi = 3$, and ϖ of the form $3m+1$. But from P.N.8, Lemma 3, we find that, when $\phi = \phi_1$, every residue $(\bmod \varpi^{\phi})$, except perhaps zero, is representable as the sum of at most $c-1$ cubic residues, which gives us

$$N(\varpi^{\phi}, n) > 0 \quad (s \geq 3, \varpi \nmid n).$$

Thus we need only consider the representations of the residue 0. When p is a cubic residue to modulus ϖ , where ϖ is of the form $3m+1$, we have

$$p^{(\varpi-1)/3} \equiv 1 \pmod{\varpi}.$$

But since ϖ is a prime it must be of the form $6m+1$, from which it follows that $\varpi - p$ is a cubic residue, and thus the residue 0 can be repre-

sented by at most two cubic residues. For the case $\varpi = 3$, since 1 is a quadratic residue and 8 a cubic residue to modulus 9 ($= \varpi^\phi$ when $\varpi = 3$), we have

$$N(\varpi^\phi, 0) > 0 \quad (s \geq 1, \varpi = 3).$$

This completes the proof of the lemma.

4.5. LEMMA 20. *We have*

$$(4.51) \quad \gamma_\varpi \leq k+1 \quad (k \text{ odd}), \quad \gamma_\varpi \leq k \quad (k \text{ even}),$$

except when

$$(a) \quad k = \varpi^{\theta_1} \epsilon \quad (\theta_1 > 0, \varpi > 2),$$

in which case

$$\gamma_\varpi \leq 2k+1 \quad (k \text{ odd}), \quad \gamma_\varpi \leq 2k \quad (k \text{ even}),$$

$$(3) \quad k = 2^{\theta_1} \quad (\theta_1 > 0), \quad \varpi = 2 \quad \text{when} \quad \gamma_\varpi \leq 4k-1,$$

$$(\gamma) \quad k = 2^{\theta_1} 3 \quad (\theta_1 > 2), \quad \varpi = 2 \quad \text{when} \quad \gamma_\varpi \leq \frac{4}{3}k-1.$$

In all cases

$$(4.52) \quad \Gamma(k) \leq 4k-1.$$

Writing $k = \varpi^{\theta_1} \epsilon k_0$, as before, we see that, if $\theta_1 = 0$, $\varpi > 2$, we have, by Lemma 18,

$$\gamma_\varpi \leq c = \epsilon + 1 \leq k+1 \quad (k \text{ odd}), \quad \gamma_\varpi \leq c-1 \leq k \quad (k \text{ even}).$$

If $\varpi = 2$ and $\theta_1 = 0$, then k is odd and $\varpi^\phi = 8$. Now $x^2 \equiv 0$ or $1 \pmod{8}$ and $x^k \equiv 0, \pm 1$ or $\pm 3 \pmod{8}$, so that $N(\varpi^\phi, n) > 0$ for all values of n if, and only if, $s \geq 1$. Also the congruence

$$x_0^2 + \varpi^{k-2u} \sum x_r^k \equiv n_0 \quad \text{or} \quad n_0 \varpi \pmod{\varpi^{3+k-2u}},$$

where $(n_0, 2) = 1$, has no solutions for certain values of n_0 . Hence

$$\gamma_2 = 1 \quad (k \text{ odd}).$$

Thus we need only consider the cases in which $\theta_1 > 0$. Suppose first that $\varpi > 2$, and that k is even. Then we have $\phi = \theta_1 + 1$, and, if $k_0 > 1$,

$$\gamma_\varpi \leq \frac{\varpi^{\theta_1+1}-1}{\varpi-1} \epsilon < \frac{2(\varpi^{\theta_1+1}-\varpi^{\theta_1})}{\varpi-1} \epsilon \leq \varpi^{\theta_1} \epsilon k_0 = k,$$

while, if $k_0 = 1$, we have

$$\gamma_\varpi < \frac{2(\varpi^{\theta_1+1}-\varpi^{\theta_1})}{\varpi-1} \epsilon = 2k.$$

Thus we have the second part of (a), and the first part follows in exactly the same way.

If $\varpi = 2$, and $\phi = \theta_1 + 2$, we have $k = 2^{\theta_1} k_0$, and

$$\gamma_2 \leq \frac{\varpi^\phi - 1}{\varpi - 1} = \varpi^\phi - 1 < k \quad (k_0 > 3).$$

Thus there remain only the cases $\varpi = 2$, $k_0 = 1$ or 3 for consideration. When $k_0 = 1$ we have

$$\gamma_\varpi \leq \varpi^\phi - 1 = 4k - 1,$$

and when $k_0 = 3$,

$$\gamma_\varpi \leq \varpi^\phi - 1 = \frac{4}{3}k - 1.$$

These are the cases (β) and (γ). The result (4.52) is an obvious corollary of the other results of the lemma.

4.6. THEOREM 4. If $s \geq \text{Max}\{\Gamma(k), k-2, 4\}$, then

$$(4.61) \quad \mathbf{S}(n) > C.$$

For, writing

$$\mathbf{S} = \mathbf{C} \prod_{\varpi \mid n} \chi_\varpi \prod_{\varpi \mid n} \chi_\varpi = \mathbf{C} \Pi_1 \Pi_2,$$

and using (3.321), we find that in Π_1 , when $s \geq 4$,

$$\chi_\varpi > 1 - C\varpi^{-\frac{1}{2}s} \geq 1 - C\varpi^{-2} > 1 - \varpi^{-\frac{7}{2}} \quad (\varpi > C).$$

On the other hand, using (3.41), we have in Π_2 , if $s \geq \text{Max}(4, k-2)$,

$$\chi_\varpi > 1 - C\varpi^{\frac{1}{2}-\frac{1}{2}s} \geq 1 - C\varpi^{-\frac{3}{2}} > 1 - \varpi^{-\frac{7}{2}} \quad (\varpi > C).$$

Hence

$$\mathbf{S} > C \prod_{\varpi < C} \chi_\varpi \prod_{\varpi \mid n} (1 - \varpi^{-\frac{7}{2}}) \prod_{\varpi \mid n} (1 - \varpi^{-\frac{7}{2}}) > C,$$

since, when $\varpi < C$, and $s \geq \Gamma(k)$, we have $\chi_\varpi > C$.

LEMMA 21. If $k = 3$, then $A_\varpi(n) = 0$ for any ϖ which is a divisor of n and any s .

We have

$$\begin{aligned} A_\varpi &= \varpi^{-s-1} \sum_p (S_{p, \varpi, 3})^s (S_{p, \varpi, 2}) e^{-(2np\pi i/\varpi)} \\ &= \varpi^{-s-\frac{1}{2}} \sum_p i^{\frac{1}{2}(\varpi-1)s} \left(\frac{p}{\varpi}\right) (S_{p, \varpi, 3})^s \quad (\varpi \mid n), \end{aligned}$$

except when $\varpi = 2$, in which case $S_{p, \varpi, 2} = 0$, and so $A_\varpi = 0$. Also, from Lemma 10, $S_{p, \varpi, 3} = 0$ except when ϖ is of the form $3m+1$. Hence

we need only consider the sum

$$(4.62) \quad \sum_p \left(\frac{p}{\varpi} \right) (S_p, \varpi, 3)^s,$$

when ϖ is of the form $12m+1$ or $12m+7$. Now in the second case it is well known that, if p is a quadratic residue, $\varpi-p$ is not. And

$$\begin{aligned} S_{p, \varpi, 3} &= \sum_{j=0}^{\varpi-1} e_{\varpi}(pj^3) = 1 + 2 \sum_{j=1}^{\frac{1}{2}(\varpi-1)} \cos \frac{2pj^3\pi}{\varpi} \\ &= 1 + 2 \sum_{j=1}^{\frac{1}{2}(\varpi-1)} \cos \frac{2(\varpi-p)j^3\pi}{\varpi} = S_{\varpi-p, p, 3}. \end{aligned}$$

Thus in this case we have $A_{\varpi} = 0$.

Next suppose that ϖ is of the form $12m+1$. In this case

$$S_{p, \varpi, 3} = \sum_{j=0}^{\varpi-1} e_{\varpi}(pj^3) = 1 + 3 \sum r^{pa},$$

where $r = e_q(1)$ and a runs through a complete set of cubic residues. But, if p is a cubic residue, so is pa , and we have

$$S_{p, \varpi, 3} = 1 + 3 \sum r^a.$$

Now there are $4m$ cubic residues*, all of which satisfy the congruence

$$(4.631) \quad p \equiv g^{3h} \pmod{\varpi},$$

where g is a primitive root to modulus ϖ . Of these $2m$ will be sextic residues satisfying the congruence

$$(4.632) \quad p \equiv g^{6h} \pmod{\varpi}.$$

These sextic residues satisfy the congruence

$$(4.633) \quad p \equiv g^{2h'} \pmod{\varpi},$$

and so are quadratic residues. The remaining $2m$ cubic residues satisfy (4.631), but not (4.632), and so must satisfy

$$p \equiv g^{6h+3} \pmod{\varpi},$$

and so are quadratic non-residues. Thus the terms of (4.62) for which p is a cubic residue vanish.

If p is not a cubic residue, it is a cubic non-residue of one or other of two different forms. We have either

$$\left(\frac{p}{\varpi} \right)_3 = e^{2\pi i/3} = \rho, \quad \text{or} \quad \left(\frac{p}{\varpi} \right)_3 = e^{4\pi i/3} = \rho^2.$$

* See G. B. Mathews, *Theory of numbers* (1892), 191, et seq.

Supposing p to be a cubic non-residue of the first kind, we have

$$S_{p, \varpi, 3} = 1 + 3 \sum r^\beta,$$

where β runs through all cubic non-residues of this type. It can be shown, as above, that there are $4m$ such cubic non-residues, of which $2m$ are quadratic residues and $2m$ quadratic non-residues. Thus these terms also vanish in (4.62). A similar argument holds when p is a cubic non-residue of the second type. Thus we have

$$A_{\varpi}(n) = 0. \quad (\varpi | n).$$

THEOREM 5. When $k = 3$, $s \geq \text{Max} \{ \Gamma(k), 3 \}$, we have

$$S(n) \geq C.$$

As in Theorem 4, we write

$$S = C \prod_{\varpi | n} \chi_{\varpi} \prod_{\varpi | n} \chi_{\varpi} = C \Pi_1 \Pi_2.$$

Now in Π_1 , when $s \geq 3$,

$$\chi_{\varpi} > 1 - C\varpi^{-\frac{1}{2}s} > 1 - C\varpi^{-\frac{3}{2}} > 1 - \varpi^{-\frac{3}{4}} \quad (\varpi > C).$$

In Π_2 we have

$$\chi_{\varpi} = 1 + \sum_{\lambda=1}^{\infty} A_{\varpi}^{\lambda} = 1 + \sum_{\lambda=2}^{\lambda_0} A_{\varpi}^{\lambda}.$$

Hence from (3.341) and (3.342), if $\beta > 0$, we have

$$\chi_{\varpi} > 1 - \varpi^{-\beta}.$$

Since $\beta = 0$ from a certain point onward, the result now follows as in Theorem 4.

4.7. LEMMA 22. If $\varpi_1, \varpi_2, \dots, \varpi_r$ is a set of C primes and Q is a number formed from these primes alone, then

$$(4.71) \quad \sum_{Q > \xi} Q^{-C} < C \xi^{-C} \quad (\xi > 1).$$

This is proved in P.N. 6, Lemma 4.

4.8. LEMMA 23. The series $\Sigma A_q(m)$ is absolutely convergent for $m > 0$, $s \geq k$. Also

$$(4.81) \quad \sum_{q > \mu} |A_q(m)| < C \mu^{-C} \{d(m)\}^{k+2} \quad (s \geq k),$$

$$(4.82) \quad |S_2(m)| < C \mu^{-C} \{d(m)\}^{k+2} \quad (s \geq k),$$

$$(4.83) \quad |S_2(m)| < C \mu^{\frac{3}{2}-sa} \quad (s > \frac{3}{2}k).$$

The last result follows from (3.29), for

$$|\mathbf{S}_2(m)| < C \sum_{q > \mu} |A_q(m)| < C \sum_{q > \mu} q^{\frac{1}{2}-as} < C\mu^{\frac{1}{2}-sa},$$

since $\frac{1}{2}-sa < -1$ if $s > \frac{3}{2}k$.

In order to prove (4.81) we write

$$(4.84) \quad q = q'Q = q_1q_2Q,$$

where now q' is the product of all odd factors of q which are prime to k , and q_1 is the product of the primes which divide q' once only, so that

$$(4.85) \quad A_q = A_{q_1}A_{q_2}A_Q.$$

In each term of (4.81) one, at least, of q_1 , q_2 , Q exceeds $\mu^{\frac{1}{2}}$; and if Σ_1 , Σ_2 , Σ_3 are the sums of the terms in which (i) $q_1 > \mu^{\frac{1}{2}}$, (ii) $q_2 > \mu^{\frac{1}{2}}$, (iii) $Q > \mu^{\frac{1}{2}}$, then

$$(4.86) \quad \sum_{q > \mu} |A_q(m)| \leq \Sigma_1 + \Sigma_2 + \Sigma_3.$$

4.9. We write $q_1 = \Pi \varpi_1$, $q_2 = \Pi \varpi_2^\lambda$, and we notice that, by (3.312),

$$|A_q| < Cq^{-c} < C$$

for all q and m , and, in particular, when $q = q_1$, q_2 , or Q . Also

$$A_{\varpi^\lambda}(m) = 0 \quad (\lambda > 1),$$

unless

$$\lambda \leq \lambda_0 = \beta k + k + \phi \quad (v + \eta \geq 2), \quad \lambda_0 = \text{Max}(\beta k + \phi, \beta k + \eta + 1) \quad (v + \eta < 2),$$

and so is certainly zero unless $q_2 | m^{k+1}$. Thus not more than $d(m^{k+1})$ and so not more than $\{d(m)\}^{k+1}$ different values of q_2 can occur in non-zero terms of (4.81). Finally we observe that

$$(4.91) \quad |A_{q_1}| = \Pi |A_{\varpi_1}| < \Pi (C\varpi_1^{\frac{1}{2}-\frac{1}{2}s}) \leq \Pi (C\varpi_1^{-1}) \leq C \Pi \varpi_1^{-\frac{1}{2}} = Cq_1^{-\frac{1}{2}}.$$

It now follows that, if $\xi > 0$, we have

$$|A_{q_1}| < Cq_1^{-\frac{1}{2}} |A_{q_1}|^{\frac{3}{2}},$$

and

$$(4.92) \quad \begin{aligned} \sum_{q_1 > \xi} |A_{q_1}| &< C\xi^{-\frac{1}{2}} \sum_{q_1 > 0} |A_{q_1}|^{\frac{3}{2}} < C\xi^{-\frac{1}{2}} \prod_{\varpi} (1 + |A_{\varpi}|^{\frac{3}{2}}) \\ &< C\xi^{-\frac{1}{2}} \prod_{\varpi | m} (1 + C\varpi^{-\frac{1}{2}}) \prod_{\varpi | m} (1 + C\varpi^{-\frac{3}{2}}) \\ &< C\xi^{-\frac{1}{2}} \prod_{\varpi < C} C \prod_{\varpi | m} 2 < C\xi^{-c} d(m). \end{aligned}$$

In particular, if we take $\xi = \frac{1}{2}$, we have

$$(4.921) \quad \sum_{q_1 > \xi} |A_q| < Cd(m).$$

We now write down upper bounds for Σ_1 , Σ_2 , and Σ_3 . Considering Σ_1 first, we note that, associated with a given q_1 , we have at most $\{d(m)\}^{k+1}$ values of q_2 . Also $|A_{q_2}| < C$, $|A_q| < CQ^{-c}$, so that the sum of these terms of Σ_1 is less than

$$C\{d(m)\}^{k+1}|A_{q_1}|\sum_Q Q^{-c} < C\{d(m)\}^{k+1}|A_{q_1}|$$

by (4.71); and

$$(4.93) \quad \Sigma_1 < C\{d(m)\}^{k+1}\sum_{q > \mu^{\frac{1}{2}}} |A_q| < C\mu^{-c}\{d(m)\}^{k+1}$$

by (4.92).

In the discussion of Σ_2 we use the inequality $|A_{q_2}| < Cq_2^{-c}$. We obtain

$$\Sigma_2 < C\mu^{-c}\{d(m)\}^{k+1}\sum_{q_1 > 0} |A_{q_1}| < C\mu^{-c}\{d(m)\}^{k+2},$$

by (4.921). The discussion of Σ_3 is like that of Σ_1 . We have

$$\begin{aligned} \Sigma_3 &< C\{d(m)\}^{k+1}\sum_{q_1 > 0} |A_{q_1}|\sum_{Q > \mu^{\frac{1}{2}}} Q^{-c} < C\{d(m)\}^{k+2}\sum_{Q > \mu^{\frac{1}{2}}} Q^{-c} \\ &< C\mu^{-c}\{d(m)\}^{k+2}, \end{aligned}$$

by (4.71).

Hence (4.81) follows and (4.82) is an obvious corollary.

PART 2.

§. 1. I collect, in the form of lemmas, the relevant results concerning the functions f and $\mathfrak{F}$.

LEMMA 24. *On a major arc M ,*

$$(5.11) \quad |\Psi| = |f - \psi| < B_\epsilon q^{\kappa+\epsilon},$$

and, on a minor arc m ,

$$(5.12) \quad |f| < B_\epsilon n^{\alpha+\epsilon}.$$

The result (5.11) is Lemma 6 of P.N.6, and the result (5.12) is Lemma 3 of P.N.1.

LEMMA 25. *We have*

$$(5.13) \quad \int_0^{2\pi} |f|^4 dx < B_\epsilon n^{2\alpha+\epsilon}$$

and

$$(5.14) \quad \Sigma \int_M |\psi|^4 |da| < B_\epsilon n^{2a+\epsilon},$$

so that

$$(5.15) \quad \Sigma \int_M |\Psi|^4 |da| < B_\epsilon n^{2a+\epsilon}.$$

The results (5.13), (5.14) can be found in P.N. 2, § 2.3, and (5.15) is an immediate corollary.

LEMMA 26. *We have*

$$(5.16) \quad |\Theta| = |\mathfrak{S} - \theta| < A_\epsilon n^{\frac{1}{2}+\epsilon}.$$

A proof of this is given in Kloosterman's dissertation*.

LEMMA 27. *On a minor arc,*

$$(5.17) \quad |\mathfrak{S}|^2 < An^{1-a}.$$

$$\text{For} \quad |\mathfrak{S}|^2 \leq A(|\theta|^2 + |\Theta|^2) \leq A|\theta|^2 + A_\epsilon(n^{\frac{1}{2}+\epsilon}),$$

and using (3.28) we obtain

$$|\theta|^2 = \pi q^{-2} |S_{p,q,2}|^2 \left| \frac{1}{n} - i\alpha \right|^{-1} \leq An^{-a} \cdot \left(\frac{1}{n^2} + \alpha^2 \right)^{-\frac{1}{2}} \leq An^{1-a},$$

whence we have the result since $1-a > \frac{1}{2} + \epsilon$.

LEMMA 28. *We have*

$$(5.18) \quad \int_0^{2\pi} |\mathfrak{S}|^4 d\alpha < An^{1+\epsilon}.$$

The proof of this can be taken from P.N. 2, § 2.3, since the proof given there and quoted in Lemma 25 (5.13) does not depend upon the fact that $k \geq 2$.

5.2. LEMMA 29. *If $|z| < 1$ and*

$$-\pi + C < \mathbf{I} \log z < \arg z = \pi - C,$$

then

$$(5.21) \quad |g(z)| \leq \left| \sum_{m=1}^{\infty} m^{sa-\frac{1}{2}} z^m \right| < C |\arg z|^{-sa-\frac{1}{2}},$$

$$(5.22) \quad \left| g(z) - \Gamma(sa + \frac{1}{2}) \left(\log \frac{1}{z} \right)^{\frac{1}{2}+sa} \right| < C.$$

* H. D. Kloosterman, *Over het splitsen van geheile positieve getallen in een som van quadraaten* (1924), 24.

These results are well known.

LEMMA 30. *We have*

$$(5.23) \quad \sum_{\xi \mu, q} |F_{p, q} - \psi^s \theta|^2 |d\alpha| < C.$$

This follows immediately, for if $z = xe_q(-p) = e^{-y}$, so that $y = \log(1/z)$, we have

$$|F_{p, q} - \psi^s \theta| = \left| \mathbf{C} \left(\frac{S_{p, q, k}}{q} \right)^s \left(\frac{S_{p, q, 2}}{q} \right) [g \{xe_q(-p)\} - \Gamma(sa + \frac{1}{2}) y^{-s\alpha - \frac{1}{2}}] \right| < C$$

by Lemma 29.

LEMMA 31. *We have*

$$(5.24) \quad \sum_M |d\alpha| < An^{2\alpha-1}.$$

This is Lemma 14 of P.N. 6.

LEMMA 32. *If $s > \frac{1}{2}k$, $\mu > 1$, then*

$$(5.25) \quad \sum_{q > \mu} \sum_p |\psi^s \theta|^2 |d\alpha| < C\mu^{1-2sa} n^{2sa}.$$

For we have

$$\int_{\xi} |\psi|^{2s} |\theta|^2 |d\alpha| = C \left| \frac{S_{p, q, k}}{q} \right|^{2s} \left| \frac{S_{p, q, 2}}{q} \right|^2 \int_{\xi} |y|^{-2sa-1} |d\alpha|$$

and

$$(5.26) \quad \int_{\xi} |y|^{-2sa-1} |d\alpha| < \int_{-\infty}^{\infty} \frac{da}{\{(1/n^2) + a^2\}^{s\alpha + \frac{1}{2}}} = n^{2sa} \int_{-\infty}^{\infty} \frac{du}{(1+u^2)^{s\alpha + \frac{1}{2}}} \\ \leq n^{2sa} \int_{-\infty}^{\infty} \frac{du}{(1+u^2)} = An^{2sa}.$$

Hence, by Lemma 12 (3.28),

$$\sum_{q > \mu} \sum_p |\psi|^{2s} |\theta|^2 |d\alpha| < Cn^{2sa} \sum_{q > \mu} q^{-2sa} < C\mu^{1-2sa} n^{2sa}.$$

LEMMA 33. *If $\mu = n^\gamma$ where $\gamma > 0$, then*

$$(5.27) \quad \int_0^{2\pi} |F_2|^2 d\omega < Cn^{2sa-c} \quad (s \geq k)$$

and

$$(5.28) \quad \int_0^{2\pi} |F_2|^2 d\omega < Cn^{2sa-\lambda_1} \quad (s > \frac{3}{2}k),$$

where

$$(5.29) \quad \lambda_1 = \gamma(2sa-3).$$

If $s \geq k$, we have, by (4.82),

$$\begin{aligned} \int_0^{2\pi} |F_2|^2 d\omega &= 2\pi \sum \{S_2(m)\}^2 m^{2sa-1} |x|^{2m} \\ &< Cn^{-2c} \sum m^{2sa-1} \{d(m)\}^{2k+4} |x|^{2m} \\ &< Cn^{-2c} \sum m^{2sa-1+\epsilon} |x|^{2m} < Cn^{2sa-2c+\epsilon}. \end{aligned}$$

The inequality (5.27) now follows if we take ϵ to be equal to the preceding c .

If $s > \frac{3}{2}k$, the integral is, by (4.83), less than

$$C\mu^{3-2sa} \sum m^{2sa-1} |x|^{2m} < C\mu^{3-2sa} n^{2sa} < Cn^{2sa-\lambda_1}.$$

5.3. Before proving the next lemma we note that the function $F_{P,Q}$ and the arc $\xi_{P,Q}$ are unaltered if P is increased or decreased by a multiple of Q . Also we get the complete set of arcs $\xi_{P,Q}$ for a given Q if P runs through a complete set of residues prime to Q . In what follows, p, q, Q being fixed, we select as our set of residues that for which

$$Q\left(\frac{p}{q} - \frac{1}{2}\right) \leq P < Q\left(\frac{p}{q} + \frac{1}{2}\right).$$

Thus it follows that

$$\left| \frac{p}{q} - \frac{P}{Q} \right| \leq \frac{1}{2}.$$

LEMMA 34. If x lies on $\xi_{p,q}$ and $\xi_{P,Q}$ is a Farey arc other than $\xi_{p,q}$, then

$$(5.31) \quad |F_{P,Q}| < CN^{sa+\frac{1}{2}} |pQ - qP|^{-sa-\frac{1}{2}}.$$

We have

$$(5.32) \quad x = |x|e^{i\omega} = |x|e_q(p)e^{-i\alpha} = |x|e_Q(P)e^{-i\Omega},$$

where

$$(5.321) \quad \Omega = 2\pi\left(\frac{P}{Q} - \frac{p}{q}\right) + \alpha,$$

so that $|\Omega| \leq \pi + |\alpha| < \frac{3}{2}\pi$ if $N \geq 4$. Hence

$$(5.33) \quad |F_{P,Q}| = \left| \mathbf{C}\left(\frac{S_{P,Q,k}}{Q}\right)^s \left(\frac{S_{P,Q,2}}{Q}\right) g(|x|e^{-i\Omega}) \right| < CQ^{-sa-\frac{1}{2}} |\Omega|^{-sa-\frac{1}{2}}$$

by (5.21) and (3.28).

If p, q, P, Q do not correspond to arcs which are adjacent in the dissection, then

$$\left| 2\pi \left(\frac{p}{q} - \frac{P}{Q} \right) \right| \geq \frac{4\pi}{qQ} \geq 2 \frac{2\pi}{qN} \geq 2|a|,$$

and so

$$(5.34) \quad |\Omega| > A \left| \frac{p}{q} - \frac{P}{Q} \right| = \frac{A}{qQ} |pQ - qP|.$$

Thus the result (5.31) follows from (5.33) and (5.34).

If the arcs are adjacent $|pQ - qP| = 1$. Also, since x is not on $\xi_{P, Q}$, we have

$$|\Omega| > \frac{A}{QN} > \frac{A}{QN} |pQ - qP|,$$

so that the conclusion follows as before.

LEMMA 35. If $G = \bar{\Sigma} |F_{P, Q}|$,

where the sign of summation $\bar{\Sigma}$ implies summation over those pairs (P, Q) distinct from (p, q) for which $Q \leq \mu = n^\gamma$, and $s \geq k$, then

$$(5.36) \quad \Sigma \int_{\xi_{p, q}} G^2 |d\alpha| < C n^{2sa - \lambda_2 + \epsilon}$$

where

$$(5.37) \quad \lambda_2 = 2sa^2 - 4\gamma.$$

For G is the sum of less than μ^2 terms, and so

$$G^2 < \mu^2 \bar{\Sigma} |F_{P, Q}|^2 < C \mu^2 N^{2sa+1} \bar{\Sigma} |pQ - qP|^{-2sa-1}$$

by Lemma 34. Thus we have

$$\begin{aligned} \Sigma \int_{\xi_{p, q}} G^2 |d\alpha| &\leq C \mu^2 N^{2sa} \Sigma_{p, q} \frac{1}{q} \bar{\Sigma}_{P, Q} |pQ - qP|^{-2sa-1} \\ &\leq C \mu^2 N^{2sa} \Sigma_{P, Q} \Sigma_{p, q} \frac{1}{q} |pQ - qP|^{-2sa-1}, \end{aligned}$$

where the inner summation is now defined by $q \leq N$, $pQ - qP \neq 0$ and the outer summation by $Q \leq \mu$. But if $Pq \equiv h \pmod{Q}$, where $0 \leq h < Q$, we have

$$\Sigma_p |pQ - qP|^{-2sa-1} < \sum'_{m=-\infty}^{\infty} |mQ + h|^{-2sa-1} < 2 + 2Q^{-2sa-1} \sum_{m=1}^{\infty} m^{-2sa-1} < A,$$

the dash in the second sum denoting that any term for which $mQ + h = 0$

must be omitted. Hence

$$\begin{aligned} \Sigma \int_{\xi} G^2 |d\alpha| &< C\mu^2 N^{2sa} \sum_{p,q} \sum_q \frac{1}{q} < C\mu^4 N^{2sa} \log(N+2) \\ &< Cn^{2sa(1-\alpha)+4\gamma} \log n < Cn^{2sa-\lambda_2+\epsilon}. \end{aligned}$$

5.4. LEMMA 36. If $s \geq k$, $\mu = n^\gamma$, and $\gamma < a$, then

$$(5.41) \quad \Sigma \int_M |F_1 - \psi^s \theta|^2 |d\alpha| < Cn^{2sa-\lambda_3+\epsilon},$$

$$(5.42) \quad \Sigma \int_m |F_1^2| |d\alpha| < Cn^{2sa-\lambda_3+\epsilon},$$

where

$$(5.43) \quad \lambda_3 = \text{Min} \{ \gamma(2sa-1), \lambda_2 \} \geq \text{Min} (\lambda_1, \lambda_2).$$

(i) We have

$$F_1 = F_{p,q} + \bar{\Sigma} F_{p,q},$$

$$(5.441) \quad |F_1 - \psi^s \theta|^2 \leq 2|F_{p,q} - \psi^s \theta|^2 + 2G^2,$$

if $q \leq \mu$; and

$$F_1 = \bar{\Sigma} F_{p,q},$$

$$(5.442) \quad |F_1 - \psi^s \theta|^2 \leq 2|\psi^s \theta|^2 + 2G^2,$$

if $q > \mu$. Hence

$$\begin{aligned} (5.45) \quad \Sigma \int_M |F_1 - \psi^s \theta|^2 |d\alpha| &\leq 2 \sum_{q \leq \mu} \int_M |F_{p,q} - \psi^s \theta|^2 |d\alpha| + 2 \sum_{q > \mu} \int_M |\psi^s \theta|^2 |d\alpha| + 2 \Sigma \int_M G^2 |d\alpha| \\ &\leq 2 \Sigma \int_{\xi} |F_{p,q} - \psi^s \theta|^2 |d\alpha| + 2 \sum_{q > \mu} \int_{\xi} |\psi^s \theta|^2 |d\alpha| + 2 \Sigma \int_{\xi} G^2 |d\alpha| \\ &\leq C + C\mu^{1-2sa} n^{2sa} + Cn^{2sa-\lambda_3+\epsilon} \\ &< C + Cn^{2sa+\gamma(1-2sa)} + Cn^{2sa-\lambda_2+\epsilon}, \end{aligned}$$

by Lemmas 30, 32, and 35. Thus we have the result (5.41).

(ii) Since $\gamma < a$, on every m , $q > \mu$. Hence

$$\Sigma \int_m |F_1|^2 |d\alpha| \leq 2 \sum_{q > \mu} \int_{\xi} |\psi^s \theta|^2 |d\alpha| + 2 \sum_{q > \nu} \int_{\xi} G^2 |d\alpha|,$$

and the conclusion follows as before.

5.5. LEMMA 37. If $s \geq 1$,

$$s_0 = \frac{1}{2}(\frac{1}{2}k-1)K+1,$$

and

$$(5.51) \quad \lambda_4 = \frac{2a}{K}(s-s_0),$$

then

$$(5.52) \quad \Sigma \int_m |f^s \mathfrak{S}|^2 |d\alpha| < C n^{2sa - \lambda_4 + \epsilon}.$$

Making use of (5.12), (5.13), and (5.18), we have

$$\begin{aligned} \Sigma \int_m |f^s \mathfrak{S}|^2 |d\alpha| &\leq \left[\Sigma \int_m |f|^{4s} d\alpha \Sigma \int_m |\mathfrak{S}|^4 d\alpha \right]^{\frac{1}{2}} \\ &\leq \left[C_\epsilon n^{4(s-1)a\kappa + \epsilon} \Sigma \int_m |f|^4 |d\alpha| \Sigma \int_m |\mathfrak{S}|^4 |d\alpha| \right]^{\frac{1}{2}} \\ &\leq C n^{2(s-1)a\kappa + a + \frac{1}{2} + \epsilon}. \end{aligned}$$

But

$$2(s-1)a\kappa + a + \frac{1}{2} = 2sa - 2aK^{-1}(s-1 + \frac{1}{2}K - \frac{1}{4}kK) = 2sa - \lambda_4,$$

which proves the lemma.

5.6. LEMMA 38. If $s \geq \frac{3}{2}k+1$, then

$$(5.611) \quad \Sigma \int_M |f^s \mathfrak{S} - \psi^s \theta|^2 |d\alpha| < C n^{2sa - \lambda_6 + \epsilon}$$

$$\text{where} \quad \lambda_6 = \text{Min}(\tfrac{1}{2}, 2a, \lambda_4) = \text{Min}(\lambda_5, \lambda_4);$$

but if $s \geq k$,

$$(5.612) \quad \Sigma \int_M |f^s \mathfrak{S} - \psi^s \theta|^2 |d\alpha| < C n^{2sa - \lambda'_6 + \epsilon},$$

$$\text{where} \quad \lambda'_6 = \text{Min}\left(\frac{2a}{K}, \lambda_4\right) = \text{Min}(\lambda'_5, \lambda_4).$$

We have

$$(5.621) \quad |f^s \mathfrak{S} - \psi^s \theta|^2 < C(|\psi^s \Theta|^2 + |\psi^{s-1} \Psi \theta|^2 + |\Psi^s \mathfrak{S}|^2) = C(\rho_1 + \rho_2 + \rho_3)$$

say. Now

$$(5.622) \quad \int_{\xi} |y|^{-t} |d\alpha| < n^{t-1} \int_{-\infty}^{\infty} \frac{du}{(1+u^2)^{\frac{t}{2}}} \leq n^{t-1} \int_{-\infty}^{\infty} \frac{du}{1+u^2} < A n^{t-1},$$

provided that $t \geq 2$. Thus, by Lemma 12 and (5.622), we have

$$(5.63) \quad \rho_1 \leq C n^{\frac{3}{2} + \epsilon} \Sigma \int_M \left| \frac{S_{p,q,k}}{q} \right|^{2s} |\mu|^{-as} |d\alpha| \leq C n^{2as - \frac{3}{2} + \epsilon} \sum_{q \leq n^a} q^{1-2as}.$$

But now $1-2as \leq -1$ if $s \geq k$, and so

$$(5.64) \quad \rho_1 \leq Cn^{2as-\frac{1}{2}+\epsilon}.$$

Similarly, using (5.11), we get

$$(5.651) \quad \begin{aligned} \rho_2 &\leq B \sum_{p, q \leq n^a} q^{2\kappa+\epsilon-1-2a(s-1)} \int |y|^{-2a(s-1)-1} |da| \\ &\leq Bn^{2a(s-1)+\epsilon} \sum_{q \leq n^a} q^{2\kappa-1-2a(s-1)}. \end{aligned}$$

When $s \geq \frac{3}{2}k+1$, it is easy to see that $2\kappa-1-2a(s-1) < -1$; but when $s \geq k$ we can replace $q^{2\kappa}$ by its upper bound $n^{2a\kappa}$. Thus

$$(5.652) \quad \rho_2 \leq Bn^{2as-2a+\epsilon} \quad (s \geq \tfrac{3}{2}k+1), \quad \rho_2 \leq Bn^{2as-(2a/K)+\epsilon} \quad (s \geq k).$$

Finally we must consider ρ_3 . We write

$$\rho_3 \leq C \left(\sum_M |\Psi^s \theta|^2 |da| + \sum_M |\Psi^s \Theta|^2 |da| \right) = C(\sigma_1 + \sigma_2)$$

say. Then

$$\sigma_1 \leq Cn^{2sa\kappa+\epsilon} \sum_M \left| \frac{S_{p, q, 2}}{q} \right|^2 |y|^{-1} |da| \leq Cn^{2sa\kappa+\epsilon} \sum_{p, q \leq n^a} \frac{1}{q} \int_{\xi} |y|^{-1} |da|,$$

by (5.11) and (3.28). Also

$$\int_{\xi} |y|^{-1} |da| \leq n \int_{\xi} \frac{da}{(1+n^2\alpha^2)^{\frac{1}{2}}} \leq An^{\epsilon}.$$

Thus

$$(5.661) \quad \sigma_1 \leq Cn^{2sa\kappa+\epsilon} \sum_{q \leq n^a} 1 = Cn^{2sa\kappa+a+\epsilon}.$$

But $2sa\kappa+a \leq 2sa-\lambda_4 = 2sa\kappa-2a\kappa+\frac{1}{2}+a$. Also

$$(5.662) \quad \sigma_2 \leq Cn^{2sa\kappa+\epsilon+\frac{1}{2}} \sum_M \int |da| \leq Cn^{2sa\kappa+2a-\frac{1}{2}+\epsilon}$$

by Lemma 31, and since $2sa\kappa+2a-\frac{1}{2} < 2sa\kappa+a$, the result follows from (5.621), (5.64), (5.651), (5.652), (5.661), and (5.662).

6.1. THEOREM 6. If

$$(6.11) \quad s \geq \tfrac{1}{2}(\tfrac{1}{2}k-1)K+2,$$

then

$$(6.12) \quad \sum_{m=1}^n \{\sigma(m)\}^2 = \sum_{m=1}^n \{r(m)-\rho(m)\}^2 = O(n^{2sa-\epsilon}).$$

We have $1 + \sum \sigma(m) x^m = f^s \mathfrak{S} - F,$

$$1 + \sum \{\sigma(m)\}^2 |x|^{2m} = \frac{1}{2\pi} \int_0^{2\pi} |f^s \mathfrak{S} - F|^2 d\omega,$$

and so

$$\begin{aligned} \sum_{m=1}^n \{\sigma(m)\}^2 &\leq e^2 \sum_{m=1}^n \{\sigma(m)\}^2 e^{-(2m/n)} < A \int_0^{2\pi} |f^s \mathfrak{S} - F|^2 d\omega \\ &\leq A \left[\sum_M |f^s \mathfrak{S} - \psi^s \theta|^2 |d\alpha| + \sum_M |F_1 - \psi^s \theta|^2 |d\alpha| \right. \\ &\quad \left. + \sum_m |f^s \mathfrak{S}|^2 |d\alpha| + \sum_m |F_1|^2 |d\alpha| + \int_0^{2\pi} |F_2|^2 d\omega \right] \\ &= A(J_1 + J_2 + J_3 + J_4 + J_5) \end{aligned}$$

say. Now, by Lemma 38, we have

$$(6.131) \quad J_1 < Cn^{2s\alpha - \lambda_6 + \epsilon} \quad (s \geq \tfrac{3}{2}k + 1),$$

$$(6.132) \quad J_1 < Cn^{2s\alpha - \lambda'_6 + \epsilon} \quad (s \geq k).$$

By Lemma 36 (5.41),

$$(6.14) \quad J_2 < Cn^{2s\alpha - \lambda_3 + \epsilon} \quad (s \geq k).$$

By Lemma 37,

$$(6.15) \quad J_3 < Cn^{2s\alpha - \lambda_4 + \epsilon} \quad (s \geq 1).$$

By Lemma 36 (5.42),

$$(6.16) \quad J_4 < Cn^{2s\alpha - \lambda_3 + \epsilon} \quad (s \geq k).$$

Finally, by Lemma 33,

$$(6.171) \quad J_5 < Cn^{2s\alpha - \lambda_1} \quad (s > \tfrac{3}{2}k),$$

$$(6.172) \quad J_5 < Cn^{2s\alpha - c} \quad (s \geq k).$$

Hence we have

$$(6.18) \quad \sum_{m=1}^n \{\sigma(m)\}^2 < Cn^{2s\alpha - \lambda + \epsilon} \quad (s \geq \tfrac{3}{2}k + 1),$$

where $\lambda = \text{Min}(\lambda_1, \lambda_3, \lambda_4, \lambda_6)$, and

$$(6.19) \quad \sum_{m=1}^n \{\sigma(m)\}^2 < Cn^{2s\alpha - \lambda' + \epsilon} \quad (s \geq k),$$

where $\lambda' = \text{Min}(c, \lambda_3, \lambda_4, \lambda'_6)$. We note that (6.18) is a much better inequality when it applies. Also, since $s_0 = \frac{1}{2}(\frac{1}{2}k-1)K+1=2$ when $k=3$, (6.18) does not apply for all values of $s \geq s_0+1$, but (6.19) does apply, for when $k \neq 3$ we have $\frac{1}{2}(\frac{1}{2}k-1)K+2 > 2^{k-2}+2 > k$.

If we write $\gamma = \frac{1}{4}a$, we see that $\lambda_3 = \text{Min}\{\gamma(2sa-1), \lambda_2\} \geq a$, $\lambda_4 = 2aK^{-1}(s-s_0) \geq 2aK^{-1}$, $\lambda_1 = \gamma(2sa-3) \geq \frac{1}{4}a$, so that $\lambda' = c$ (since it is independent of s). Hence we have the result.

6.2. THEOREM 7. *If $k > 4$ almost all numbers are sums of $\frac{1}{2}(\frac{1}{2}k-1)K+2$ non-negative k -th powers and a square; in particular, the number of 5-th powers sufficient for such a representation is 14, of 6-th powers is 34, of 7-th powers is 82, of 8-th powers is 194, of 9-th powers is 450, and of 10-th powers is 1026. The number of non-representable numbers is in all cases $O(n^{1-c})$.*

If

$$(6.21) \quad s \geq \text{Max}\{\Gamma(k), 4, k-2\},$$

then, by Theorem 3,

$$(6.22) \quad \rho(m) > Cm^{sa}.$$

Now $\frac{1}{2}(\frac{1}{2}k-1)K+1 \geq 4k > \Gamma(k)$ except when $k=3, 4$, or 5 . But $k=5$ is only exceptional in the sense of Lemma 20 if $\varpi=5$, so that $\Gamma(5) \leq 2k+1=11$, while $\frac{1}{2}(\frac{1}{2}k-1)K+1=13$ for $k=5$, and thus (6.22) holds for all cases considered here.

If $r(m)=0$, then $|\sigma(m)| = \rho(m) > Cm^{sa}$. So, if $\mu(n)$ is the number of non-representable numbers between $\frac{1}{2}n$ and n (excluding the $\frac{1}{2}n$, but including the n), we have

$$\sum_{\frac{1}{2}n}^n \{\sigma(m)\}^2 \geq C(\frac{1}{2}n)^{2sa} \mu(n),$$

and thus

$$\mu(n) \leq C \cdot 2^{2sa} n^{-2sa} \sum_1^n \{\sigma(m)\}^2 < Cn^{1-c}$$

by (6.12). Hence the total number of non-representable numbers less than n is less than

$$Cn^{1-c} [1 + (\frac{1}{2})^{1-c} + (\frac{1}{4})^{1-c} + \dots] < Cn^{1-c},$$

which proves the theorem.

6.3. THEOREM 8. *Almost all numbers are the sums of one square and six biquadrates.*

When $k = 4$ the argument of the preceding theorem fails. Suppose that

$$s = 6, \quad t > 0, \quad \frac{1}{2}n < m \leq n,$$

and write $m = m'(16^t | m), \quad m = m''(16^t | m).$

Further, suppose that a zero suffix attached to a number implies that it is not representable by one square and six biquadrates, and that $\mu(n), \mu'(n), \mu''(n)$ are the numbers of the classes m_0, m'_0, m''_0 .

We have, by Lemma 17,

$$(6.31) \quad \rho(m') > A_t m',$$

and so $|\sigma(m'_0)| > A_t m'_0 > A_t n.$

It follows that

$$A_t n^2 \mu'(n) \leq \sum_{\frac{1}{2}n}^n \{\sigma(m)\}^2 < A n^{3-c}.$$

$$(6.32) \quad \mu(n) = \mu'(n) + \mu''(n) < n(A_t n^{-c} + 16^{-t}).$$

If $\nu(n)$ is the total number of non-representable numbers greater than n , then, by (6.32),

$$\nu(n) = \mu(n) + \mu(\frac{1}{2}n) + \mu(\frac{1}{4}n) + \dots < n \left(\frac{2^{1-c}}{2^{1-c}-1} + 2 \cdot 16^{-t} \right) < \epsilon n,$$

by a choice, first of t and then of n . This proves the theorem.

THEOREM 9. *Almost all numbers are the sums of one square and three cubes.*

The proof of this is very like the proof of Theorem 7, though it is not a particular case of that general result since, when $k = 3$, $\frac{1}{2}(\frac{1}{2}k-1)K+1 = 2$. It was proved, however, in Lemma 19 that, when $k = 3$, we have $\gamma_{\infty} \leq 3$, so that $\Gamma(3) \leq 3$, and hence the result follows as in Theorem 7.

7.1. In order to prove the last theorems we need several new functions. We denote by

$$\bar{N}_s(n) = \bar{N}_{k,s}(n)$$

the number of distinct numbers not exceeding n which are representable as the sum of s or fewer k -th powers, and by

$$\nu_s(n) = \nu_{k,s}(n)$$

the number of distinct numbers *not* representable as the sum of one square and s powers, and by

$$a_s = \alpha_s(k)$$

the largest number ξ such that

$$\bar{N}_s(n) > C_\epsilon n^{\xi-\epsilon},$$

for $n \geq n_0(k, s, \epsilon)$.

LEMMA 39. *We have*

$$(7.11) \quad \alpha_s \geq 1 - (1 - 2a)(1 - a)^{s-2}.$$

This is Lemma 22 of P.N. 6.

LEMMA 40. *If*

$$(7.12) \quad s \geq \text{Max} \left\{ \Gamma(k), \frac{3}{2}k + 1, s_0 + 1 \right\},$$

$$\text{then} \quad \nu_s(n) < C_\epsilon n^{1-\lambda_s+\epsilon},$$

$$\text{where} \quad \lambda_s = \text{Min}(\lambda_1, \lambda_3, \lambda_4, \lambda_6), \quad \lambda_1 = \frac{2sa^2(2sa-3)}{2sa+1}.$$

The value of λ_1 used is the value obtained by choosing γ so that $\lambda_1 = \lambda_2$. We then have

$$\gamma = \frac{2sa^2}{2sa+1}.$$

To prove the lemma we note that, if m is non-representable as the sum of one square and s k -th powers, we have

$$\{\sigma_s(m)\}^2 = \{r_s(m) - \rho_s(m)\}^2 = \{\rho_s(m)\}^2 > Cm^{2sk-1},$$

$$\nu_s(n) - \nu_s\left(\frac{1}{2}n\right) < Cn^{1-2sk} \sum_{\frac{1}{2}n}^n \{\sigma_s(m)\}^2.$$

$$< Cn^{1-2sk} C_\epsilon n^{2sk-\lambda_s+\epsilon} < C_\epsilon n^{1-\lambda_s+\epsilon},$$

$$\nu_s(n) < C_\epsilon n^{1-\lambda_s+\epsilon} \left\{ 1 + \left(\frac{1}{2}\right)^{1-\lambda_s+\epsilon} + \left(\frac{1}{4}\right)^{1-\lambda_s+\epsilon} + \dots \right\} < C_\epsilon n^{1-\lambda_s+\epsilon}.$$

LEMMA 41. *If s satisfies (7.12) and if $\lambda_s > 1 - a_s$, and so certainly if*

$$(7.13) \quad \lambda_s > (1 - 2a)(1 - a)^{s'-2},$$

we have

$$(7.14) \quad G(k) \leq s + s'.$$

For if n is not representable by one square and $(s+s')$ k -th powers there are at least $\bar{N}_s(n)$ numbers less than the n which are not representable by s k -th powers. That is

$$\nu_s(n) \geq \bar{N}_s(n),$$

which gives us $1 - \lambda_s \geq \alpha_{s'}$,

a contradiction to the hypothesis $\lambda_s > 1 - \alpha_{s'}$.

7.2. THEOREM 10. *If $k > 4$, then*

$$G(k) \leq \frac{1}{2}(\frac{1}{2}k-1)K + k + 4 + [\xi_k],$$

where
$$[\xi_k] = \frac{(k-2) \log 2 + \log(k-2) - \log k}{\log k - \log(k-1)}.$$

In particular, all large numbers are sums of one square and 28 fifth powers, 54 sixth powers, 112 seventh powers, 232 eighth powers, 500 ninth powers, 1088 tenth powers.

Clearly it is sufficient to verify that numbers s and s' exist which satisfy the conditions

$$(7.21) \quad s + s' = \frac{1}{2}(\frac{1}{2}k-1)K + k + 4 + [\xi_k],$$

$$(7.22) \quad s \geq \Gamma(k), \quad s \geq [\frac{3}{2}k] + 1, \quad s > s_0, \quad s' \geq 2,$$

$$(7.23) \quad \lambda_s > (1-2a)(1-a)^{s'-2}.$$

We take $s = \frac{1}{2}(\frac{1}{2}k-1)K + k = s_0 + k - 1,$

$$s' = [\xi_k] + 4.$$

Then it is obvious that our first condition is satisfied. Next we notice that

$$s_0 = \frac{1}{2}(\frac{1}{2}k-1)K + 1 > \frac{1}{4}(k-2)K > 2^{k-4}k.$$

But it now follows that $s_0 > 4k$ if $k > 5$, and $s_0 > 2k$ when $k = 5$. Hence, in either case,

$$s_0 > \text{Max} \{ \Gamma(k), [\frac{3}{2}k] + 1 \},$$

and since ξ_k is positive when $k > 4$, conditions (7.22) are satisfied.

In order to see that the last condition is satisfied we must find the value of λ_s . We have

$$\lambda_1 = \lambda_2 = \frac{2(s_0 + k - 1)(2s_0 - k - 2)}{k^2(2s_0 + 3k - 2)}, \quad \lambda_4 = \frac{2a(k-1)}{K}, \quad \lambda_6 = \text{Min}(\frac{1}{2}, 2a, \lambda_4).$$

It is easy to see that of these the least is λ_4 , for, first,

$$\lambda_4 = 2^{2-k}(1-a) < 2^{2-k} < \frac{1}{2} \leq 2a \quad (k > 3), \quad \lambda_4 = \frac{1}{2} \cdot \frac{2}{3} < 2a \quad (k = 3).$$

Next, $s_0 \geq 2k+1$ if $s > 4$, and so

$$\lambda_1 > \frac{2}{k} \cdot \frac{9k}{7k} > \frac{2a(k-1)}{K}.$$

Hence we have only to verify the inequality

$$\lambda = \frac{2a(k-1)}{K} > (1-2a)(1-a)^{[\zeta_k]+2},$$

which is obviously satisfied if

$$\left(\frac{k}{k-1}\right)^{[\zeta_k]+2} > \frac{(k-2)K}{2(k-1)},$$

$$\text{or if} \quad [\zeta_k]+1 > \frac{(k-2) \log 2 + \log(k-2) - \log k}{\log k - \log(k-1)} = \zeta_k,$$

which is certainly true.

THEOREM 11. *All large numbers are the sums of one square and 14 biquadrates.*

In this case the general result is untrue, and we have to find different values of s and s' to satisfy the conditions (7.22) and (7.23). We now have $s_0 = 5$, $\Gamma(k) = 7$, $\frac{3}{2}k = 6$, so conditions (7.22) are satisfied if $s \geq 7$, $s' \geq 2$. Taking $s = 10$, we have

$$\lambda = \text{Min} \left[\frac{5}{12}, \frac{5}{16}, \frac{1}{2} \right] = \frac{5}{16},$$

and in order to satisfy (7.23) we need

$$\frac{5}{16} > \frac{1}{2} \left(\frac{3}{4} \right)^{s'-2},$$

i.e. $s' \geq 4$. Hence it follows that $G(4) \leq 14$.

7.3. THEOREM 12. *All large numbers are sums of one square and seven cubes.*

The method of Theorems 8 and 9 do not give the best result here. We find that $\lambda_s = \lambda_1$, and obtain the result $G(3) \leq 9$. If, however, we apply the method of P.N. 1 we obtain the result given. Thus we evaluate

$$\int_{\Gamma} \frac{f^s \mathfrak{G}}{x^{n+1}} dx,$$

as in our previous work. Straightforward application of the method gives

$$(7.31) \quad r_{3,s}(n) = n^{\frac{3}{2}s - \frac{1}{2}} \mathbf{S}(n) + \Sigma \int_M |(f^s \mathfrak{S} - \psi^s \theta)| d\alpha + \Sigma \int_m |f^s \mathfrak{S}| d\alpha \\ = n^{\frac{3}{2}s - \frac{1}{2}} \mathbf{S}(n) + I_1 + I_2,$$

say. Now, by (5.12), (5.13), and (5.18), we have

$$(7.32) \quad I_2 \leq \left\{ \Sigma \int_m |f^{\frac{3}{2}s}| d\alpha \right\}^{\frac{3}{4}} \left\{ \Sigma \int_m |\mathfrak{S}^4| d\alpha \right\} \\ = O \left\{ n^{\frac{1}{4} + \epsilon + \frac{3}{4}\alpha\kappa(\frac{3}{4}s - 4)} \left(\Sigma \int_m |f^4| d\alpha \right)^{\frac{3}{4}} \right\} = O(n^{\frac{3}{2}s - (s/12) + \epsilon}),$$

provided that $s \geq 3$. Now

$$(7.33) \quad I_1 \leq A_s \Sigma \int_M \{ |\psi^s \theta| + |\Psi^s \theta| + |\theta \Psi \psi^{s-1}| + |\theta \Psi^s| \} d\alpha \\ = A(J_1 + J_2 + J_3 + J_4)$$

say. Then, by (5.16),

$$J_1 = O \left(n^{\frac{1}{4} + \epsilon} \Sigma \int_M q^{-\alpha s} y^{-\alpha s} d\alpha \right) = O \left(n^{\alpha s - \frac{3}{4} + \epsilon} \sum_{q \leq n^\alpha} q^{1 - \frac{3}{4}s} \right) \\ = O(n^{\frac{3}{2}s - \frac{3}{4} + \epsilon}),$$

when $s \geq 6$. Also, by (5.16), (5.11), and (5.15), we have

$$J_2 = O(n^{\frac{1}{4} + \epsilon + (s-4)\alpha\kappa + 2\alpha}) = O(n^{\frac{3}{2}s - \frac{1}{2}(s+1) + \epsilon}),$$

provided that $s \geq 4$. In the same way we have

$$J_3 = O \left(\Sigma \int_M q^{\kappa + \epsilon - \alpha(s-1) - \frac{1}{2}} y^{-\alpha(s-1) - \frac{1}{2}} |d\alpha| \right) = O(n^{\frac{1}{2}(s-1) - \frac{1}{2}})$$

if $s \geq 6$. Finally, by (5.11) and (5.24),

$$J_4 \leq \left[\Sigma \int_M |\Psi^{2s}| |d\alpha| \Sigma \int_M |\theta^2| |d\alpha| \right]^{\frac{1}{2}} \\ = O \left(n^{\alpha\kappa s + \frac{1}{2}(2\alpha-1) + \epsilon} \left\{ \sum_{p, q < n^\alpha} q^{-1} \int |y|^{-1} |d\alpha| \right\}^{\frac{1}{2}} \right) = O(n^{\frac{3}{2}s - (s/12) + \epsilon}).$$

Thus, if $s \geq 7$, we have

$$r_{3,s}(n) = n^{\frac{3}{2}s - \frac{1}{2}} \mathbf{S}(n) + O(n^{\frac{3}{2}s - \frac{1}{2} - \lambda}),$$

where $\lambda > 0$, which proves the theorem.

7.4. In a similar way we can obtain results when we have two squares and s k -th powers. If we suppose that $\Gamma(k)$ denotes the corresponding function for this case we can state a theorem corresponding to Theorem 7.

THEOREM 13. *Almost all numbers can be expressed as the sums of two squares and s k -th powers, where*

$$s \geq \text{Max} \{ \Gamma(k), 1 \}.$$

And exactly as in our previous working we obtain

THEOREM 14. *All large numbers can be expressed as the sums of 2 squares and 5 cubes, 9 biquadrates, 15 fifth powers, 22 sixth powers, 30 seventh powers, 40 eighth powers, 51 ninth powers, or 63 tenth powers.*

As Theorem 13 suggests, the difficulty in this case is that of showing that the singular series does not vanish for small values of s .

INDEX.

INDEX OF PAPERS PUBLISHED IN THIS VOLUME ARRANGED ACCORDING TO NAMES OF AUTHORS.

[The numbers refer to pages.]

- Bailey, W. N. Some definite integrals involving Bessel functions, 200.
 ——— An extension of Whipple's theorem on well-poised hypergeometric series, 505.
 Bosanquet, L. S. On Abel's integral equation and fractional integrals, 134.
 ——— On the summability of Fourier series, 144.
 Bromwich, T. J. I'A. The application of operational methods to some electrical problems in diffusion, 209.
 Cartwright, M. L. On the relation between the different types of Abel summation, 81.
 Das Gupta, P. N. The simultaneous system of a quadric surface and two linear complexes, 487.
 Estermann, T. On the representations of a number as the sum of two products, 123.
 Grimshaw, M. E. A convergence theorem for non-absolutely convergent integrals, 439.
 Hille, E. On functions of bounded deviation, 165.
 Hirst, D. M. On expansions which are formally reduced to zero by the operator $\sinh D - cD$, 174.
 Jacob, S. M. The enumeration of the Latin rectangle of depth three by means of a formula of reduction, with other theorems relating to non-clashing substitutions and Latin squares, 329.
 Jeffery, R. L. The continuity of a function defined by a definite integral, 18.
 Kourensky, M. A method of integrating the general form of a system of partial differential equations of the first order in two dependent and two independent variables, 407.
 Lightfoot, N. M. H. The solidification of molten steel, 97.
 Littlewood, D. E. The quadratic equation in quaternions, 40.
 Lob, H., and Richmond, H. W. On a neglected principle in elementary trigonometry, 355.
 Marković, Ž. Sur les solutions de l'équation différentielle linéaire du second ordre à coefficient périodique, 417.
 Morton, V. C. On Darboux lines, 445.
 Naylor, V. A type of infinite matrix, 23.
 Paley, R. E. A. C. A proof of a theorem on averages, 289.
 ——— On some theorems connected with Weierstrass's non-differentiable function, 301.
 Price, A. T. Electromagnetic induction in a conducting sphere, 217.
 Richardson, A. R. Congruences in real quaternions, 117.
 Room, T. G. An extension of the theorem of the fifth associated line, 455.
 Ruse, H. S. Some theorems in the tensor calculus, 225.
 Sayers, K. I. Cesàro summation of the differentiated series of Fourier-Lebesgue series and their allied series, 29.

- Stanley, G. K. The representation of a number as the sum of one square and a number of k -th powers, 512.
- Steen, S. W. P. Divisor functions: their differential equations and recurrence formulae, 47.
- Temple, G. Doubly orthogonal systems of functions. Part I, 231.
——— Doubly orthogonal systems of functions. Part II, 243.
- Verblunsky, S. On summable trigonometric series, 370.
——— The generalized third derivative and its application to the theory of trigonometric series, 387.
- Wilton, J. R. Congruence properties of Ramanujan's function $\tau(n)$, 1.
——— An approximate functional equation for the product of two ζ -functions, 11.
- Young, A. On quantitative substitutional analysis. Fourth paper, 253.
——— On quantitative substitutional analysis. Fifth paper, 273.

INDEX ACCORDING TO SUBJECTS.

[The numbers refer to pages.]

- Bessel Functions: Bailey, W. N., 200.
 Combinatory Analysis: Jacob, S. M., 329.
 Differential Equations: Kourensky, M., 407; Marković, Ž., 417; Steen, S. W. P., 47.
 Electricity: Bromwich, T. J. I'A., 209; Price, A. T., 217.
 Equations of Mathematical Physics: Bromwich, T. J. I'A., 209; Lightfoot, N. M. H., 97; Price, A. T., 217.
 Fourier Series: Bosanquet, L. S., 144; Sayers, K. I., 29; Verblunsky, S., 370, 387.
 Functions of a Complex Variable: Steen, S. W. P., 47.
 Functions of a Real Variable: Hille, E., 165; Hirst, D. M., 174; Jeffery, R. L., 18; Paley, R. E. A. C., 301; Temple, G., 231, 243.
 Geometry: Das Gupta, P. N., 487; Morton, V. C., 445; Room, T. G., 455.
 Heat, Conduction of: Lightfoot, N. M. H., 97.
 Hypergeometric Series: Bailey, W. N., 505.
 Integral Calculus: Bosanquet, L. S., 134; Grimshaw, M. E., 439; Jeffery, R. L., 18.
 Integral Equations: Bosanquet, L. S., 134; Temple, G., 231, 243.
 Invariants: Young, A., 253, 273.
 Mathieu Functions: Marković, Ž., 417.
 Matrices: Naylor, V., 23.
 Modular Functions: Wilton, J. R., 1.
 Numbers, Theory of: Estermann, T., 123; Stanley, G. K., 512; Steen, S. W. P., 47; Wilton, J. R., 1.
 Operational Methods: Bromwich, T. J. I'A., 209; Hirst, D. M., 174.
 Probability: Paley, R. E. A. C., 289.
 Quantics: Young, A., 253, 273.
 Quaternions: Littlewood, D. E., 40; Richardson, A. R., 117.
 Series, Theory of: Bosanquet, L. S., 144; Cartwright, M. L., 81; Hirst, D. M., 174; Paley, R. E. A. C., 301; Sayers, K. I., 29; Verblunsky, S., 370, 387.
 Substitutions: Young, A., 253, 273.
 Tensor Calculus: Ruse, H. S., 225.
 Trigonometry: Lob, H., and Richmond, H. W., 355.
 Zeta-function: Wilton, J. R., 11.

CORRIGENDA.

The following notes have been received from the authors of the papers concerned.

ON THE SUMMABILITY OF FOURIER SERIES.

Certain statements in connection with Theorem 3 of my paper on p. 147, ll. 16 and 22-25, and on p. 151, ll. 4-11, of this volume need emendation. In the proof of Theorem 3, I quoted from Miss Grimshaw the necessary and sufficient condition that the Fourier integral should be summable (C, β) . Since the publication of my paper, however, Miss Grimshaw has kindly pointed out to me that her proof is valid only for $\beta \geq 1$. Therefore in the enunciation of Theorem 3 the condition $\beta \geq 1$ should be added, in contradiction of the remark following it. However, it has not been shown in this case, as in that of Theorem 2, that the condition is essential.

P. 149, l. 14. For $\kappa = \text{Min}(\alpha, 1)$ read $\kappa = \text{Min}(\beta, 1)$.

P. 150, l. 3. For $\omega\eta$ read $\omega\delta$.

P. 150, l. 7. For $\int_{\lambda/\omega}^{\mu/\omega} \dots$ read $\left| \int_{\lambda/\omega}^{\mu/\omega} \dots \right|$.

L. S. BOSANQUET.

ON QUANTITATIVE SUBSTITUTIONAL ANALYSIS.

In my paper printed in pp. 273-288 of this volume, on p. 288, l. 18,

for $R = (A_1 B_1 A_2 B_2 A_3 B_3)(A_4 B_4)$ read $R = (A_1 B_3 A_2 B_1 A_3 B_2)(A_4 B_4)$.

A. YOUNG.



PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

SERIES 2.—VOL. 31.—PART 1.

CONTENTS.

PAGE

CONGRUENCE PROPERTIES OF RAMANUJAN'S FUNCTION $\tau(n)$: J. R. WILTON ...	1
AN APPROXIMATE FUNCTIONAL EQUATION FOR THE PRODUCT OF TWO ζ -FUNCTIONS: J. R. WILTON ...	11
THE CONTINUITY OF A FUNCTION DEFINED BY A DEFINITE INTEGRAL: R. L. JEFFERY ...	18
A TYPE OF INFINITE MATRIX: V. NAYLOR ...	23
CESÀRO SUMMATION OF THE DIFFERENTIATED SERIES OF FOURIER-LEBESGUE SERIES AND THEIR ALLIED SERIES: K. I. SAYERS ...	29
THE QUADRATIC EQUATION IN QUATERNIONS: D. E. LITTLEWOOD ...	40
DIVISOR FUNCTIONS: THEIR DIFFERENTIAL EQUATIONS AND RECURRENCE FORMULAE: S. W. P. STEEN ...	47

Issued May 28th, 1930.

LONDON:

PRINTED BY C. F. HODGSON & SON, LTD.,
2 NEWTON STREET, KINGSWAY, W.C.2;

AND PUBLISHED FOR THE SOCIETY BY

FRANCIS HODGSON, LTD., 15 ST. BRIDE STREET, E.C.4.

THE QUARTERLY JOURNAL OF MATHEMATICS

(OXFORD SERIES)

By arrangement with the proprietors of *The Quarterly Journal of Mathematics* and *The Messenger of Mathematics* the OXFORD UNIVERSITY PRESS will, after 31 March, 1930, continue both journals by a single successor in new format ($10'' \times 6\frac{1}{4}''$) to be called *The Quarterly Journal of Mathematics (Oxford Series)*, and to appear quarterly, commencing in April, 1930.

Messrs. T. W. Chaundy, M.A., E. G. C. Poole, M.A., D.Phil., and W. L. Ferrar, M.A., have consented to act as editors, and will be assisted by an Advisory Board composed of:—

Professor A. L. Dixon, M.A., F.R.S.

Professor E. B. Elliott, M.A., F.R.S.

Professor G. H. Hardy, M.A., F.R.S.

Professor A. E. H. Love, M.A., D.Sc., F.R.S.

Professor E. A. Milne, M.A., M.Sc., F.R.S.

F. B. Pidduck, M.A., D.Sc.

Professor E. C. Titchmarsh, M.A.

The new journal will include articles both in Pure and Applied Mathematics. The price for a single number will be 7s. 6d. net, with an annual subscription (four numbers) of 27s. 6d., post free.

The following are the contents of No. 1, Volume I:—

E. A. Milne: The Motion of a Fluid in a Field of Radiation.

G. Pólya: Some Problems connected with Fourier's work on Transcendental Equations.

F. B. Pidduck: Electrical Notes.

Miss M. L. Cartwright: The Zeros of Certain Integral Functions.

O. Veblen: A Generalization of the Quadratic Differential Form.

All inquiries relating to *The Quarterly Journal of Mathematics (Oxford Series)* should be addressed to the Publisher, Mr. Humphrey Milford, Oxford University Press, Amen House, Warwick Square, London, E.C.4.

OXFORD UNIVERSITY PRESS
LONDON : HUMPHREY MILFORD

THE LONDON MATHEMATICAL SOCIETY is instituted for the promotion and extension of Mathematical Knowledge.

It was founded in 1865, and incorporated under Section 23 of the Companies Act 1867 in 1894.

Every Candidate for Membership must be proposed and recommended, according to a form which the Secretaries will supply, by not less than three Members, of whom one at least, except in special cases to be submitted for the decision of the Council, must certify his personal knowledge of the Candidate.

This form is read at one of the Ordinary or Annual General Meetings, and the Candidate is balloted for at the next ensuing meeting, provided that seven Members are present thereat.

The Candidate, if elected, is informed of his election by one of the Secretaries, and supplied with a copy of the Memorandum and Articles of Association and By-Laws. He must pay the contribution which is due from him within six months after the day of his election, otherwise his election shall be void.

An entrance fee of two guineas is required to be paid by each newly elected Member.

The Annual Subscription to be paid by each Member is two guineas. The subscription due from a newly elected member for his first year of membership is one guinea if his election takes place between March and June inclusive: any Member may compound for his annual subscriptions by the payment of twenty-five guineas in one sum, or in two equal annual instalments. In certain cases the composition fee is reduced. Members of the Society who are also members of the American Mathematical Society are entitled to a reduction of subscription as regards one of the Societies. Details of the arrangements may be obtained from the Secretaries or the Treasurer of either Society.

Every Member is considered liable for his annual subscription until he has signified in writing his desire to resign, and has returned all books and property belonging to the Society.

Annual subscriptions are due at the beginning of each Session (the first Thursday in November of each year), and may be paid to the account of the London Mathematical Society, London County, Westminster and Parr's Bank, 1 St. James's Square, London, S.W.1, or to the Treasurer, Dr. A. E. Western, 35 Essex Street, Strand, London, W.C.2.

The affairs of the Society are directed by the Council and Officers.

The Council consists of sixteen Members, including the Officers, and is chosen from among the Ordinary Members of the Society at the Annual General Meeting in November.

The Officers are a President, Vice-Presidents, a Treasurer, a Librarian, and Secretaries.

The Ordinary Meetings of the Society are now held at Burlington House, W.1, and begin at 5 o'clock in the evening. The Meetings for the Session 1929-30 are on November 14 and December 12, 1929, and January 16, February 6, March 13, April 24, May 15, and June 19, 1930. Cards giving the Programmes of the meetings can be had by Members a few days beforehand on application to C. F. Hodgson & Son, Ltd., 2 Newton Street, Kingsway, London, W.C.2.

At these meetings papers are read and communications made: upon each paper or communication the Chairman invites discussion.

The Council alone decides whether any paper proposed for reading shall or shall not be read.

After a paper has been presented to the Society, it is referred by the Council to one or more Members, who report to the Council on its fitness for publication in the *Proceedings* or the *Journal*. After hearing the reports, the Council decides by ballot whether it shall be printed or not.

Authors of Papers of more than about eight pages in length intended for communication to the Society are requested to furnish to the Secretaries short Abstracts of their Papers, indicating the nature of the methods employed and the character of the results obtained, for the use of the Council and Referees, not for publication. Such an Abstract should be extremely brief; as a rule its length should not exceed the length of half a page of the *Proceedings*.

Communications for the Secretaries may be forwarded to them at the following addresses:—

The University, Birmingham—G. N. WATSON, Sc.D., F.R.S.

St. John's College, Cambridge—F. P. WHITE, M.A.

London Mathematical Society, Burlington House, W.1.

Mathematical Books and Journals

¶ We have a very efficient department for finding the unusual, the rare, and the out-of-print book.

¶ We have a very highly organised department for the supply to Librarians, Institutions, and others, of New Books and Periodicals, both English and Foreign. We can guarantee promptitude and intelligent handling of all orders entrusted to us. Librarians, Executors, and others, desirous of either disposing of or buying Scientific Books or Periodicals, or Books in General Literature, are invited to communicate with us. Valuers for Probate. Catalogues frequently issued and sent post free.

We will purchase the following:

BOOKS:

Archimedes, all Works by.	Newton. Principia. All editions.
Dickson's History of Numbers.	Recordes, all Books by.
Dodgson's Euclid.	Routh's Dynamics of a Particle.
Euclid, Geometrie, Billingsley.	Russell's Principles of Mathematics.
Galilei, all Works by.	Vol. I.
Gilbert. On the Magnet. All editions.	Salmon. Higher Algebra.
Hamilton. Quaternions.	" Higher Plane Curves.
Harkness & Morley. Functions.	Thomson's Vortex Rings.
Mathew's Theory of Numbers.	Todhunter. Theory of Probability.
Maxwell. Scientific Papers.	" Laplace's Functions.
Meyer's Kinetic Theory of Gases.	" Elasticity.
Montucla. Histoire des Mathématiques.	" Attraction.
Napier (John), all Books by.	Wallis, Opera Mathematica.
Newton. Opera Omnia.	Whittaker's Aether and Electricity.
	Young's Sets of Points.

PERIODICALS: Complete Sets or Long Runs.

Acta Mathematica.	LONDON MATHEMATICAL
Annalen der Physik.	SOCIETY PROCEEDINGS.
Archiv der Mathematik und Physik.	Mathematical Gazette.
Astrophysical Journal.	Maths. from "The Educational Times."
Cambridge Philosophical Society	Mathematische Annalen.
Transactions.	Mathematische Encyclopädie.
Comptes Rendus de l'Académie.	Mathematische Zeitschrift.
Jahrbuch der Mathematik.	Messenger of Mathematics.
Journal de Mathématiques. Liouville.	Philosophical Magazine.
Journal de Physique.	Physical Society Proceedings.
Journal für Mathematik. Crelle.	R.S. Philosophical Transactions.
Journal (Quarterly) of Mathematics.	Science Abstracts. Series A and B.
Zeitschrift für Mathematik und Physik. And all standard Mathematical Publications.	

Catalogue 340 Scientific Books and Publications
of Learned Societies. *Free on request.*

A large number of Portraits of famous Scientists are for sale. Lists on application.

HEFFER'S PETTY CURY BOOKSHOP,

W. HEFFER & SONS, LTD.,
CAMBRIDGE, ENGLAND.

Phone: 862. Telegrams & Cables: "Heffer, Cambridge."

